Stochastic Averaging in Continuous Time and Its Applications to Extremum Seeking

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***Abstract*—We investigate stochastic averaging theory in continuous time for locally Lipschitz systems and the applications of this theory to stability analysis of stochastic extremum seeking algorithms. First, we establish a general stochastic averaging principle and some related stability theorems for a class of continuous-time nonlinear systems with stochastic perturbations and remove or weaken several significant restrictions present in existing results: global Lipschitzness of the nonlinear vector field, equilibrium preservation under the stochastic perturbation, global exponential stability of the average system, and compactness of the state space of the perturbation process. Then, we propose a continuous-time extremum seeking algorithm with stochastic excitation signals instead of deterministic periodic signals. We analyze the stability of stochastic extremum seeking for static maps and for general nonlinear dynamic systems.**

***Index Terms*—Extremum seeking, stochastic averaging, stochastic nonlinear control.**

I. Introduction

Background—Extremum Seeking:

EXTREMUM seeking is a non-model based real-time optimization approach for dynamic problems where only limited knowledge of a system is available, such as, that the system has a nonlinear equilibrium map which has a local minimum or maximum. Popular in applications around the middle of the twentieth century, extremum seeking was nearly dormant for several decades until the emergence of a proof of its stability [15], with a subsequent resurgence of interest in extremum seeking for applications [23], [25], [26], [32], including adaptive fluid flow control in [11], control of a tunable thermoacoustic cooler in [18], and control of plasmas in fusion reactors [27], [40], and further theoretical developments [1], [6], [37], [38], including a proof of non-local stability in [39].

Why Stochastic Extremum Seeking?: In existing perturbation-based extremum seeking algorithms, periodic (sinusoidal) excitation signals are primarily used to probe the nonlinearity and estimate its gradient. Biological systems (such as bacterial chemotaxis) do not use periodic probing in locating optima. In man-made source seeking systems, the nearly random motion of the stochastic seeker has its advantage in applications where the seeker itself may be pursued by another pursuer. A seeker, which successfully performs the source finding task but with an unpredictable, nearly random trajectory, is a more challenging target, and is hence less vulnerable, than a deterministic seeker. Furthermore, if the system has high dimensionality, the orthogonality requirements on the elements of the periodic perturbation vector pose an implementation challenge. Thus there is merit in investigating the use of stochastic perturbations within the ES architecture. The first results in that direction were achieved in the discrete-time case [24], using the existing theory of stochastic averaging in the discrete-time case. Source seeking results employing deterministic perturbations in the presence of stochastic noise have been reported in [37], [38], also in discrete time.

Analysis Tools—Stochastic Averaging: With many applications of extremum seeking involving mechanical systems and vehicles, which are naturally modeled by nonlinear continuous-time systems, much need exists for continuous-time extremum seeking algorithms and stability theory. Unfortunately, existing stochastic averaging theorems in continuous time are too restrictive to be applicable to extremum seeking algorithms. Such algorithms violate the global Lipschitz assumptions, do not possess an equilibrium at the extremum, the average system is only locally exponentially stable. In this paper we supply the needed extensions to the stochastic averaging theory and then present stochastic continuous-time extremum seeking algorithms and prove theirs stability.

The basic idea of averaging—be it deterministic or stochastic—is to approximate the original system (periodic, almost periodic, or randomly perturbed) by a simpler "average" system (time-invariant and deterministic) or some approximating diffusion system (a stochastic system simpler than the original one). Starting with applied considerations, averaging principle has been developed in mechanics/dynamics applications [5], [29], [30], [34], [41] as well as within a general mathematical framework [4], [7], [9], [33], for both deterministic dynamics [5], [9], [30], [31] and stochastic dynamics [4], [7], [8], [33]. Stochastic averaging has been the cornerstone of many control and optimization methods, such as in stochastic approximation and adaptive algorithms [3], [17], [22], [35], [36].

Although there exist results on stability analysis based on stochastic averaging for nonlinear systems with stochastic perturbations [4], [12]–[14], all these results are established under

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all or almost all of the following conditions: a) the average system or approximating diffusion system is globally exponentially stable; b) the nonlinear vector field of the original system has bounded derivative or is dominated by some forms of Lyapunov function of the average system; c) the nonlinear vector field of the original system vanishes at the origin for any value of perturbation process (equilibrium condition); and d) the state space of the perturbation process is a compact space. In our recent companion work [21] we weaken some of these conditions and develop stochastic averaging stability theorems on infinite time interval for locally Lipschitz systems, which extends the deterministic general averaging for aperiodic functions to the stochastic case. However, in [21] a uniform convergence condition on the perturbation process or equilibrium condition on the original system is required, which makes those stochastic averaging theorems inapplicable for a stability study of stochastic extremum seeking algorithms.

Results and Organization of the Paper: In this paper, we present new stochastic averaging theorems (Section II and Appendix) that relax the key limiting conditions in the existing stochastic averaging theory. We first introduce the notion of weak stability under random perturbation for general nonlinear systems. This stability notion is a stability robustness property for a deterministic system, relative to perturbations involving a stochastic process, and in the presence of a small parameter. Then we study some stability like properties of the solution of the original system by investigating the weak stability under the random perturbation of the equilibrium of the average system. We present the proofs for the general theorems in Appendix. Then, using the averaging theorems we develop in Section II, in Section III we present stochastic extremum seeking algorithms for static maps and for general nonlinear dynamic systems and prove their stability. We offer some concluding remarks in Section IV.

Notation: X_t^ε denotes a process $X^\varepsilon(t,\omega)$ that depends on the parameter ε, $Y_{t/\varepsilon}$ denotes a process dependent on the parameter ε, defined by $Y_{t/\varepsilon}(\omega)=Y(t/\varepsilon,\omega)$, where $Y_t(\omega)=Y(t,\omega)$ is a given process, and $t\wedge T=\min\{t,T\}$. I denotes the identity matrix; I_A denotes the indicator function of the set A, i.e., $I_A(x)=1$ when $x\in A$, otherwise, $I_A(x)=0$.

II. General Stochastic Averaging

A. Problem Formulation

Consider the following system:

$$\frac{dX_t^\varepsilon}{dt}=a\left(X_t^\varepsilon,Y_{t/\varepsilon}\right),\quad X_0^\varepsilon=x \tag{1}$$

where $X_t^\varepsilon\in\mathbb{R}^n$, $Y_t\in\mathbb{R}^m$ is a time homogeneous continuous Markov process defined on a complete probability space $(\Omega,\mathcal{F},P)$, where Ω is the sample space, $\mathcal{F}$ is the σ-field, and P is the probability measure. The initial condition $X_0^\varepsilon=x$ is deterministic. ε is a small parameter in $(0,\varepsilon_0)$ with fixed $\varepsilon_0>0$. Let $S_Y\subset\mathbb{R}^m$ be the living space of the perturbation process $(Y_t,t\geq 0)$ and note that S_Y may be a proper (e.g., compact) subset of $\mathbb{R}^m$.

The following assumptions will be considered.

Assumption 1: The vector field $a(x,y)$ is a continuous function of (x,y), and for any $x\in\mathbb{R}^n$, it is a bounded function of y. Further it satisfies the locally Lipschitz condition in $x\in\mathbb{R}^n$ uniformly in $y\in S_Y$, i.e., for any compact subset $D\subset\mathbb{R}^n$, there is a constant k_D such that for all $x_1,x_2\in D$ and all $y\in S_Y$, $|a(x_1,y)-a(x_2,y)|\leq k_D|x_1-x_2|$.

Assumption 2: The perturbation process $(Y_t,t\geq 0)$ is ergodic with invariant distribution μ.

Assumption 2 is in contrast to most of the stochastic averaging theory, where, in addition to this assumption, the perturbation process is required to satisfy some form of a strong mixing property. The meaning of ergodicity, in simple terms, is that the time average of a function of the process along the trajectories exists almost surely and equals the space average: $\lim_{T\to\infty}(1/T)\int_0^T f(Y_s)ds=\int_{S_Y}f(y)\mu(dy)$, a.s. for any integrable function $f(\cdot)$. The following are two examples of ergodic stochastic processes (one is a 1-D process and the other is a 2-D process):

1) The Ornstein-Uhlenbeck (OU) process $(Y_t,t\geq 0)$

$$dY_t=-pY_tdt+qdW_t \tag{2}$$

where W_t is a 1-D standard Brownian motion on some probability space $(\Omega,\mathcal{F},P)$. It is known that the OU process is ergodic with invariant distribution $\mu(dx)=(\sqrt{p}/\sqrt{\pi}q)e^{-(px^2/q^2)}dx$ ([28]).

2) Brownian motion on the unit circle $(Y_t,t\geq 0)$

$$Y_t=e^{jW_t}=[\cos(W_t),\sin(W_t)]^T \tag{3}$$

where j is the imaginary unit and W_t is a 1-D Brownian motion which is not necessarily standard in the form $W_0=0$. By Ito's formula, its coordinates Y_{1t} and Y_{2t} satisfy

$$\begin{cases} dY_{1t}=-\frac{1}{2}\cos(W_t)dt-\sin(W_t)dW_t\\ dY_{2t}=-\frac{1}{2}\sin(W_t)dt+\cos(W_t)dW_t.\end{cases}$$

Thus the process $Y_t=[Y_{1t},Y_{2t}]^T$ is the solution of the following stochastic differential equations with initial condition $Y_{10}=\cos(W_0)$ and $Y_{20}=\sin(W_0)$

$$\begin{cases} dY_{1t}=-\frac{1}{2}Y_{1t}dt-Y_{2t}dW_t,\\ dY_{2t}=-\frac{1}{2}Y_{2t}dt+Y_{1t}dW_t,\end{cases}$$

or in matrix notation

$$dY_t=-\frac{1}{2}Y_tdt+BY_tdW_t \tag{4}$$

where $B=\begin{bmatrix}0&-1\\1&0\end{bmatrix}$. On the other hand, the solution of (4) with initial value $Y_0=[\cos(\vartheta),\sin(\vartheta)]^T$ $(\vartheta\in\mathbb{R})$ is

$$\begin{aligned} Y_t&=e^{\left(-\frac{1}{2}I-\frac{1}{2}B^2\right)t+BW_t}Y_0\\ &=e^{BW_t}Y_0=\sum_{k=0}^{\infty}\frac{B^kW^k(t)}{k!}Y_0\quad (B^2=-I)\\ &=\left(I\cos(W_t)+B\sin(W_t)\right)Y_0\\ &=[\cos(\vartheta+W_t),\sin(\vartheta+W_t)]^T\\ &=e^{j(\vartheta+W_t)}.\end{aligned}$$

Therefore Brownian motion on the unit circle $Y_t = [\cos(W_t), \sin(W_t)]^T$ is equivalent to the solution of stochastic differential equation

$$dY_t = -\frac{1}{2}Y_t dt + BY_t d\check{W}_t \tag{5}$$

with initial condition $Y_0 = [\cos(W_0), \sin(W_0)]^T$, where $\check{W}_t$ is a 1-D standard Brownian motion with $\check{W}_0 = 0$. It is known that Brownian motion on the unit circle $(Y_t, t \geq 0)$ is exponentially ergodic and its invariant distribution μ is the uniform measure on $T = \{(x,y) \in \mathbb{R}^2 | x^2 + y^2 = 1\}$ ([2]), i.e., $\mu(S) = l(S)/2\pi$ for any set $S \subset T$, and $l(S)$ denotes the length (Lebesgue measure) of S.

In the extremum seeking applications in this paper, we will use the ergodic processes (2) and (5) as the excitation signals to develop stochastic extremum seeking algorithm.

Assumption 3: For any $x \in \mathbb{R}^n$ and the perturbation process $(Y_t, t \geq 0)$, system (1) has a unique (almost surely) continuous solution on $[0, \infty)$.

Since Y_t is a time homogeneous continuous Markov process, if $a(x,y)$ is globally Lipschitz in (x,y), then the solution of system (1) exists with probability 1 for any $x \in \mathbb{R}^n$ and it is defined uniquely for all $t \geq 0$ (see Section 2 of Chapter 7 of [7]). In this paper, we don't emphasize how to guarantee or prove the existence of the solution of system (1) but just assume that system (1) has a unique (almost surely) continuous solution on $[0, \infty)$. In fact, by Assumption 1, we know that for any trajectory of the perturbing process $(Y_t, t \geq 0)$ and for any $\varepsilon > 0$, system (1) has a unique solution up to a possible explosion time. Assumption 3 implies that there is no finite explosion time for system (1), so that (1) has a continuous solution defined on the whole time interval $[0, +\infty)$.

Under Assumption 2, we obtain the average system of system (1) as follows:

$$\frac{d\overline{X}_t}{dt} = \overline{a}(\overline{X}_t), \qquad \overline{X}_0 = x \tag{6}$$

where

$$\overline{a}(x) = \int_{S_Y} a(x,y)\mu(dy). \tag{7}$$

By Assumption 1, $a(x,y)$ is bounded with respect to y, thus $y \to a(x,y)$ is μ-integrable. So $\overline{a}$ is well defined. For the average system (6), we consider the following assumption.

