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ADAPTIVE CONTROL OF AN ANTI-STABLE WAVE PDE

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Abstract. Adaptive control of PDEs is a problem of nonlinear dynamic feedback design for an infinite-dimensional system. The problem is nonlinear even when the PDE is linear. Past papers on adaptive control of unstable PDEs with unmatched parametric uncertainties have considered only parabolic PDEs and first-order hyperbolic PDEs. In this paper we introduce several tools for approaching adaptive control problems of second-order-in-time PDEs. We present these tools through a benchmark example of an unstable wave equation with an unmatched (non-collocated) anti-damping term, which serves both as a source of instability and of parametric uncertainty. This plant has infinitely many eigenvalues arbitrarily far to the right of the imaginary axis and they reside on a vertical line whose position is completely unknown. The key effort in the design is to avoid the appearance of the second time derivative of the parameter estimate in the error system.

Keywords: Adaptive control, distributed parameter systems, backstepping, boundary control.

1 Introduction

Background. Adaptive control of infinite-dimensional systems is a challenging topic to which several researchers have contributed over the last two decades [4, 5, 8, 12, 13, 14, 16, 17, 18, 19, 20, 21, 27, 28, 34, 36]. The results have either allowed plant instability but required distributed actuation, or allowed boundary control but required that the plant be at least neutrally stable.

Recently we introduced several designs for non-adaptive [30] and adaptive [26, 31, 32] boundary control of unstable parabolic PDEs. Subsequently, in [6] we also tackled systems with unknown input delay, i.e., an important class of infinite-dimensional systems with first-order hyperbolic PDE dynamics. The remaining major class of PDEs for which adaptive boundary control results have not been developed yet, at least not in the case where the plant is unstable, are second-order hyperbolic PDEs, namely wave equations. Wave (and beam) equations have been tackled in [5, 8, 12, 18, 20, 21, 28, 34, 36], however, not *unstable* ones.

Contributions of the Paper. In this paper we present the first adaptive control design for an unstable wave equation controlled from a boundary, and where the source of instability is not collocated (matched) with control. We focus on the (notationally) simplest problem, but a problem that, among all basic wave equation problems with constant coefficients, is the most challenging. We introduce tools for dealing with boundary control of *second-order-in-time* PDEs, which require parameter estimate-dependent state transformations, and which, if approached in the same way as parabolic or first-order hyperbolic PDEs, would give rise to perturbation terms involving the second derivative (in time) of the parameter estimate. With standard update law choices such perturbations would not be a priori bounded, whereas alternative choices that would make those perturbations bounded would require overparametrization.

The wave equation example that we focus on is with an 'anti-damping' type of boundary condition in the boundary opposite from the controlled boundary. This PDE has all of its infinitely many eigenvalues in the right half plane, with arbitrary positive real parts. It is exponentially stable in negative time, thus we refer to it as 'anti-stable.' Even in the non-adaptive case, this PDE has been an open problem until a recent breakthrough by A. Smyshlyaev [33] who constructed a novel 'backstepping transformation' for boundary control of this PDE. The phenomenon of anti-damping in wave equations occurs in combustion instabilities where heat release is modeled as proportional to the pressure rate, or in the case of electrically amplified stringed instruments where an electromagnetic pickup measures the string velocity and a high-gain amplifier provides forcing on the string which is proportional to the velocity.

Our adaptive control approach employs parameter projection. Projection is not used in this paper as a standard robustness tool for non-parametric uncertainties, but for enabling a certainty equivalence controller design in combination with a normalized Lyapunov update law. The rate of variation of the parameter estimate acts as a perturbation on the error system. The size of this perturbation can be made small by choosing the adaptation gain to be small. However, the restriction on the adaptation gains depends on the size of the transients of the parameter estimates. Parameter projection is used to make it possible to predict bounds on the size of the transients of the parameter estimate, and hence, to choose a sufficiently small adaptation gain for achieving stability. However, parameter projection has a somewhat undesirable property that the right-hand side of the projection operator is discontinuous. This does not result in discontinuous evolution of the state of the parameter estimate in time, but it does raise concerns regarding existence and uniqueness of solutions. While the bulk of our paper employs the standard discontinuous projection operator, to alleviate the concerns regarding the discontinuity, in Section 7 we present an alternative, Lipschitz continuous version of projection, and establish its properties and the properties of the closed-loop system with an update law employing the Lipschitz projector.