Assumption 4: The average system (6) has a solution on $[0, \infty)$.

For the original system (1) and the average system (6), we introduce the following definitions.

Definition 1: A solution X_t^ε of system (1) is said to satisfy the property of

1) *weak boundedness* if there exists a constant $M > 0$ such that $\lim_{\varepsilon \to 0} \inf\{t \geq 0 : |X_t^\varepsilon| > M\} = +\infty$, a.s.
2) *weak attractivity* if there exists a point $x^* \in \mathbb{R}^n$ such that for any $\delta > 0$, there exists a constant $T_\delta > 0$ such that $\lim_{\varepsilon \to 0} \inf\{t \geq T_\delta : |X_t^\varepsilon - x^*| > \delta\} = +\infty$, a.s.

By convention, $\inf \emptyset = +\infty$.

Since it is not assumed that system (1) has an equilibrium, we cannot necessarily study the stability of an equilibrium solution of system (1). However, the average system (6) may have stable equilibria. We consider system (1) as a perturbation of the average system (6) and analyze suitably defined stability properties by studying equilibrium stability of (6). To this end, we rewrite system (1) as

$$\frac{dX_t^\varepsilon}{dt} = \overline{a}\left(X_t^\varepsilon\right) + R\left(X_t^\varepsilon, Y_{t/\varepsilon}\right), \quad X_0^\varepsilon = x \tag{8}$$

where $R(X_t^\varepsilon, Y_{t/\varepsilon}) = a(X_t^\varepsilon, Y_{t/\varepsilon}) - \overline{a}(X_t^\varepsilon)$, and consider system (8) as a random perturbation of the average system (6). We assume that $\overline{a}(0) = 0$, and $\overline{X}_t \equiv 0$ is a stable (respectively, asymptotically stable, exponentially stable) solution of system (6).

Definition 2: The solution $\overline{X}_t \equiv 0$ of system (6) is called

1) *weakly stable* under random perturbation $R(\cdot, Y_{t/\varepsilon})$, if for any $\delta > 0$, there exists a constant $r_\delta > 0$ such that for any initial condition $x \in \{\check{x} \in \mathbb{R}^n : |\check{x}| < r_\delta\}$, the solution of system (1) satisfies $\lim_{\varepsilon \to 0} \inf\{t \geq 0 : |X_t^\varepsilon| > \delta\} = +\infty$, a.s.
2) *weakly asymptotically stable* under random perturbation $R(\cdot, Y_{t/\varepsilon})$, if it is weakly stable under random perturbation $R(\cdot, Y_{t/\varepsilon})$ and there exists $r > 0$ such that for any initial condition $x \in \{\check{x} \in \mathbb{R}^n : |\check{x}| < r\}$, the solution X_t^ε of system (1) is weakly attracted to the point 0.
3) *weakly exponentially stable* under random perturbation $R(\cdot, Y_{t/\varepsilon})$, if there exist constants $r > 0$, $c > 0$ and $\gamma > 0$ such that for any initial condition $x \in \{\check{x} \in \mathbb{R}^n : |\check{x}| < r\}$ and any $\delta > 0$, the solution of system (1) satisfies $\lim_{\varepsilon \to 0} \inf\{t \geq 0 : |X_t^\varepsilon| > c|x|e^{-\gamma t} + \delta\} = +\infty$, a.s.

In Definitions 1 and 2, we use the term "weakly" because the properties in question involve $\lim_{\varepsilon \to 0}$ and are defined through the first exit time from a set. In [10], stability concepts that are similarly defined under random perturbations are introduced for a nonlinear system perturbed by a stochastic process. In this paper, the system perturbation also comes from a small parameter ε.

B. Statements of General Results on Stochastic Averaging

Lemma 1: Consider system (1) under Assumptions 1, 2, 3 and 4. Then for any $T > 0$

$$\lim_{\varepsilon \to 0} \sup_{0 \leq t \leq T} \left|X_t^\varepsilon - \overline{X}_t\right| = 0, \quad a.s. \tag{9}$$

This result extends the stochastic averaging for globally Lipschitz systems [20] to locally Lipschitz systems. The result (9) means that $\sup_{0 \leq t \leq T} |X_t^\varepsilon - \overline{X}_t|$ converges to 0 almost surely as $\varepsilon \to 0$, and thus it converges to 0 in probability as $\varepsilon \to 0$, i.e., for any $\delta > 0$, $\lim_{\varepsilon \to 0} P\{\sup_{0 \leq t \leq T} |X_t^\varepsilon - \overline{X}_t| > \delta\} = 0$, which is a stochastic averaging result on finite time in [7] for globally Lipschitz systems. Here we obtain a stronger result (9) for locally Lipschitz systems by using ergodic perturbation process but assuming the existence and uniqueness of the solution.

Let ρ be the metric in the space $\mathbb{C}([0,\infty), \mathbb{R}^n)$ of all the continuous vector functions $f, g \in \mathbb{C}([0,\infty), \mathbb{R}^n)$, defined as $\rho(f,g) = \sum_{k=1}^{\infty}(1/2^k)(1 \wedge (\sup_{0 \leq t \leq k} |f(t) - g(t)|))$. Suppose that the conditions of Lemma 1 hold, and denote $X^\varepsilon(\omega) = (X_t^\varepsilon(\omega), t \geq 0)$, $\overline{X} = (\overline{X}_t, t \geq 0)$. Then by (9) we have $\lim_{\varepsilon \to 0} \rho(X^\varepsilon(\omega), \overline{X}) = 0$, a.s., i.e., X^ε converges almost surely to $\overline{X}$ as $\varepsilon \to 0$. By [16], X^ε also converges weakly to $\overline{X}$ as $\varepsilon \to 0$.

Next, we extend the finite-time approximation result in Lemma 1 to arbitrarily long time intervals.

Theorem 1: Consider system (1) under Assumptions 1, 2, 3 and 4. Then (i) for any $\delta > 0$

$$\lim_{\varepsilon \to 0} \inf \left\{ t \geq 0 : \left| X_t^\varepsilon - \overline{X}_t \right| > \delta \right\} = +\infty, \quad a.s. \tag{10}$$

(ii) there exists a function $T(\varepsilon) : (0, \varepsilon_0) \to \mathbb{N}$ such that for any $\delta > 0$

$$\lim_{\varepsilon \to 0} P \left\{ \sup_{0 \leq t \leq T(\varepsilon)} \left| X_t^\varepsilon - \overline{X}_t \right| > \delta \right\} = 0 \tag{11}$$

where

$$\lim_{\varepsilon \to 0} T(\varepsilon) = +\infty. \tag{12}$$

This is our "approximation theorem" of stochastic averaging for locally Lipschitz systems: as ε tends to zero, the solutions to the original and average systems will remain δ-close for arbitrarily long time in the sense of both almost surely (10) and in probability (11). Based on this result, we investigate the solution property of the original system (1) under the stability of the average system (6).

Theorem 2: Consider system (1) under Assumptions 1, 2, 3 and 4. Then

i) (Boundedness) if the solution of the average system (6) with initial condition $\overline{X}_0 = x$ is bounded, then the solution of system (1) with $X_0^\varepsilon = x$ is weakly bounded, more precisely, for any $c > 0$

$$\lim_{\varepsilon \to 0} \inf \left\{ t \geq 0 : |X_t^\varepsilon| > M + c \right\} = +\infty, \quad a.s. \tag{13}$$

where $M = \sup_{t \geq 0} |\overline{X}_t| < +\infty$.

ii) (Attractivity) if the solution of the average system (6) with initial condition $\overline{X}_0 = x$ converges to $x^* \in \mathbb{R}^n$, i.e., $\lim_{t \to \infty} \overline{X}_t = x^*$, then for the system (1) with $X_0^\varepsilon = x$, whose solution is X_t^ε, the point x^* is weakly attractive, i.e., for any $\delta > 0$, there exists a constant $T_\delta > 0$ such that the solution of system (1) satisfies

$$\lim_{\varepsilon \to 0} \inf \left\{ t \geq T_\delta : |X_t^\varepsilon - x^*| > \delta \right\} = +\infty, \quad a.s. \tag{14}$$

iii) (Stability) if the equilibrium $\overline{X}_t \equiv 0$ of the average system (6) is stable, then it is weakly stable under random perturbation $R(\cdot, Y_{t/\varepsilon})$, i.e., for any $\delta > 0$, there exists a constant $r_\delta > 0$ such that for any initial condition $x \in \{\breve{x} \in \mathbb{R}^n : |\breve{x}| < r_\delta\}$, the solution of system (1) satisfies

$$\lim_{\varepsilon \to 0} \inf \left\{ t \geq 0 : |X_t^\varepsilon| > \delta \right\} = +\infty, \quad a.s. \tag{15}$$

iv) (Asymptotic stability) if the equilibrium $\overline{X}_t \equiv 0$ of the average system (6) is asymptotically stable, then it is weakly asymptotically stable under random perturbation, i.e., for any $\delta > 0$, there exists a constant $r_\delta > 0$, such that for any initial condition $x \in \{\breve{x} \in \mathbb{R}^n : |\breve{x}| < r_\delta\}$, the solution of system (1) satisfies

$$\lim_{\varepsilon \to 0} \inf \left\{ t \geq 0 : |X_t^\varepsilon| > \delta \right\} = +\infty, \quad a.s. \tag{16}$$

and moreover, for any $0 < c < \delta$, there exists a constant $T_\delta^c > 0$ such that

$$\lim_{\varepsilon \to 0} \inf \left\{ t \geq T_\delta^c : |X_t^\varepsilon| > c \right\} = +\infty, \quad a.s. \tag{17}$$

v) (Exponential stability) if the equilibrium $\overline{X}_t \equiv 0$ of the average system (6) is exponentially stable, then it is weakly exponentially stable under random perturbation $R(\cdot, Y_{t/\varepsilon})$, i.e., there exist constants $r > 0$, $c > 0$ and $\gamma > 0$ such that for any initial condition $x \in \{\breve{x} \in \mathbb{R}^n : |\breve{x}| < r\}$, and any $\delta > 0$, the solution of system (1) satisfies

$$\lim_{\varepsilon \to 0} \inf \left\{ t \geq 0 : |X_t^\varepsilon| > c|x|e^{-\gamma t} + \delta \right\} = +\infty, \quad a.s. \tag{18}$$

Moreover, there exists a function $T(\varepsilon) : (0, \varepsilon_0) \to \mathbb{N}$ such that under the conditions of of (i)–(v), the respective results (13)–(18) can be replaced by i) the boundedness result

$$\lim_{\varepsilon \to 0} P \left\{ \sup_{0 \leq t \leq T(\varepsilon)} |X_t^\varepsilon| > M + c \right\} = 0 \tag{19}$$

ii) the attractivity result

$$\lim_{\varepsilon \to 0} P \left\{ \sup_{T_\delta \leq t \leq T(\varepsilon)} |X_t^\varepsilon - x^*| > \delta \right\} = 0 \tag{20}$$

iii) the stability result

$$\lim_{\varepsilon \to 0} P \left\{ \sup_{0 \leq t \leq T(\varepsilon)} |X_t^\varepsilon| > \delta \right\} = 0 \tag{21}$$

iv) the asymptotic stability result

$$(21) \text{ and } \lim_{\varepsilon \to 0} P \left\{ \sup_{T_\delta^c \leq t \leq T(\varepsilon)} |X_t^\varepsilon| > c \right\} = 0 \tag{22}$$

and v) the exponential stability result

$$\lim_{\varepsilon \to 0} P \left\{ \sup_{0 \leq t \leq T(\varepsilon)} \left\{ |X_t^\varepsilon| - c|x|e^{-\gamma t} \right\} > \delta \right\} = 0. \tag{23}$$

Furthermore, (23) is equivalent to

$$\lim_{\varepsilon \to 0} P \left\{ |X_t^\varepsilon| \leq c|x|e^{-\gamma t} + \delta, \forall t \in [0, T(\varepsilon)] \right\} = 1.$$

According to the approximation result (10), we obtain the almost sure stabilities: i)–v) in Theorem 2, while by the approximation result (11), we obtain the stabilities in probabilities: (19)–(23) in Theorem 2. It should be pointed out that the two approximation results (10), (11) together with the corresponding two kinds of stability results in Theorem 2 are independent, but to make the paper more compact, we combine them in one theorem.

The stability results in Theorem 2 are weaker than the stability in probability results in our recent companion work [21], where stronger conditions, not satisfied in stochastic extremum seeking applications, are imposed. Compared with other results on stochastic averaging on infinite time interval [4], [12], [14], we remove or weaken the following restrictions: global Lipschitzness of the nonlinear vector field, equilibrium condition, global exponential stability of the average system, and compactness of the state space of the perturbation process, but impose the assumption of the existence and uniqueness of the solution of the original system.

III. Stochastic Extremum Seeking

A. Extremum Seeking for a Static Map

Consider the quadratic function

$$\varphi(\theta) = \varphi^* + \frac{\varphi''}{2}(\theta - \theta^*)^2 \tag{24}$$

where θ^*, φ^*, and φ'' are unknown. Any $\mathbb{C}^2$ function $\varphi(\theta)$ with an extremum at $\theta = \theta^*$ and with $\varphi'' \neq 0$ can be locally approximated by (24). Without loss of generality, we assume that $\varphi'' > 0$. In this subsection, we design an algorithm to make $\theta - \theta^*$ as small as possible, so that the output $y = \varphi(\theta)$ is driven to its minimum φ^*.

Denote $\hat{\theta}(t)$ as the estimate of the unknown optimal input θ^*. Let

$$\tilde{\theta}(t) = \theta^* - \hat{\theta}(t) \tag{25}$$

denote the estimation error. Instead of the deterministic periodic perturbation [1], here we use a stochastic perturbation to develop a gradient estimate. Let

$$\theta(t) = \hat{\theta}(t) + a\sin(\eta(t)) \tag{26}$$

where $a > 0$ and $(\eta(t), t \geq 0)$ is a stochastic process satisfying

$$\eta = \frac{\sqrt{\varepsilon}q}{\varepsilon s + 1}[\dot{W}], \quad \text{or} \quad \varepsilon d\eta = -\eta dt + \sqrt{\varepsilon}q dW$$

where $q > 0$, $W(t)$, $t \geq 0$ is a 1-D standard Brownian motion defined on some complete probability space $(\Omega, \mathcal{F}, P)$ and $(\sqrt{\varepsilon}q/(\varepsilon s + 1))[\dot{W}]$ denotes a time domain signal obtained as the output of the transfer function $\sqrt{\varepsilon}q/(\varepsilon s + 1)$ when the input is $\dot{W}(t)$. Thus, by (25) and (26), we have

$$\theta - \theta^* = a\sin(\eta) - \tilde{\theta}. \tag{27}$$

Substituting (27) into (24), we have the output

$$y = \varphi^* + \frac{\varphi''}{2}\left(a\sin(\eta) - \tilde{\theta}\right)^2.$$

Now, similar to the deterministic case [1], we design the parameter update law as follows:

$$\frac{d\hat{\theta}}{dt} = -k\sin(\eta)(y - \zeta) \tag{28}$$

$$\frac{d\zeta}{dt} = -h\zeta + hy \tag{29}$$

$$\varepsilon d\eta = -\eta dt + \sqrt{\varepsilon}q dW \tag{30}$$

where $k > 0, h > 0$ are scalar design parameters.

From (30), we have $\eta(t) = \eta(0) - \int_0^t (1/\varepsilon)\eta(s)ds + \int_0^t (q/\sqrt{\varepsilon})dW(s)$. Thus it holds that $\eta(\varepsilon t) = \eta(0) - \int_0^{\varepsilon t}(1/\varepsilon)\eta(s)ds + \int_0^{\varepsilon t}(q/\sqrt{\varepsilon})dW(s) = \eta(0) - \int_0^t \eta(\varepsilon u)du + \int_0^t (q/\sqrt{\varepsilon})dW(\varepsilon u)$. Define $\chi(t) = \eta(\varepsilon t)$ and $B(t) = (1/\sqrt{\varepsilon})W(\varepsilon t)$. Then we have $d\chi(t) = -\chi(t)dt + qdB(t)$, where $B(t)$ is a 1-D standard Brownian motion.

Define the output error variable $e = (h/(s+h))[y] - \varphi^*$. Then we have the following error dynamics:

$$\frac{d\tilde{\theta}^\varepsilon}{dt} = -\dot{\hat{\theta}} = k\sin(\chi(t/\varepsilon)) \cdot \left(\frac{\varphi''}{2}\left(a\sin(\chi(t/\varepsilon)) - \tilde{\theta}^\varepsilon\right)^2 - e^\varepsilon\right) \tag{31}$$

$$\frac{de^\varepsilon}{dt} = h\left(\frac{\varphi''}{2}\left(a\sin(\chi(t/\varepsilon)) - \tilde{\theta}^\varepsilon\right)^2 - e^\varepsilon\right). \tag{32}$$

Now we calculate the average system. It is known that the stochastic process $(\chi(t), t \geq 0)$ is ergodic and has invariant distribution $\mu(dx) = (1/\sqrt{\pi}q)e^{-(x^2/q^2)}dx$. Notice that $e^{-(x^2/q^2)}$ is an even function and $\int_{-\infty}^{+\infty}\cos(2xt)e^{-bt^2}dt = \sqrt{\pi/b}e^{-(x^2/b)}$ (x, b are parameters). Thus we have $\int_{\mathbb{R}}\sin^{2k+1}(x)\mu(dx) = \int_{-\infty}^{+\infty}\sin^{2k+1}(x)(1/\sqrt{\pi}q)e^{-(x^2/q^2)}dx = 0$, $k = 0, 1, 2, \ldots$

$$\int_{\mathbb{R}}\sin^2(x)\mu(dx) = \int_{-\infty}^{+\infty}\sin^2(x)\frac{1}{\sqrt{\pi}q}e^{-\frac{x^2}{q^2}}dx = \frac{1}{2}\left(1 - e^{-q^2}\right). \tag{33}$$

Therefore, by (7), we obtain that the average system of (31), (32) is

$$\frac{d\tilde{\theta}^{ave}}{dt} = -\frac{k\varphi'' a}{2}\left(1 - e^{-q^2}\right)\tilde{\theta}^{ave},$$
$$\frac{de^{ave}}{dt} = h\left(\frac{\varphi'' a^2}{4}\left(1 - e^{-q^2}\right) + \frac{\varphi''}{2}\tilde{\theta}^{ave^2} - e^{ave}\right).$$

By simple calculation, we get the following equilibrium of the above average system

$$\tilde{\theta}^{a,e} = 0, \quad e^{a,e} = \frac{a^2\varphi''}{4}\left(1 - e^{-q^2}\right)$$

with the corresponding Jacobian matrix

$$\begin{bmatrix} -\frac{k\varphi'' a}{2}\left(1 - e^{-q^2}\right) & 0 \\ 0 & -h \end{bmatrix}.$$

Noticing that $\varphi'' > 0$, $k > 0$, $a > 0$, and $h > 0$, we know that the above Jacobian is Hurwitz, i.e., the equilibrium $(0, (a^2\varphi''/4)(1 - e^{-q^2}))$ of the average system is exponentially stable.

According to Theorem 2, for the stochastic extremum seeking algorithm in Fig. 1, we have the following result.

Theorem 3: Consider the static map (24) under the parameter update law (28)–(30). Suppose system (31), (32) has a unique (almost surely) continuous solution on $[0, \infty)$. Then there exist

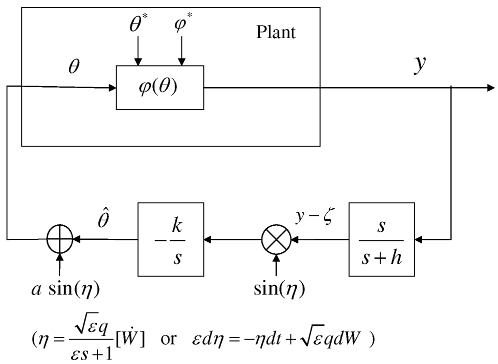

Fig. 1. Stochastic extremum seeking scheme for a static map.

constants $r > 0, c > 0, \gamma > 0$ and a function $T(\varepsilon) : (0, \varepsilon_0) \to \mathbb{N}$ such that for any initial condition $|\Lambda^\varepsilon(0)| < r$ and any $\delta > 0$

$$\lim_{\varepsilon\to 0} \inf\left\{t \geq 0 : |\Lambda^\varepsilon(t)| > c\,|\Lambda^\varepsilon(0)|\,e^{-\gamma t} + \delta\right\} = \infty, \text{ a.s.} \tag{34}$$

and

$$\lim_{\varepsilon\to 0} P\left\{|\Lambda^\varepsilon(t)| \leq c\,|\Lambda^\varepsilon(0)|\,e^{-\gamma t} + \delta, \forall t \in [0, T(\varepsilon)]\right\} = 1 \quad \text{with } \lim_{\varepsilon\to 0} T(\varepsilon) = \infty \tag{35}$$

where $\Lambda^\varepsilon(t) \triangleq (\tilde{\theta}^\varepsilon(t), e^\varepsilon(t)) - (0, (a^2\varphi''/4)(1 - e^{-q^2}))$.

These two results imply that the norm of the error vector $\Lambda^\varepsilon(t)$ exponentially converges, both almost surely and in probability, to below an arbitrarily small residual value δ, over an arbitrarily long time interval, which tends to infinity as ε goes to zero. In particular, the $\tilde{\theta}^\varepsilon(t)$-component of the error vector converges to below δ. To quantify the output convergence to the extremum, for any $\varepsilon > 0$, define a stopping time

$$\tau_\varepsilon^\delta = \inf\left\{t \geq 0 : |\Lambda^\varepsilon(t)| > c\,|\Lambda^\varepsilon(0)|\,e^{-\gamma t} + \delta\right\}.$$

Then by (34) and the definition of $\Lambda^\varepsilon(t)$, we know that $\lim_{\varepsilon\to 0} \tau_\varepsilon^\delta = \infty$, a.s. and

$$\left|\tilde{\theta}^\varepsilon(t)\right| \leq c\,|\Lambda^\varepsilon(0)|\,e^{-\gamma t} + \delta, \quad \forall t \leq \tau_\varepsilon^\delta. \tag{36}$$

Since $y(t) = \varphi(\theta^* + \tilde{\theta}^\varepsilon(t) + a\sin(\eta(t)))$ and $\varphi'(\theta^*) = 0$, we have

$$y(t) - \varphi(\theta^*) = \frac{\varphi''(\theta^*)}{2}\left(\tilde{\theta}^\varepsilon(t) + a\sin(\eta(t))\right)^2 + O\left(\left(\tilde{\theta}^\varepsilon(t) + a\sin(\eta(t))\right)^3\right).$$

Thus by (36), it holds that

$$|y(t) - \varphi(\theta^*)| \leq O(a^2) + O(\delta^2) + C\,|\Lambda^\varepsilon(0)|^2\,e^{-2\gamma t}, \ \forall t \leq \tau_\varepsilon^\delta \tag{37}$$

for some positive constant C. Similarly, by (35)

$$\lim_{\varepsilon\to 0} P\left\{|y(t) - \varphi(\theta^*)| \leq O(a^2) + O(\delta^2) + C\,|\Lambda^\varepsilon(0)|^2\,e^{-2\gamma t}, \ \forall t \in [0, T(\varepsilon)]\right\} = 1 \tag{38}$$

where $T(\varepsilon)$ is a deterministic function with $\lim_{\varepsilon\to 0} T(\varepsilon) = \infty$.

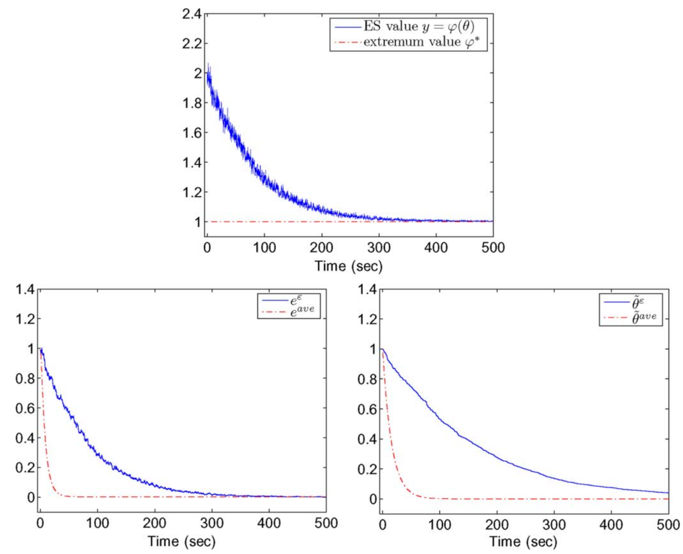

Fig. 2. Stochastic extremum seeking with an OU process perturbation. Top: output and extremum values. Bottom: solutions of the error system and average system.

Inequalities (37) and (38) characterize the asymptotic performance of extremum seeking in Fig. 1 and explain why it is not only important that the perturbation parameter ε be small but also that the perturbation gain a be small.

In the gradient-based estimator (28), stochastic excitation is chosen in the form of $\sin(\eta(t))$. The use of the sinusoidal nonlinearity should not be confused with the use of sinusoidal perturbation signals in deterministic extremum seeking [1]. In the present stochastic design, the sinusoidal nonlinearity is simply used as a bounded function whose role is to guarantee that the vector field of the error system (31), (32) is a bounded function of the perturbation process. We can choose other bounded odd functions to replace sinusoidal functions, such as, $g(x) = xe^{-x^2}$. Corresponding to (33) in calculating the average system, the following integral is computed: $\int_{-\infty}^{+\infty} x^2 e^{-2x^2}(1/\sqrt{\pi}q)e^{-(x^2/q^2)}dx = 1/(2q(2 + (1/q^2))^{3/2}$.