Physical Motivation for the Wave Equation with Anti-Damping. The 1D wave equation is a good model for acoustic dynamics in ducts and vibration of strings. The phenomenon of *anti-damping* that we include in our study of wave dynamics represents injection of energy in proportion to the velocity field, akin to a damper with a negative damping coefficient. Such a process arises in combustion dynamics, where the pressure field is disturbed in proportion to varying *heat release* rate,

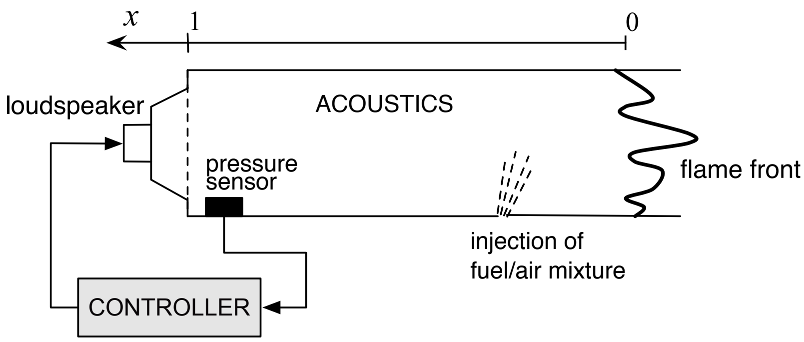

Figure 1: Control of a thermoacoustic instability in a Rijke tube [1] (a duct-type combustion chamber).

which, in turn, is proportional to the rate of change of pressure. Figure 1 shows the classical Rijke tube control experiment [1]. For the sake of our study, we assume that the system is controlled using a loudspeaker, with a pressure sensor (microphone) near the speaker. Both the actuator and the sensor are placed as far as possible from the flame front, for their thermal protection. Alternatively, instead of the loudspeaker actuation depicted in Figure 1, the combustion instability can be controlled using modulation of the fuel injection rate [2, 22, 25].

The process of anti-damping is located mostly at the flame front. In a Rijke tube the flame front may not be at the end of the tube, however, the leaner the fuel/air mixture, the longer it takes for the mixture to ignite, and the further the flame is from the injection point. In the limit, for the leanest mixture for which combustion is sustained, the flame is near the exit of the tube, which corresponds to a situation with boundary anti-damping.

Organization of the Paper. We begin with a problem statement in Section 2, followed by the presentation of the adaptive control law and a stability statement in Section 3. The main stability theorem is proved through a series of lemmas established in Sections 4, 5, and 6. In Section 7 we address the question of discontinuity in the parameter projection operator employed in the design and present a Lipschitz alternative to the projection operator. In Section 8 we present conclusions.

2 Problem Statement

Consider the system

$$u_{tt}(x,t) = u_{xx}(x,t) \tag{1}$$

$$u_x(0,t) = -qu_t(0,t) \tag{2}$$

$$u_x(1,t) = U(t)\,, \tag{3}$$

where $U(t)$ is the input and $(u,u_t) \in H_1(0,1) \times L_2(0,1)$ is the system state. Our goal is to design a feedback law for the input $U(t)$, employing the measurement of the variables $u(0,t), u(1,t), u(x,t), x \in (0,1)$, as well as their time derivatives, if needed, to stabilize the anti-stable wave equation system. The key challenge is the large uncertainty in the antidamping coefficient $q \geq 0$, which will be dealt with by employing an estimate $\hat{q}(t)$ in the adaptive controller, and by designing an update law for $\hat{q}(t)$.

Assumption 2.1 *Non-negative constants $\underline{q}$ and $\bar{q}$ are known such that $q \in [\underline{q}, \bar{q}]$ and either $\bar{q} < 1$ or $\underline{q} > 1$.*

This is simply a stabilizability assumption. When $q = 1$, the real part of all the plant eigenvalues is $+\infty$, requiring infinite control gains for stabilization.

While in this paper we approach the control problem for the system (1), (2) using Neumann actuation (3), the problem can also be solved using Dirichlet actuation, $u(1,t) = U(t)$.

What is the significance of the adaptive boundary control problem for the system (1), (2)? We highlight the following five aspects of the problem:

1. *Infinitely many unstable eigenvalues.* The significance of the *non*-adaptive boundary control problem is that the uncontrolled system (1), (2) has infinitely many unstable eigenvalues, with arbitrarily large positive real parts. For $q \in [0,1)$ the eigenvalues are

$$\{0\} \cup \left\{ \frac{1}{2} \ln \left| \frac{1+q}{1-q} \right| + j\pi n, \quad n \in \mathbb{Z} \setminus \{0\} \right\}$$

and for $q > 1$ the eigenvalues are

$$\{0\} \cup \left\{ \frac{1}{2} \ln \left| \frac{1+q}{1-q} \right| + j\pi \left(n + \frac{1}{2} \right), \quad n \in \mathbb{Z} \right\}. \tag{4}$$

Figures 2 and 3 show graphically the distribution of the eigenvalues and their dependence on q and n. As q grows from 0 to $+1$, the eigenvalues move rightward, all the way to $+\infty$. As q further grows from $+1$ to $+\infty$, the eigenvalues move leftward from $+\infty$ towards the imaginary axis, whereas their imaginary parts drop down by $\pi/2$. For a fixed q, the eigenvalues are always distributed on a vertical line. They depend linearly on n, namely, they are equidistant along the vertical line. The system is not well posed when $q = 1$. Hence, we assume that

$$q \in (0,1) \cup (1,\infty). \tag{5}$$

The key challenge is that the source of instability, the anti-damping term in (2), is on the opposite boundary from the boundary that is controlled. To deal with this challenge, we employ a transformation invented by A. Smyshlyaev [33] (for known q).

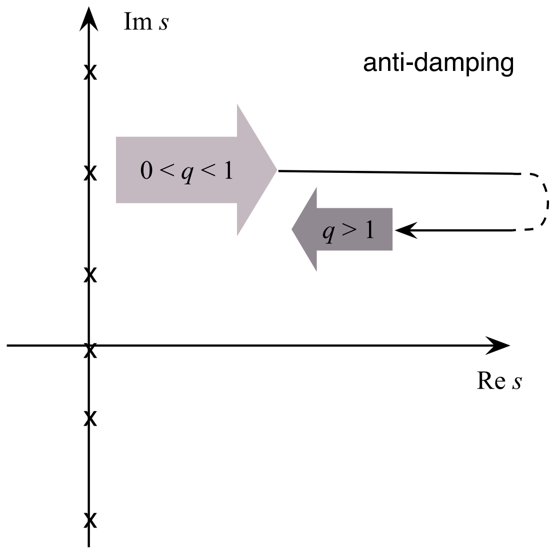

Figure 2: Eigenvalues of the wave PDE with boundary anti-damping. As q grows from 0 to $+1$, the eigenvalues move rightward, all the way to $+\infty$. As q further grows from $+1$ to $+\infty$, the eigenvalues move leftward from $+\infty$ towards the imaginary axis, whereas their imaginary parts drop down by $\pi/2$.

2. *Nonlinear dependence of the feedback law on the parameter estimate.* The adaptive boundary control law for (1), (2) depends in a non-trivial (non-linear) manner on the estimate of the uncertain parameter q (this would not be the case if we dealt with a simpler problem where the unknown parameter is the propagation speed coefficient and the wave equation is of the standard kind, with eigenvalues on the real axis). So, this problem is a good non-routine example of adaptive control for second-order hyperbolic PDEs.

3. *Second-order-in-time character of the wave PDE requires different handling of the perturbation effect of the parameter estimation rate than for parabolic PDEs.* If we proceed to differentiate (11) twice with respect to time, to get $w_{tt}(x,t)$, a second derivative of the parameter estimate $\ddot{\hat{q}}(t)$ would arise, as opposed to only the first derivative as in the parabolic PDE case [26]. This requires a particular choice of a backstepping transformation, one for the displacement variable and the other for the velocity variable, such that the second derivative does not arise.

4. *Arbitrarily high uncertainty affecting infinitely many unstable eigenvalues.* The most striking aspect of the problem is that the plant at hand not only has

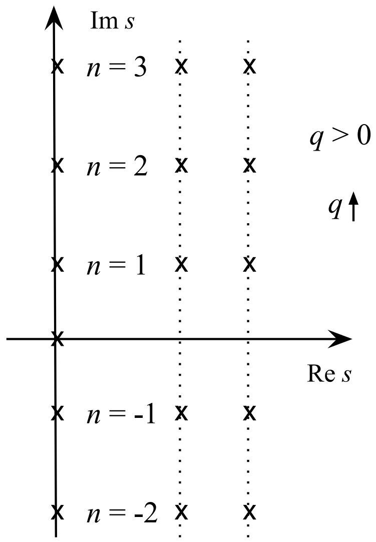

Figure 3: Eigenvalues of the wave PDE with boundary anti-damping, continued from Figure 2. For a fixed q, the eigenvalues are always distributed on a vertical line. They depend linearly on n, namely, they are equidistant along the vertical line.

infinitely many unstable eigenvalues in the right-half plane, which are arbitrarily to the right of the imaginary axis, but the location of the eigenvalues is unknown (the eigenvalues could be on any vertical line in the right half-plane). We are not aware of any other work on control of linear dynamic systems of any kind that contains simultaneously this level of instability and uncertainty.