Fig. 2 displays the simulation results with $\varphi^* = 1$, $\varphi'' = 2$, $\theta^* = 0$ in the static map (24) and $a = 0.1$, $h = k = q = 1$, $\varepsilon = 0.25$ in the parameter update law (28)–(30) and initial condition $\tilde{\theta}^\varepsilon(0) = 1$, $e^\varepsilon(0) = 0.99$, $\hat{\theta}(0) = -1$, $\zeta(0) = 1.99$. The simulation result is robust to design parameters, and similar results are obtained for values on this order of magnitude.

The only requirements on the perturbation process in our averaging theorems are ergodicity and the bounded dependence of the vector field on the perturbation. The OU process satisfies these requirements. Brownian motion on the unit circle can also be used as the excitation signal. In the extremum seeking algorithm, we replace the bounded signal $\sin(\eta(t)) = \sin(\chi(t/\varepsilon))$ with the signal $H^T\check{\eta}(t/\varepsilon)$, where $\check{\eta}(t) = [\cos(W(t)), \sin(W(t))]^T$ is Brownian motion on the unit circle and $H = [h_1, h_2]^T$ is a constant vector. By a similar analysis, we obtain results as in (34) and (35), where $\Lambda^\varepsilon(t) \triangleq (\tilde{\theta}^\varepsilon(t), e^\varepsilon(t)) - (0, (a^2\varphi''/4)(h_1^2 + h_2^2))$.

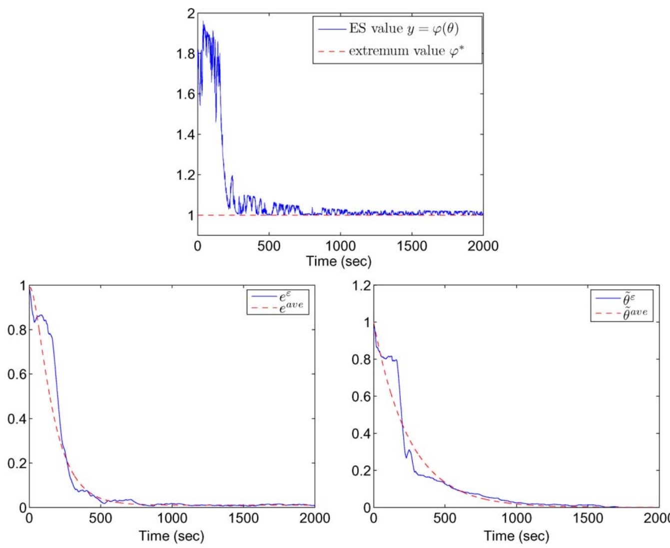

Fig. 3. Stochastic extremum seeking with perturbation based on the Brownian motion on the unit circle. Top: output and extremum values. Bottom: solutions of the error system and average system.

For Brownian motion on the unit circle as the stochastic perturbation, Fig. 3 shows the simulation results with $\varphi^* = 1$, $\varphi'' = 2$, $\theta^* = 0$ in the static map (24), $a = 0.1$, $h = k = h_1 = h_2 = 1$, $\varepsilon = 0.02$ in the parameter update law (28)–(30) and initial condition $\tilde\theta^\varepsilon(0) = 1$, $e^\varepsilon(0) = 0.99$, $\hat\theta(0) = -1$, $\zeta(0) = 1.99$. The simulation is made under the time scale $s = t/\varepsilon$.

By comparing Figs. 2 and 3, we observe that faster convergence is obtained with the Brownian motion on the unit circle as compared to the convergence rate of the average system, whereas with the OU process the actual convergence is poorer than predicted with the average system (this observation is generic and independent of the fact that different parameters were used for the two perturbation processes). The difference between the effects of the two perturbation processes may be due to the "exponentially decaying form" of the invariant distribution of the OU process, in contrast to the uniform distribution of Brownian motion on the unit circle.

B. Stability of Stochastic Extremum Seeking Feedback for General Nonlinear Dynamic Systems

Consider a general SISO nonlinear model

$$\dot x = f(x,u),\qquad y = h(x)$$

where $x \in \mathbb{R}^n$ is the state, $u \in \mathbb{R}$ is the input, $y \in \mathbb{R}$ is the output, and $f : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$ and $h : \mathbb{R}^n \to \mathbb{R}$ are smooth. Suppose that we know a smooth control law

$$u = \alpha(x,\theta)$$

parameterized by a scalar parameter θ. Then the closed-loop system

$$\dot x = f(x,\alpha(x,\theta)) \tag{39}$$

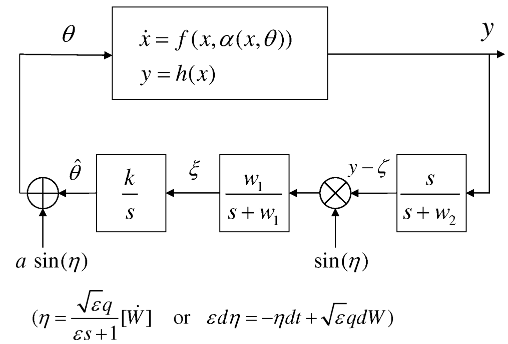

Fig. 4. Stochastic extremum seeking scheme for nonlinear dynamics.

has equilibria parameterized by θ. As the deterministic case [1], we make the following assumptions about the closed-loop system.

Assumption 5: There exists a smooth function $l : \mathbb{R} \to \mathbb{R}^n$ such that

$$f(x,\alpha(x,\theta)) = 0 \text{ if and only if } x = l(\theta).$$

Assumption 6: For each $\theta \in \mathbb{R}$, the equilibrium $x = l(\theta)$ of system (39) is exponentially stable with decay and overshoot constant uniform in θ.

Assumption 7: There exists $\theta^* \in \mathbb{R}$ such that

$$(h\circ l)'(\theta^*) = 0,\qquad (h\circ l)''(\theta^*) < 0.$$

Thus, we assume that the output equilibrium map $y = h(l(\theta))$ has a local maximum at $\theta = \theta^*$.

Our objective is to develop a feedback mechanism which makes the output equilibrium map $y(h(l(\theta)))$ as close as possible to the maximum $y(h(l(\theta^*)))$ but without requiring the knowledge of either θ^* or the functions h and l.

We use a stochastic rather than deterministic perturbation signal and choose the parameter update law as (Fig. 4)

$$\frac{d\hat\theta}{dt} = k\xi,\qquad \frac{d\xi}{dt} = -w_1\xi + w_1(y-\zeta)\sin(\eta) \tag{40}$$
$$\frac{d\zeta}{dt} = -w_2\zeta + w_2 y,$$

$$\varepsilon d\eta = -\eta dt + \sqrt{\varepsilon}q dW \tag{41}$$

where $k > 0$, $w_1 > 0$, $w_2 > 0$, $\epsilon > 0$, and $q > 0$ are design parameters and $(W(t), t \ge 0)$ is a 1-D standard Brownian motion on some probability space $(\Omega, \mathcal{F}, P)$.

Remark 3.1: As in the deterministic case [1], the parameters k, w_1, w_2 need to be chosen as $O(\delta)$, where $0 < \delta \ll \varepsilon$. This yields a decomposition into three time scales (in contrast to two time scales encountered with the static map in Section III-A). The fastest of the three time scales, the time scale associated with the plant $\dot x = f(x,\alpha(x,\theta))$, requires the employment of a singular perturbation argument, whereas averaging analysis is applied to the two lower time scales. Since we do not have a suitable infinite-time stochastic singular perturbation theorem at our disposal, we apply the singular perturbation reduction without invoking a formal theorem, though the reduced and boundary layer systems do satisfy the usual local exponential stability assumptions. In addition, the low-pass filter (40), together with the high-pass filter (29) in Section III-A, is introduced for improved asymptotic performance but is not essential for achieving stability [39].

We define

$$\theta = \hat{\theta} + a\sin(\eta(t))$$

with $a > 0$ and obtain the closed-loop system as

$$\begin{aligned}\frac{dx}{dt} &= f\left(x, \alpha\left(x, \hat{\theta} + a\sin(\eta(t))\right)\right),\\ \frac{d\hat{\theta}}{dt} &= k\xi,\\ \frac{d\xi}{dt} &= -w_1\xi + w_1(y-\zeta)\sin(\eta(t)),\\ \frac{d\zeta}{dt} &= -w_2\zeta + w_2 y,\\ \varepsilon d\eta(t) &= -\eta(t)dt + \sqrt{\varepsilon}q dW(t).\end{aligned}$$

Define $\chi(t) = \eta(\varepsilon t)$ and $B(t) = (1/\sqrt{\varepsilon})W(\varepsilon t)$. Then with the error variables

$$\begin{aligned}\tilde{\theta} &= \hat{\theta} - \theta^*,\\ \tilde{\zeta} &= \zeta - h\circ l(\theta^*)\end{aligned}$$

the closed-loop system is rewritten as

$$\frac{dx}{dt} = f\left(x, \alpha\left(\theta^* + \tilde{\theta} + a\sin(\chi(t/\varepsilon))\right)\right) \quad (42)$$

$$\frac{d}{dt}\begin{bmatrix}\tilde{\theta}\\ \xi\\ \tilde{\zeta}\end{bmatrix} = \tilde{E} \quad (43)$$

where $\tilde{E} \triangleq [k\xi, -w_1\xi + w_1(h(x) - h\circ l(\theta^*) - \tilde{\zeta})\sin(\chi(t/\varepsilon)), -w_2\tilde{\zeta} + w_2(h(x) - h\circ l(\theta^*))]^T$, and $d\chi(t) = -\chi(t)dt + qdB(t)$.

As indicated in Remark 3.1, we employ a singular perturbation reduction, freeze x in (42) at its quasi-steady state value as

$$x = l\left(\theta^* + \tilde{\theta} + a\sin(\chi(t/\varepsilon))\right)$$

and substitute it into (43), and then get the reduced system

$$\frac{d}{dt}\begin{bmatrix}\tilde{\theta}_r\\ \xi_r\\ \tilde{\zeta}_r\end{bmatrix} = \tilde{E}_r$$

where $\tilde{E}_r \triangleq [k\xi_r, -w_1\xi_r + w_1(v(\tilde{\theta}_r + a\sin(\chi(t/\varepsilon))) - \tilde{\zeta}_r)\sin(\chi(t/\varepsilon)), -w_2\tilde{\zeta}_r + w_2 v(\tilde{\theta}_r + a\sin(\chi(t/\varepsilon)))]^T$, and $v(\tilde{\theta}_r + a\sin(\chi(t/\varepsilon))) = h\circ l(\theta^* + \tilde{\theta}_r + a\sin(\chi(t/\varepsilon))) - h\circ l(\theta^*)$. With Assumption 7, we have

$$v(0) = 0 \quad (44)$$

$$v'(0) = (h\circ l)'(\theta^*) = 0 \quad (45)$$

$$v''(0) = (h\circ l)''(\theta^*) < 0. \quad (46)$$

Now we use our stochastic averaging theorems to analyze system (44). According to (7), we obtain that the average system of (44) is

$$\frac{d}{dt}\begin{bmatrix}\tilde{\theta}_r^{ave}\\ \xi_r^{ave}\\ \tilde{\zeta}_r^{ave}\end{bmatrix} = \tilde{E}_r^{ave} \quad (47)$$

where $\tilde{E}_r^{ave} \triangleq [k\xi_r^{ave}, -w_1\xi_r^{ave} + w_1(1/\sqrt{\pi}q)\int_{-\infty}^{+\infty} v(\tilde{\theta}_r^{ave} + a\sin(y))\sin(y)e^{-(y^2/q^2)}dy, -w_2\tilde{\zeta}_r^{ave} + w_2(1/\sqrt{\pi}q)\int_{-\infty}^{+\infty} v(\tilde{\theta}_r^{ave} + a\sin(y))e^{-(y^2/q^2)}dy]^T$.

First, we determine the average equilibrium $(\tilde{\theta}_r^{a,e}, \xi_r^{a,e}, \tilde{\eta}_r^{a,e})$ which satisfies

$$\begin{aligned}&\xi_r^{a,e} = 0\\ &\int_{-\infty}^{+\infty} v\left(\tilde{\theta}_r^{a,e} + a\sin(y)\right)\frac{\sin(y)}{\sqrt{\pi}q}e^{-\frac{y^2}{q^2}}dy = 0\\ &\tilde{\zeta}_r^{a,e} = \frac{1}{\sqrt{\pi}q}\int_{-\infty}^{+\infty} v\left(\tilde{\theta}_r^{a,e} + a\sin(y)\right)e^{-\frac{y^2}{q^2}}dy.\end{aligned}$$

Assume that $\tilde{\theta}_r^{a,e}$ has the form

$$\tilde{\theta}_r^{a,e} = b_1 a + b_2 a^2 + O(a^3) \quad (48)$$

and by (44), (45), define

$$v(x) = \frac{v''(0)}{2}x^2 + \frac{v'''(0)}{3!}x^3 + O(x^4). \quad (49)$$

Then substituting (48) and (49) into (48), we have

$$\begin{aligned}&\int_{-\infty}^{+\infty} v\left(b_1 a + b_2 a^2 + O(a^3) + a\sin(y)\right)\cdot\sin(y)\frac{1}{\sqrt{\pi}q}e^{-\frac{y^2}{q^2}}dy\\ &= \int_{-\infty}^{+\infty}\left[\frac{v''(0)}{2}\left(b_1 a + b_2 a^2 + O(a^3) + a\sin(y)\right)^2\right.\\ &\qquad + \frac{v'''(0)}{3!}\left(b_1 a + b_2 a^2 + O(a^3) + a\sin(y)\right)^3\\ &\qquad \left. + O\left(\left(b_1 a + b_2 a^2 + O(a^3) + a\sin(y)\right)^4\right)\right]\\ &\qquad \cdot\sin(y)\frac{1}{\sqrt{\pi}q}e^{-\frac{y^2}{q^2}}dy\\ &= O(a^4) + v''(0)b_1\left(\frac{1}{2} - \frac{1}{2}e^{-q^2}\right)a^2\\ &\quad + \left[\left(b_2 v''(0) + \frac{v'''(0)}{2}b_1^2\right)\left(\frac{1}{2} - \frac{1}{2}e^{-q^2}\right)\right.\\ &\qquad \left. + \frac{v'''(0)}{6}\left(\frac{3}{8} - \frac{1}{2}e^{-q^2} + \frac{1}{8}e^{-4q^2}\right)\right]a^3 = 0 \quad (50)\end{aligned}$$

where the following facts are used:

$$\frac{1}{\sqrt{\pi}q}\int_{-\infty}^{+\infty}\sin^{2k+1}(y)e^{-\frac{y^2}{q^2}}dy = 0, \quad k = 0, 1, 2, \ldots,$$

$$\frac{1}{\sqrt{\pi}q}\int_{-\infty}^{+\infty}\sin^2(y)e^{-\frac{y^2}{q^2}}dy = \frac{1}{2}-\frac{1}{2}e^{-q^2},$$
$$\frac{1}{\sqrt{\pi}q}\int_{-\infty}^{+\infty}\sin^4(y)e^{-\frac{y^2}{q^2}}dy = \frac{3}{8}-\frac{1}{2}e^{-q^2}+\frac{1}{8}e^{-4q^2}.$$

Comparing the coefficients of the powers of a on the right-hand and left-hand sides of (50), we have

$$b_1 = 0,$$
$$b_2 = -\frac{v'''(0)\left(3-4e^{-q^2}+e^{-4q^2}\right)}{24v''(0)\left(1-e^{-q^2}\right)}$$

and thus by (48), we have

$$\tilde{\theta}_r^{a,e} = -\frac{v'''(0)\left(3-4e^{-q^2}+e^{-4q^2}\right)}{24v''(0)\left(1-e^{-q^2}\right)}a^2+O(a^3).$$

From this equation, together with (48), we have

$$\begin{aligned}\tilde{\zeta}_r^{a,e} &= \int_{-\infty}^{+\infty} v\left(\tilde{\theta}_r^{a,e}+a\sin(y)\right)\frac{1}{\sqrt{\pi}q}e^{-\frac{y^2}{q^2}}dy\\ &= \int_{-\infty}^{+\infty} v\left(b_2a^2+O(a^3)+a\sin(y)\right)\frac{e^{-\frac{y^2}{q^2}}}{\sqrt{\pi}q}dy\\ &= \int_{-\infty}^{+\infty}\left[\frac{v''(0)}{2}\left(b_2a^2+O(a^3)+a\sin(y)\right)^2\right.\\ &\qquad +\frac{v'''(0)}{3!}\left(b_2a^2+O(a^3)+a\sin(y)\right)^3\\ &\qquad \left.+O\left(\left(b_2a^2+O(a^3)+a\sin(y)\right)^4\right)\right]\frac{e^{-\frac{y^2}{q^2}}}{\sqrt{\pi}q}dy\\ &= \frac{a^2v''(0)}{2}\int_{-\infty}^{+\infty}\sin^2(y)\frac{1}{\sqrt{\pi}q}e^{-\frac{y^2}{q^2}}dy+O(a^3)\\ &= \frac{v''(0)\left(1-e^{-q^2}\right)}{4}a^2+O(a^3).\end{aligned}$$

Thus the equilibrium of the average system (47) is

$$\begin{bmatrix}\tilde{\theta}_r^{a,e}\\ \xi_r^{a,e}\\ \zeta_r^{a,e}\end{bmatrix} = \begin{bmatrix}-\frac{v'''(0)\left(3-4e^{-q^2}+e^{-4q^2}\right)}{24v''(0)\left(1-e^{-q^2}\right)}a^2+O(a^3)\\ 0\\ \frac{v''(0)\left(1-e^{-q^2}\right)}{4}a^2+O(a^3)\end{bmatrix}.$$

The Jacobian matrix of the average system (47) at the equilibrium $(\tilde{\theta}_r^{a,e}, \xi_r^{a,e}, \zeta_r^{a,e})$ is

$$J_r^a = \begin{bmatrix}0 & k & 0\\ J_{r21}^a & -w_1 & 0\\ J_{r31}^a & 0 & -w_2\end{bmatrix} \tag{51}$$

where $J_{r21}^a = (w_1/\sqrt{\pi}q)\int_{-\infty}^{+\infty} v'(\tilde{\theta}_r^{a,e} + a\sin(y))\sin(y)e^{-(y^2/q^2)}dy$, and $J_{r31}^a = (w_2/\sqrt{\pi}q)\int_{-\infty}^{+\infty} v'(\tilde{\theta}_r^{a,e} + a\sin(y))e^{-(y^2/q^2)}dy$.

Since J_r^a is block-lower triangular we see that it will be Hurwitz if and only if

$$\int_{-\infty}^{+\infty} v'\left(\tilde{\theta}_r^{a,e}+a\sin(y)\right)\sin(y)e^{-\frac{y^2}{q^2}}dy<0.$$

With Taylor expansion and by calculating the integral, we get

$$\int_{-\infty}^{+\infty} v'\left(\tilde{\theta}_r^{a,e}+a\sin(y)\right)\sin(y)e^{-\frac{y^2}{q^2}}dy = a\sqrt{\pi}qv''(0)\left(\frac{1}{2}-\frac{1}{2}e^{-q^2}\right)+O(a^2). \tag{52}$$

By substituting (52) into (51) we get

$$\det(\lambda I - J_r^a) = \left(\lambda^2+w_1\lambda-\frac{w_1k}{2}v''(0)a \times\left(1-e^{-q^2}\right)+O(a^2)\right)(\lambda+w_2)$$

which proves that J_r^a is Hurwitz for sufficiently small a. This implies that the equilibrium of the average system is exponentially stable for sufficiently small a. Then according to Theorem 2, we have the following result for stochastic extremum seeking algorithm in Fig. 4.

Theorem 4: Consider system (44) under Assumption 7. Suppose system (44) has a unique continuous solution on $[0,\infty)$. Then there exists a constant $a^* > 0$ such that for any $0 < a < a^*$ there exist constants $r > 0, c > 0, \gamma > 0$ and a function $T(\varepsilon) : (0,\varepsilon_0) \to \mathbb{N}$ such that for any initial condition $|\Delta^{\varepsilon,a}(0)| < r$, and any $\delta > 0$

$$\liminf_{\varepsilon\to 0}\left\{t\geq 0 : |\Delta^{\varepsilon,a}(t)| > c|\Delta^{\varepsilon,a}(0)|e^{-\gamma t}+\delta\right\} = \infty, \text{ a.s.} \tag{53}$$

and

$$\lim_{\varepsilon\to 0}P\left\{|\Delta^{\varepsilon,a}(t)|\leq c|\Delta^{\varepsilon,a}(0)|e^{-\gamma t}+\delta, \forall t\in[0,T(\varepsilon)]\right\} = 1 \quad \text{with } \lim_{\varepsilon\to 0}T(\varepsilon)=\infty \tag{54}$$

where

$$\Delta^{\varepsilon,a}(t) \triangleq \left(\tilde{\theta}_r(t),\xi_r(t),\tilde{\zeta}_r(t)\right) - \left(-\frac{v'''(0)\left(3-4e^{-q^2}+e^{-4q^2}\right)}{24v''(0)\left(1-e^{-q^2}\right)}a^2+O(a^3), 0, \frac{v''(0)\left(1-e^{-q^2}\right)}{4}a^2+O(a^3)\right).$$

These results imply that the norm of the error vector $\Delta^{\varepsilon,a}(t)$ exponentially converges, both almost surely and in probability, to below an arbitrarily small residual value δ over an arbitrary large time interval, which tends to infinity as the perturbation parameter ϵ goes to zero. In particular, the $\tilde{\theta}^{\epsilon}(t)$-component of the error vector converges to below δ. To quantify the output convergence to the extremum, we define a stopping time

$$\tau_\varepsilon^\delta = \inf\left\{t\geq 0 : |\Delta^{\varepsilon,a}(t)| > c|\Delta^{\varepsilon,a}(0)|e^{-\gamma t}+\delta\right\}.$$

Then by (53) and the definition of $\Delta^{\varepsilon,a}(t)$, we know that $\lim_{\varepsilon\to 0}\tau_\varepsilon^\delta = \infty$, a.s. and $|\tilde{\theta}_r(t) - (-(v'''(0)(3-4e^{-q^2}+e^{-4q^2})/24v''(0)(1-e^{-q^2}))a^2+O(a^3))| \le c|\Delta^{\varepsilon,a}(0)|e^{-\gamma t}+\delta$, $\forall t \le \tau_\varepsilon^\delta$, which implies that

$$\left|\tilde{\theta}_r(t)\right| \le O(a^2) + c\left|\Delta^{\varepsilon,a}(0)\right| e^{-\gamma t} + \delta,\ \forall t \le \tau_\varepsilon^\delta. \quad (55)$$

Since $y(t) = h(l(\theta^*+\tilde{\theta}_r(t)+a\sin(\eta(t))))$ and $(h \circ l)'(\theta^*) = 0$, we have

$$y(t) - h\circ l(\theta^*) = \frac{(h\circ l)''(\theta^*)}{2}\left(\tilde{\theta}_r(t) + a\sin(\eta(t))\right)^2 + O\left(\left(\tilde{\theta}_r + a\sin(\eta(t))\right)^3\right).$$

Thus by (55), it holds that

$$|y(t) - h\circ l(\theta^*)| \le O(a^2) + O(\delta^2) + C\left|\Delta^{\varepsilon,a}(0)\right|^2 e^{-2\gamma t}, \quad \forall t \le \tau_\varepsilon^\delta$$

for some positive constant C. Similarly, by (54)

$$\lim_{\varepsilon\to 0} P\left\{|y(t) - h\circ l(\theta^*)| \le O(a^2) + O(\delta^2) + C\left|\Delta^{\varepsilon,a}(0)\right|^2 e^{-2\gamma t},\ \forall t\in[0,T(\varepsilon)]\right\} = 1$$

where $T(\varepsilon)$ is a deterministic function with $\lim_{\varepsilon\to 0} T(\varepsilon) = \infty$.

IV. Conclusion

In this paper, we proposed continuous-time extremum seeking algorithms that employ stochastic excitation signals instead of deterministic periodic signals. Since there are no existing stochastic averaging theorems applicable for the analysis of stability of the extremum seeking algorithms, we developed a set of general stochastic averaging theorems for a class of nonlinear systems with stochastic perturbations. These theorems characterize the behavior of the original system by investigating the weak stability under random perturbation of the equilibrium of the average system. They guarantee both almost sure stability and stability in probability. Our stochastic averaging theorems represent the stochastic analogs to the deterministic *general averaging* theorems for systems with aperiodic vector fields. Compared with our companion work [21], the present paper provides weaker but more practically usable stability results, under much weaker conditions. With them we prove the stability of stochastic extremum seeking algorithms for a static maps and for general nonlinear dynamic systems. In future work, we will explore more specific applications of general stochastic averaging theorems established in this paper, including stochastic source seeking for nonholonomic vehicle.

Appendix

A. *Proof of Lemma 1*

Fix $T > 0$, and denote

$$M' = \sup_{0\le t\le T} |\overline{X}_t|. \quad (A1)$$

Since $(\overline{X}_t, t \ge 0)$ is continuous and $[0,T]$ is a compact set, we have that $M' < +\infty$. Denote $M = M'+1$. For any $\varepsilon \in (0,\varepsilon_0)$, define a stopping time τ_ε by

$$\tau_\varepsilon = \inf\{t \ge 0 : |X_t^\varepsilon| > M\}.$$

By the definition of M (noting that $|x| = |\overline{X}_0| \le M'$) and the continuity of the sample path of $(X_t^\varepsilon, t\ge 0)$, we know that $0 < \tau_\varepsilon \le +\infty$, and if $\tau_\varepsilon < +\infty$, then

$$\left|X_{\tau_\varepsilon}^\varepsilon\right| = M. \quad (A2)$$

From (1) and (6), we have that for any $t \ge 0$

$$\begin{aligned} X_t^\varepsilon - \overline{X}_t &= \int_0^t \left[a\left(X_s^\varepsilon, Y_{s/\varepsilon}\right) - \overline{a}(\overline{X}_s)\right] ds \\ &= \int_0^t \left[a\left(X_s^\varepsilon, Y_{s/\varepsilon}\right) - a(\overline{X}_s, Y_{s/\varepsilon})\right] ds \\ &\quad + \int_0^t \left[a(\overline{X}_s, Y_{s/\varepsilon}) - \overline{a}(\overline{X}_s)\right] ds. \end{aligned} \quad (A3)$$

By Assumption 1, we obtain that for any $s \le \tau_\varepsilon \wedge T$

$$\left|a\left(X_s^\varepsilon, Y_{s/\varepsilon}\right) - a(\overline{X}_s, Y_{s/\varepsilon})\right| \le k_M \left|X_s^\varepsilon - \overline{X}_s\right| \quad (A4)$$

where k_M is the Lipschitz constant of $a(x,y)$ with respect to the compact subset $\{x\in\mathbb{R}^n : |x| \le M\}$ of $\mathbb{R}^n$.

Thus by (A3) and (A4), we have that if $t \le \tau_\varepsilon \wedge T$, then

$$\left|X_t^\varepsilon - \overline{X}_t\right| \le k_M \int_0^t \left|X_s^\varepsilon - \overline{X}_s\right| ds + \left|\int_0^t \left[a(\overline{X}_s, Y_{s/\varepsilon}) - \overline{a}(\overline{X}_s)\right] ds\right|. \quad (A5)$$

Define

$$\begin{aligned} \Delta_t^\varepsilon &= \left|X_t^\varepsilon - \overline{X}_t\right|, \\ \alpha(\varepsilon) &= \sup_{0\le t\le T}\left|\int_0^t \left[a(\overline{X}_s, Y_{s/\varepsilon}) - \overline{a}(\overline{X}_s)\right] ds\right|. \end{aligned} \quad (A6)$$

Then by (A5) and Gronwall's inequality, we have

$$\sup_{0\le t\le \tau_\varepsilon\wedge T} \Delta_t^\varepsilon \le \alpha(\varepsilon) e^{k_M(\tau_\varepsilon\wedge T)} \le \alpha(\varepsilon) e^{k_M T}. \quad (A7)$$

Since $(\overline{X}_t, t \ge 0)$ is a deterministic continuous function, by Assumption 1 and Birkhoff ergodic theorem (see e.g. Liptser and Shiryaev [19]), we have that

$$\lim_{\varepsilon\to 0}\alpha(\varepsilon) = 0,\ a.s. \quad (A8)$$

For the reader's convenience, we give the detailed proof of (A8) in Appendix F.

It follows from (A6), (A7) and (A8) that

$$\limsup_{\varepsilon\to 0}\sup_{0\le t\le\tau_\varepsilon\wedge T}\left|X_t^\varepsilon-\overline{X}_t\right|=0,\ a.s. \tag{A9}$$

Thus by (A1) and (A9), we have

$$\begin{aligned}\limsup_{\varepsilon\to 0}&\sup_{0\le t\le\tau_\varepsilon\wedge T}|X_t^\varepsilon|\\ &\le\limsup_{\varepsilon\to 0}\left[\sup_{0\le t\le\tau_\varepsilon\wedge T}\left|X_t^\varepsilon-\overline{X}_t\right|+\sup_{0\le t\le\tau_\varepsilon\wedge T}|\overline{X}_t|\right]\\ &\le\limsup_{\varepsilon\to 0}\sup_{0\le t\le\tau_\varepsilon\wedge T}\left|X_t^\varepsilon-\overline{X}_t\right|+M'\\ &=M'<M,\ a.s.\end{aligned} \tag{A10}$$

By (A2) and (A10), we obtain that for almost every $\omega\in\Omega$, there exists an $\varepsilon_0(\omega)>0$ such that for any $0<\varepsilon<\varepsilon_0(\omega)$

$$\tau_\varepsilon(\omega)>T. \tag{A11}$$

Thus by (A9) and (A11), we obtain that $\limsup_{\varepsilon\to 0}\sup_{0\le t\le T}|X_t^\varepsilon-\overline{X}_t|=0$, a.s. Hence (9) holds. The proof is completed.

B. *Proof of Approximation Result (10) of Theorem 1*

Define

$$\Omega'=\left\{\omega:\limsup_{\varepsilon\to 0}\sup_{0\le t\le T}\left|X_t^\varepsilon(\omega)-\overline{X}_t\right|=0,\ \forall T\in\mathbb{N}\right\}. \tag{A12}$$

Then by Lemma 1, we have

$$P(\Omega')=1. \tag{A13}$$

Let $\delta>0$. For $\varepsilon\in(0,\varepsilon_0)$, define a stopping time τ_ε^δ by

$$\tau_\varepsilon^\delta=\inf\left\{t\ge 0:\left|X_t^\varepsilon-\overline{X}_t\right|>\delta\right\}. \tag{A14}$$

By the fact that $X_0^\varepsilon-\overline{X}_0=0$, and the continuity of the sample paths of $(X_t^\varepsilon,t\ge 0)$ and $(\overline{X}_t,t\ge 0)$, we know that $0<\tau_\varepsilon^\delta\le+\infty$, and if $\tau_\varepsilon^\delta<+\infty$, then

$$\left|X_{\tau_\varepsilon^\delta}^\varepsilon-\overline{X}_{\tau_\varepsilon^\delta}\right|=\delta. \tag{A15}$$

For any $\omega\in\Omega'$, by (A12) and (A15), we get that for any $T\in\mathbb{N}$, there exists $\varepsilon_0(\omega,\delta,T)>0$ such that for any $0<\varepsilon<\varepsilon_0(\omega,\delta,T)$

$$\tau_\varepsilon^\delta(\omega)>T$$

which implies that

$$\lim_{\varepsilon\to 0}\tau_\varepsilon^\delta(\omega)=+\infty. \tag{A16}$$

Thus it follows from (A13) and (A16) that $\lim_{\varepsilon\to 0}\tau_\varepsilon^\delta=+\infty$, a.s. The proof is completed.

C. *Preliminary Lemmas for the Proof of Approximation Result (11) of Theorem 1*

Lemma 2: Consider system (1) under Assumptions 1, 2, 3 and 4. Then for any $\delta>0$, $0<\check{\delta}<1$, there exists a decreasing sequence $\{\varepsilon_T\}_{T\in\mathbb{N}}$ of positive real numbers satisfying $\varepsilon_T\downarrow 0$ as $T\to\infty$, such that

$$P\left(\bigcap_{T=1}^{\infty}\bigcap_{\varepsilon\in(0,\varepsilon_T]}\left\{\sup_{0\le t\le T}\left|X_t^\varepsilon-\overline{X}_t\right|\le\delta\right\}\right)>1-\check{\delta} \tag{A17}$$

or equivalently

$$P\left\{\sup_{T\in\mathbb{N}}\sup_{0<\varepsilon\le\varepsilon_T}\sup_{0\le t\le T}\left|X_t^\varepsilon-\overline{X}_t\right|>\delta\right\}<\check{\delta}.$$

Proof: Let τ_ε^δ be defined by (A14). Since

$$\left\{\lim_{\varepsilon\to 0}\tau_\varepsilon^\delta=+\infty\right\}=\bigcap_{T=1}^{+\infty}\bigcup_{\check{\varepsilon}\in(0,\varepsilon_0)}\bigcap_{\varepsilon\in(0,\check{\varepsilon}]}\left\{\tau_\varepsilon^\delta\ge T\right\}$$

by Theorem 1, we have

$$P\left(\bigcup_{T=1}^{+\infty}\bigcap_{\check{\varepsilon}\in(0,\varepsilon_0)}\bigcup_{\varepsilon\in(0,\check{\varepsilon}]}\left\{\tau_\varepsilon^\delta<T\right\}\right)=0. \tag{A18}$$

We show that the set $\bigcup_{\varepsilon\in(0,\check{\varepsilon}]}\{\tau_\varepsilon^\delta<T\}$ is measurable. Let Q denote the set of all rational numbers. Then by the definition of τ_ε^δ, and the continuity of X_t^ε and $\overline{X}_t$ with respect to ε and t, we have

$$\begin{aligned}\bigcup_{\varepsilon\in(0,\check{\varepsilon}]}&\left\{\tau_\varepsilon^\delta<T\right\}\\ &=\left\{\exists\varepsilon\in(0,\check{\varepsilon}]\text{s.t.}\tau_\varepsilon^\delta<T\right\}\\ &=\left\{\exists\varepsilon\in(0,\check{\varepsilon}],\exists t\in[0,T)\text{s.t.}\left|X_t^\varepsilon-\overline{X}_t\right|>\delta\right\}\\ &=\left\{\exists\varepsilon\in(0,\check{\varepsilon}]\cap Q,\exists t\in[0,T)\cap Q\text{s.t.}\left|X_t^\varepsilon-\overline{X}_t\right|>\delta\right\}\\ &=\bigcup_{(0,\check{\varepsilon}]\cap Q}\bigcup_{[0,T)\cap Q}\left\{\left|X_t^\varepsilon-\overline{X}_t\right|>\delta\right\}\end{aligned}$$

which is measurable. Since the set $\bigcup_{\varepsilon\in(0,\check{\varepsilon}]}\{\tau_\varepsilon^\delta<T\}$ is increasing relative to $\check{\varepsilon}$, we have $\bigcup_{\varepsilon\in(0,\check{\varepsilon}]}\{\tau_\varepsilon^\delta<T\}=\bigcap_{\check{\varepsilon}\in(0,\varepsilon_0)\cap Q}\bigcup_{\varepsilon\in(0,\check{\varepsilon}]}\{\tau_\varepsilon^\delta<T\}$, and hence the set $\bigcap_{\check{\varepsilon}\in(0,\varepsilon_0)}\bigcup_{\varepsilon\in(0,\check{\varepsilon}]}\{\tau_\varepsilon^\delta<T\}$ is also measurable. Thus by (A18), we obtain that for any $T\in\mathbb{N}$

$$P\left(\bigcap_{\check{\varepsilon}\in(0,\varepsilon_0)}\bigcup_{\varepsilon\in(0,\check{\varepsilon}]}\left\{\tau_\varepsilon^\delta<T\right\}\right)=0$$

which implies that for any $T\in\mathbb{N}$

$$\lim_{\check{\varepsilon}\to 0}P\left(\bigcup_{\varepsilon\in(0,\check{\varepsilon}]}\left\{\tau_\varepsilon^\delta<T\right\}\right)=0$$

and thus there exists $\varepsilon_T \in (0, \varepsilon_0)$ (without loss of generality, we assume that ε_T decreases to 0, as $T \to \infty$) such that

$$P\left(\bigcup_{\varepsilon\in(0,\varepsilon_T]}\{\tau_\varepsilon^\delta < T\}\right) < \frac{\check{\delta}}{2^T}. \tag{A19}$$

Define $N = \bigcup_{T=1}^{+\infty}\bigcup_{\varepsilon\in(0,\varepsilon_T]}\{\tau_\varepsilon^\delta < T\}$. Then by (A19), we have $P(N) < \check{\delta}$, and thus $P(N^c) > 1-\check{\delta}$, where $N^c = \bigcap_{T=1}^{+\infty}\bigcap_{\varepsilon\in(0,\varepsilon_T]}\{\tau_\varepsilon^\delta \ge T\}$. By the definition of τ_ε^δ, we have

$$\left\{\sup_{0\le t\le T}\left|X_t^\varepsilon - \overline{X}_t\right| \le \delta\right\} \supseteq \left\{\tau_\varepsilon^\delta \ge T\right\}.$$

Hence (A17) holds. The proof is completed.

Lemma 3: Consider system (1) under Assumptions 1, 2, and 3. Then for any $\delta > 0$, there exists a function $T_\delta(\varepsilon) : (0, \varepsilon_0) \to \mathbb{N}$ such that

$$\lim_{\varepsilon\to 0} P\left\{\sup_{0\le t\le T_\delta(\varepsilon)}\left|X_t^\varepsilon - \overline{X}_t\right| > \delta\right\} = 0 \tag{A20}$$

and

$$\lim_{\varepsilon\to 0} T_\delta(\varepsilon) = +\infty. \tag{A21}$$

Proof: For $\delta > 0, 0 < \check{\delta} < 1$, we use $\varepsilon_T(\delta, \check{\delta})$ instead of ε_T in Lemma 2. Now fix $\delta > 0$. For any $k = 2, 3, \ldots$, by Lemma 2 we obtain a decreasing sequence $\{\varepsilon_T(\delta, (1/k))\}_{T\in\mathbb{N}}$ of positive real numbers, $\varepsilon_T(\delta, (1/k)) \downarrow 0$ as $T \to \infty$, such that

$$P\left\{\sup_{T\in\mathbb{N}}\ \sup_{0<\varepsilon\le\varepsilon_T\left(\delta,\frac{1}{k}\right)}\ \sup_{0\le t\le T}\left|X_t^\varepsilon - \overline{X}_t\right| > \delta\right\} < \frac{1}{k}. \tag{A22}$$

By the proof of Lemma 2, we assume that for any $T \in \mathbb{N}$, $\varepsilon_T(\delta, (1/k))$ is a nonincreasing function of k, and thus for any $k = 2, 3, \ldots$

$$\begin{aligned} 0 < \varepsilon_{k+1}\left(\delta, \frac{1}{k+1}\right) < \varepsilon_k\left(\delta, \frac{1}{k+1}\right) \le \varepsilon_k\left(\delta, \frac{1}{k}\right) \\ \le \varepsilon_k\left(\delta, \frac{1}{2}\right). \end{aligned} \tag{A23}$$

It follows from (A23) and $\lim\limits_{k\to+\infty}\varepsilon_k(\delta, (1/2)) = 0$ that:

$$\varepsilon_k\left(\delta, \frac{1}{k}\right) \downarrow 0, \text{ as } k \to +\infty. \tag{A24}$$

Now we define the desired function $T_\delta(\varepsilon)$ as follows:

$$T_\delta(\varepsilon) := \begin{cases} 1, & \text{if } \varepsilon \in \left(\varepsilon_2\left(\delta, \frac{1}{2}\right), \varepsilon_0\right), \\ k, & \text{if } \varepsilon \in \left(\varepsilon_{k+1}\left(\delta, \frac{1}{k+1}\right), \varepsilon_k\left(\delta, \frac{1}{k}\right)\right]. \\ & k = 2, 3, \cdots. \end{cases} \tag{A25}$$

Then for any $k = 2, 3, \ldots$, by (A22) and (A25), we get that

$$\sup_{\varepsilon_{k+1}\left(\delta,\frac{1}{k+1}\right)<\varepsilon\le\varepsilon_k\left(\delta,\frac{1}{k}\right)} P\left\{\sup_{0\le t\le T_\delta(\varepsilon)}\left|X_t^\varepsilon - \overline{X}_t\right| > \delta\right\} \le \frac{1}{k} \tag{A26}$$

and for $j = k+1, k+2, \ldots$, we have

$$\sup_{\varepsilon_{j+1}\left(\delta,\frac{1}{j+1}\right)<\varepsilon\le\varepsilon_j\left(\delta,\frac{1}{j}\right)} P\left\{\sup_{0\le t\le T_\delta(\varepsilon)}\left|X_t^\varepsilon - \overline{X}_t\right| > \delta\right\} \le \frac{1}{j} < \frac{1}{k}. \tag{A27}$$

By (A24), (A26) and (A27), we get that for any $k = 2, 3, \ldots$

$$\sup_{0<\varepsilon\le\varepsilon_k\left(\delta,\frac{1}{k}\right)} P\left\{\sup_{0\le t\le T_\delta(\varepsilon)}\left|X_t^\varepsilon - \overline{X}_t\right| > \delta\right\} \le \frac{1}{k}$$

which implies (A20). By (A24) and (A25), we obtain (A21). The proof is completed.

D. Proof of Approximation Result (11) of Theorem 1

For $k = 1, 2, \ldots$, by Lemma 3 there exists a function $T_{1/k}(\varepsilon) : (0, \varepsilon_0) \to \mathbb{N}$ such that

$$\lim_{\varepsilon\to 0} P\left\{\sup_{0\le t\le T_{\frac{1}{k}}(\varepsilon)}\left|X_t^\varepsilon - \overline{X}_t\right| > \frac{1}{k}\right\} = 0 \tag{A28}$$

and $\lim_{\varepsilon\to 0} T_{1/k}(\varepsilon) = +\infty$. Without loss of generality, we assume that for any $k \in \mathbb{N}$, we have

$$T_{\frac{1}{k+1}}(\varepsilon) \le T_{\frac{1}{k}}(\varepsilon),\ \forall \varepsilon \in (0, \varepsilon_0). \tag{A29}$$

In fact, we can replace the function $T_{1/(k+1)}(\varepsilon)$ by $T_{1/(k+1)}(\varepsilon) \wedge T_{1/k}(\varepsilon)$. Let $\varepsilon_1 = 1$. For $k = 2, 3, \ldots$, define

$$\varepsilon_k := \sup\left\{\varepsilon \in (0, \varepsilon_{k-1}) : T_{\frac{1}{k}}(\varepsilon) = k\right\}. \tag{A30}$$

Now we define the desired function $T(\varepsilon) : (0, \varepsilon_0) \to \mathbb{N}$ as follows:

$$T(\varepsilon) = \begin{cases} T_1(\varepsilon), & \text{if } \varepsilon \in \left(\varepsilon_2 \wedge \frac{1}{2}, \varepsilon_0\right), \\ T_{\frac{1}{k}}(\varepsilon), & \text{if } \varepsilon \in \left(\varepsilon_{k+1} \wedge \frac{1}{k+1}, \varepsilon_k \wedge \frac{1}{k}\right] \\ & k = 2, 3, \ldots. \end{cases} \tag{A31}$$

Since $\lim_{k\to\infty} \varepsilon_k \wedge (1/k) = 0$, the function $T(\varepsilon)$ is defined on $(0, \varepsilon_0)$. By (A30) and the definition of $T_{1/k}(\varepsilon)(k \in \mathbb{N})$ stated in the proof of Lemma 3 ($T_{1/k}(\varepsilon)$ is increasing when ε decreases to 0), we have that for any $0 < \varepsilon \leq \varepsilon_k \wedge (1/k), T(\varepsilon) \geq k$, and thus (12) holds.

Next, we prove (11). For any $\delta > 0$, take $\check{k} \in \mathbb{N}$ such that $(1/\check{k}) \leq \delta$. Then for $j = \check{k}, \check{k}+1, \check{k}+2, \ldots$, by (A29) and (A31), we get that

$$\begin{aligned}
&\sup_{\varepsilon\in\left(\varepsilon_{j+1}\wedge\frac{1}{j+1},\varepsilon_j\wedge\frac{1}{j}\right]} P\left\{\sup_{0\leq t\leq T(\varepsilon)}\left|X_t^\varepsilon-\overline{X}_t\right|>\delta\right\}\\
&\quad=\sup_{\varepsilon\in\left(\varepsilon_{j+1}\wedge\frac{1}{j+1},\varepsilon_j\wedge\frac{1}{j}\right]} P\left\{\sup_{0\leq t\leq T_{\frac{1}{j}}(\varepsilon)}\left|X_t^\varepsilon-\overline{X}_t\right|>\delta\right\}\\
&\quad\leq\sup_{\varepsilon\in\left(\varepsilon_{j+1}\wedge\frac{1}{j+1},\varepsilon_j\wedge\frac{1}{j}\right]} P\left\{\sup_{0\leq t\leq T_{\frac{1}{\check{k}}}(\varepsilon)}\left|X_t^\varepsilon-\overline{X}_t\right|>\delta\right\}\\
&\quad\leq\sup_{\varepsilon\in\left(\varepsilon_{j+1}\wedge\frac{1}{j+1},\varepsilon_j\wedge\frac{1}{j}\right]} P\left\{\sup_{0\leq t\leq T_{\frac{1}{\check{k}}}(\varepsilon)}\left|X_t^\varepsilon-\overline{X}_t\right|>\frac{1}{\check{k}}\right\}\\
&\quad\leq\sup_{\varepsilon\in\left(0,\varepsilon_j\wedge\frac{1}{j}\right]} P\left\{\sup_{0\leq t\leq T_{\frac{1}{\check{k}}}(\varepsilon)}\left|X_t^\varepsilon-\overline{X}_t\right|>\frac{1}{\check{k}}\right\} \qquad \text{(A32)}
\end{aligned}$$

and thus for any $l = j+1, j+2, \ldots$

$$\begin{aligned}
&\sup_{\varepsilon\in\left(\varepsilon_{l+1}\wedge\frac{1}{l+1},\varepsilon_l\wedge\frac{1}{l}\right]} P\left\{\sup_{0\leq t\leq T(\varepsilon)}\left|X_t^\varepsilon-\overline{X}_t\right|>\delta\right\}\\
&\quad\leq\sup_{\varepsilon\in\left(0,\varepsilon_l\wedge\frac{1}{l}\right]} P\left\{\sup_{0\leq t\leq T_{\frac{1}{\check{k}}}(\varepsilon)}\left|X_t^\varepsilon-\overline{X}_t\right|>\frac{1}{\check{k}}\right\}\\
&\quad\leq\sup_{\varepsilon\in\left(0,\varepsilon_j\wedge\frac{1}{j}\right]} P\left\{\sup_{0\leq t\leq T_{\frac{1}{\check{k}}}(\varepsilon)}\left|X_t^\varepsilon-\overline{X}_t\right|>\frac{1}{\check{k}}\right\} \qquad \text{(A33)}
\end{aligned}$$

where in the second inequality of (A33), we use the fact that $\varepsilon_l \wedge (1/l) \leq \varepsilon_j \wedge (1/j)$ for any $l = j+1, j+2, \ldots$. Hence by (A32), (A33) and the fact that $\lim_{k\to\infty} \varepsilon_k \wedge (1/k) = 0$, we obtain that for $j = \check{k}, \check{k}+1, \check{k}+2, \ldots$

$$\begin{aligned}
&\sup_{\varepsilon\in\left(0,\varepsilon_j\wedge\frac{1}{j}\right]} P\left\{\sup_{0\leq t\leq T(\varepsilon)}\left|X_t^\varepsilon-\overline{X}_t\right|>\delta\right\}\\
&\quad\leq\sup_{\varepsilon\in\left(0,\varepsilon_j\wedge\frac{1}{j}\right]} P\left\{\sup_{0\leq t\leq T_{\frac{1}{\check{k}}}(\varepsilon)}\left|X_t^\varepsilon-\overline{X}_t\right|>\frac{1}{\check{k}}\right\}. \qquad \text{(A34)}
\end{aligned}$$

By the fact that $\lim_{k\to\infty} \varepsilon_k \wedge (1/k) = 0$, (A28) and (A34), we obtain that for any $\delta > 0$

$$\lim_{\varepsilon\to 0} P\left\{\sup_{0\leq t\leq T(\varepsilon)}\left|X_t^\varepsilon-\overline{X}_t\right|>\delta\right\}=0.$$

The proof is completed.

E. Proof of Theorem 2

i) We prove the boundedness. Notice that $M = \sup_{t\geq 0}|\overline{X}_t|$ and

$$\{|X_t^\varepsilon| > M + c\} \subseteq \left\{\left|X_t^\varepsilon - \overline{X}_t\right| > c\right\}.$$

Then by the continuity of the sample path of $(X_t^\varepsilon, t \geq 0)$ (we don't mention this fact in the following proofs again), we have

$$\inf\{t\geq 0: |X_t^\varepsilon| > M + c\} \geq \inf\left\{t \geq 0: \left|X_t^\varepsilon - \overline{X}_t\right| > c\right\}.$$

Thus by Theorem 1, (13) holds.

ii) We prove the attractivity. Since $\lim_{t\to\infty} \overline{X}_t = x^*$, we have $\lim_{t\to\infty} |\overline{X}_t - x^*| = 0$, and thus for any $\delta > 0$, there exists a constant $T_\delta > 0$ such that

$$\sup_{t\geq T_\delta} |\overline{X}_t - x^*| < \frac{\delta}{2}$$

by which, we obtain that for any $t \geq T_\delta$

$$\begin{aligned}
\{|X_t^\varepsilon - x^*| > \delta\} &= \left\{\left|\left(X_t^\varepsilon - \overline{X}_t\right) + \left(\overline{X}_t - x^*\right)\right| > \delta\right\}\\
&\subseteq \left\{\left|X_t^\varepsilon - \overline{X}_t\right| > \frac{\delta}{2}\right\}
\end{aligned}$$

and thus

$$\begin{aligned}
&\inf\,\{t \geq T_\delta : |X_t^\varepsilon - x^*| > \delta\}]\\
&\quad\geq \inf\left\{t \geq T_\delta : \left|X_t^\varepsilon - \overline{X}_t\right| > \frac{\delta}{2}\right\}\\
&\quad\geq \inf\left\{t \geq 0 : \left|X_t^\varepsilon - \overline{X}_t\right| > \frac{\delta}{2}\right\}
\end{aligned}$$

which together with Theorem 1 implies (14).

iii) We prove the stability. If $\overline{X}_t \equiv 0 \in \mathbb{R}^n$ is a stable equilibrium of the average system (6), then for any $\delta > 0$, there exists a constant $r_\delta > 0$ such that

$$|\overline{X}_0| < r_\delta \Rightarrow \sup_{t\geq 0}|\overline{X}_t| < \frac{\delta}{2}$$

which together with Theorem 1, implies that for $|x| < r_\delta$

$$\begin{aligned}
&\lim_{\varepsilon\to 0}\inf\,\{t\geq 0: |X_t^\varepsilon| > \delta\}\\
&\quad= \lim_{\varepsilon\to 0}\inf\left\{t\geq 0: \left|\left(X_t^\varepsilon - \overline{X}_t\right) + \overline{X}_t\right| > \delta\right\}\\
&\quad\geq \lim_{\varepsilon\to 0}\inf\left\{t\geq 0: \left|X_t^\varepsilon - \overline{X}_t\right| > \frac{\delta}{2}\right\} = +\infty,\ a.s.
\end{aligned}$$

Hence (15) holds.

iv) For asymptotic stability, the proof follows directly from ii) and iii) above.

v) We prove the exponential stability. Since the equilibrium $\overline{X}_t = 0$ of the average system is exponentially stable, there exist constants $r > 0, c > 0, \gamma > 0$ such that for any $|x| < r$

$$|\overline{X}_t| < c|x|e^{-\gamma t}, \quad \forall t > 0.$$

Thus for any $\delta > 0$, we have

$$\left\{|X_t^\varepsilon| > c|x|e^{-\gamma t} + \delta\right\} \subseteq \left\{\left|X_t^\varepsilon - \overline{X}_t\right| > \delta\right\}$$

which together with Theorem 1 implies that

$$\lim_{\varepsilon\to 0} \inf\left\{t \geq 0 : |X_t^\varepsilon| > c|x|e^{-\gamma t} + \delta\right\} \geq \lim_{\varepsilon\to 0} \inf\left\{t \geq 0 : \left|X_t^\varepsilon - \overline{X}_t\right| > \delta\right\} = +\infty,\ a.s.$$

Hence (18) holds.

Let the function $T(\varepsilon)$ be defined in Theorem 1. Thus $\lim_{\varepsilon\to 0} T(\varepsilon) = +\infty$. For the stabilities (19)–(23) with respect to the approximation result (11), we only prove (23). The proofs for (19)–(22) are similar.

Since the equilibrium $\overline{X}_t = 0$ of the average system is exponentially stable, there exist constants $r > 0, c > 0, \gamma > 0$ such that for any $|x| < r$

$$|\overline{X}_t| < c|x|e^{-\gamma t}, \quad \forall t > 0.$$

Thus for any $\delta > 0$, we have that for any $|x| < r$

$$\begin{aligned}\left\{\sup_{0\leq t\leq T(\varepsilon)} \left\{|X_t^\varepsilon| - c|x|e^{-\gamma t}\right\} > \delta\right\} &= \bigcup_{0\leq t\leq T(\varepsilon)} \left\{|X_t^\varepsilon| - c|x|e^{-\gamma t} > \delta\right\}\\ &\subseteq \bigcup_{0\leq t\leq T(\varepsilon)} \left\{\left|X_t^\varepsilon - \overline{X}_t\right| > \delta\right\}\\ &= \left\{\sup_{0\leq t\leq T(\varepsilon)} \left|X_t^\varepsilon - \overline{X}_t\right| > \delta\right\}\end{aligned}$$

which together with result (11) of Theorem 1 gives that

$$\limsup_{\varepsilon\to 0} P\left\{\sup_{0\leq t\leq T(\varepsilon)} \left\{|X_t^\varepsilon| - c|x|e^{-\gamma t}\right\} > \delta\right\} \leq \lim_{\varepsilon\to 0} P\left\{\sup_{0\leq t\leq T(\varepsilon)} \left|X_t^\varepsilon - \overline{X}_t\right| > \delta\right\} = 0.$$

Hence (23) holds. The whole proof is completed.

F. Proof of (A8)

We give a detailed proof of (A8), i.e.

$$\lim_{\varepsilon\to 0} \sup_{0\leq t\leq T} \left|\int_0^t \left(a(\overline{X}_s, Y_{s/\varepsilon}) - \overline{a}(\overline{X}_s)\right) ds\right| = 0,\ a.s.$$

Proof: We follow the proof of Theorem 5 of Chapter 3 of [34] for the globally Lipschitz case. Notice that

$$M' = \sup_{0\leq t\leq T} |\overline{X}_t|, \quad M = M' + 1$$

and k_M is the Lipschitz constant of $a(x,y)$ with respect to the compact subset $D_M := \{x \in \mathbb{R}^n : |x| \leq M\}$ of $\mathbb{R}^n$, i.e., for any $x, \check{x} \in D_M$ and any $y \in S_Y$ (see Assumption 1)

$$|a(x,y) - a(\check{x},y)| \leq k_M|x - \check{x}|. \tag{A35}$$

Then by (7) and (A35), we have that for any $x, \check{x} \in D_M$

$$|\overline{a}(x) - \overline{a}(\check{x})| \leq k_M|x - \check{x}|.$$

For any $n \in \mathbb{N}$, define a function $\overline{X}_s^n$, $s \geq 0$, by

$$\overline{X}_s^n = \sum_{k=0}^{\infty} \overline{X}_{\frac{k}{n}} I_{\left\{\frac{k}{n} \leq s < \frac{k+1}{n}\right\}}.$$

Then for any $n \in \mathbb{N}$, we have

$$\sup_{0\leq s\leq T} \left|\overline{X}_s^n\right| \leq \sup_{0\leq s\leq T} |\overline{X}_s| \leq M' < M. \tag{A36}$$

By (A35), (A36), we obtain that

$$\begin{aligned}&\sup_{0\leq t\leq T} \left|\int_0^t \left(a(\overline{X}_s, Y_{s/\varepsilon}) - \overline{a}(\overline{X}_s)\right) ds\right|\\ &= \sup_{0\leq t\leq T} \left|\int_0^t \left[\left(a(\overline{X}_s, Y_{s/\varepsilon}) - a\left(\overline{X}_s^n, Y_{s/\varepsilon}\right)\right) + \left(a\left(\overline{X}_s^n, Y_{s/\varepsilon}\right) - \overline{a}\left(\overline{X}_s^n\right)\right) + \left(\overline{a}\left(\overline{X}_s^n\right) - \overline{a}(\overline{X}_s)\right)\right] ds\right|\\ &\leq \sup_{0\leq t\leq T} \int_0^t \left|a(\overline{X}_s, Y_{s/\varepsilon}) - a\left(\overline{X}_s^n, Y_{s/\varepsilon}\right)\right| ds\\ &\quad + \sup_{0\leq t\leq T} \left|\int_0^t \left(a\left(\overline{X}_s^n, Y_{s/\varepsilon}\right) - \overline{a}\left(\overline{X}_s^n\right)\right) ds\right|\\ &\quad + \sup_{0\leq t\leq T} \int_0^t \left|\overline{a}\left(\overline{X}_s^n\right) - \overline{a}(\overline{X}_s)\right| ds\\ &\leq 2k_M T \sup_{0\leq s\leq T} \left|\overline{X}_s - \overline{X}_s^n\right|\\ &\quad + \sup_{0\leq t\leq T} \left|\int_0^t \left(a\left(\overline{X}_s^n, Y_{s/\varepsilon}\right) - \overline{a}\left(\overline{X}_s^n\right)\right) ds\right|.\end{aligned} \tag{A37}$$

Next, we focus on the second term on the right-hand side of (A37). We have

$$\begin{aligned}&\sup_{0\leq t\leq T} \left|\int_0^t \left(a\left(\overline{X}_s^n, Y_{s/\varepsilon}\right) - \overline{a}\left(\overline{X}_s^n\right)\right) ds\right|\\ &= \sup_{0\leq t\leq T} \left|\int_0^t \left(a\left(\overline{X}_s^n, Y_{s/\varepsilon}\right) - \overline{a}\left(\overline{X}_s^n\right)\right) \cdot \sum_{k=0}^{\infty} I_{\left\{\frac{k}{n} \leq s < \frac{k+1}{n}\right\}} ds\right|\\ &= \sup_{0\leq t\leq T} \left|\int_0^t \sum_{k=0}^{\infty} \left(a\left(\overline{X}_{\frac{k}{n}}, Y_{s/\varepsilon}\right) - \overline{a}\left(\overline{X}_{\frac{k}{n}}\right)\right) \cdot I_{\left\{\frac{k}{n} \leq s < \frac{k+1}{n}\right\}} ds\right|\end{aligned}$$

$$
\begin{aligned}
&= \sup_{0\le t\le T}\left|\sum_{k=0}^{n([t]+1)}\int_0^t \left(a\left(\overline{X}_{\frac{k}{n}},Y_{s/\varepsilon}\right)-\overline{a}\left(\overline{X}_{\frac{k}{n}}\right)\right)\right.\\
&\qquad\qquad\left.\cdot I_{\left\{\frac{k}{n}\le s<\frac{k+1}{n}\right\}}ds\right|\\
&= \sup_{0\le t\le T}\left|\sum_{k=0}^{n([t]+1)}\int_{\frac{k}{n}\wedge t}^{\frac{k+1}{n}\wedge t}\left(a\left(\overline{X}_{\frac{k}{n}},Y_{s/\varepsilon}\right)-\overline{a}\left(\overline{X}_{\frac{k}{n}}\right)\right)ds\right|\\
&\le \sup_{0\le t\le T}\sum_{k=0}^{n([t]+1)}\left|\int_{\frac{k}{n}\wedge t}^{\frac{k+1}{n}\wedge t}\left(a\left(\overline{X}_{\frac{k}{n}},Y_{s/\varepsilon}\right)-\overline{a}\left(\overline{X}_{\frac{k}{n}}\right)\right)ds\right|
\end{aligned}
\tag{A38}
$$

where $[t]$ is the largest integer not greater than t. For fixed n and k with $k\le n([T]+1)$, we have

$$
\begin{aligned}
&\sup_{0\le t\le T}\left|\int_{\frac{k}{n}\wedge t}^{\frac{k+1}{n}\wedge t}\left(a\left(\overline{X}_{\frac{k}{n}},Y_{s/\varepsilon}\right)-\overline{a}\left(\overline{X}_{\frac{k}{n}}\right)\right)ds\right|\\
&\le \sup_{0\le t\le T}\left(\left|\int_0^{\frac{k+1}{n}\wedge t}\left(a\left(\overline{X}_{\frac{k}{n}},Y_{s/\varepsilon}\right)-\overline{a}\left(\overline{X}_{\frac{k}{n}}\right)\right)ds\right|\right.\\
&\qquad\left.+\left|\int_0^{\frac{k}{n}\wedge t}\left(a\left(\overline{X}_{\frac{k}{n}},Y_{s/\varepsilon}\right)-\overline{a}\left(\overline{X}_{\frac{k}{n}}\right)\right)ds\right|\right)\\
&\le 2\sup_{0\le t\le \frac{k+1}{n}}\left|\int_0^t\left(a\left(\overline{X}_{\frac{k}{n}},Y_{s/\varepsilon}\right)-\overline{a}\left(\overline{X}_{\frac{k}{n}}\right)\right)ds\right|\\
&= 2\sup_{0\le t\le \frac{k+1}{n}}\varepsilon\left|\int_0^{\frac{t}{\varepsilon}}\left(a\left(\overline{X}_{\frac{k}{n}},Y_s\right)-\overline{a}\left(\overline{X}_{\frac{k}{n}}\right)\right)ds\right|.
\end{aligned}
\tag{A39}
$$

Then by Birkhoff ergodic theorem and [20, Problem 5.3.2], we obtain that

$$
\lim_{\varepsilon\to 0}\sup_{0\le t\le \frac{k+1}{n}}\varepsilon\left|\int_0^{\frac{t}{\varepsilon}}\left(a\left(\overline{X}_{\frac{k}{n}},Y_s\right)-\overline{a}\left(\overline{X}_{\frac{k}{n}}\right)\right)ds\right| = 0,\ a.s.
$$

which together with (A38) and (A39) gives that for any $n\in\mathbb{N}$

$$
\lim_{\varepsilon\to 0}\sup_{0\le t\le T}\left|\int_0^t\left(a\left(\overline{X}_s^n,Y_{s/\varepsilon}\right)-\overline{a}\left(\overline{X}_s^n\right)\right)ds\right| = 0,\ a.s.
\tag{A40}
$$

Thus by (A37), (A40), and

$$
\lim_{n\to\infty}\sup_{0\le s\le T}\left|\overline{X}_s-\overline{X}_s^n\right| = 0
$$

we obtain that

$$
\lim_{\varepsilon\to 0}\sup_{0\le t\le T}\left|\int_0^t\left(a(\overline{X}_s,Y_{s/\varepsilon})-\overline{a}(\overline{X}_s)\right)ds\right| = 0,\ a.s.
$$

The proof is completed.

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