5. *The unmatched, anti-collocated unknown parameter.* The control input $U(t)$ and the unknown parameter q appear on the opposite boundaries of the PDE domain. This introduces a challenge for the design of the parameter estimator. If the objective is to estimate the unknown q using an estimator whose dynamic order is one, namely, without adding filters or observers (whose order would have to be infinite), the parameter estimator design needs to be Lyapunov-based. However, a Lyapunov design involves a complicated infinite-dimensional (albeit non-dynamic) transformation into an error system based on which the update law can be selected. This transformation depends on the parameter estimate, which is time varying. As a result, the error system is perturbed not only by the parameter estimation error, but also by the parameter estimation rate (the time derivative of the parameter estimate). The update law is chosen in such a way as to cancel the effect of the parameter estimation error in the Lyapunov analysis, however, the effect of

the parameter estimation rate remains uncompensated. Robustness with respect to the parameter estimation rate perturbation is achieved by making the adaptation gain sufficiently small. However, the bound on the gain requires the knowledge of the bound on the parameter estimate transients. To make such a bound available, parameter projection is employed, assuming an a priori known interval for the unknown parameter q. However, the parameter projection operator is discontinuous. To alleviate possible concerns regarding regularity of solutions, it is desirable to develop a projection operator which is Lipschitz and which achieves stability and regulation of the closed-loop adaptive system. The main development in the paper is conducted for the standard, discontinuous projection operator, however, in Section 7 a Lipschitz projector is introduced and its properties, and the properties of the resulting closed-loop system, are established.

3 Control Design and Stability Statement

We propose the following adaptive control law

$$\begin{aligned} U(t) = & c_1\hat{q}(t)\frac{\hat{q}(t)+c}{1+\hat{q}(t)c}u(0,t) - c_1 u(1,t) \\ & -\frac{\hat{q}(t)+c}{1+\hat{q}(t)c}u_t(1,t) - c_1\frac{\hat{q}(t)+c}{1+\hat{q}(t)c}\int_0^1 u_t(y,t)dy \end{aligned} \tag{6}$$

and parameter update law

$$\dot{\hat{q}}(t) = \gamma \mathrm{Proj}\left\{\frac{(1-c^2)\omega(0,t)+c_1(\hat{q}(t)+c)w(1,t)}{(1+\hat{q}(t)c)(1+E(t))}u_t(0,t)\right\}, \tag{7}$$

where Proj is the standard projection operator

$$\mathrm{Proj}_{[\underline{q},\bar{q}]}\{\tau\} = \tau\begin{cases} 0, & \hat{q} \le \underline{q} \text{ and } \tau < 0 \\ 0, & \hat{q} \ge \bar{q} \text{ and } \tau > 0 \\ 1, & \text{else}, \end{cases} \tag{8}$$

the initial condition for the parameter estimate is restricted to

$$\hat{q}(0) \in [\underline{q}, \bar{q}], \tag{9}$$

the normalization function is given by

$$\begin{aligned} E(t) \quad = \quad & \frac{1}{2}\left(\int_0^1 \omega^2(x,t)dx + \int_0^1 w_x^2(x,t)dx + c_1 w^2(1,t)\right) \\ & +\delta\int_0^1 (-2+x)\omega(x,t)w_x(x,t)dx. \end{aligned} \tag{10}$$

and the variables (w,ω), namely, the transformed displacement and velocity, are defined as

$$w(x,t) \triangleq u(x,t) + \frac{\hat{q}(t)+c}{1+\hat{q}(t)c}\left(-\hat{q}(t)u(0,t) + \int_0^x u_t(y,t)dy\right) \tag{11}$$

$$\omega(x,t) \triangleq u_t(x,t) + \frac{\hat{q}(t)+c}{1+\hat{q}(t)c} u_x(x,t)\,. \tag{12}$$

For implementation of the update law (7), we point out that $\omega(0,t)$ and $w(1,t)$ are given by

$$\omega(0,t) = u_t(0,t) + \frac{\hat{q}(t)+c}{1+\hat{q}(t)c} u_x(0,t) \tag{13}$$

$$w(1,t) = u(1,t) + \frac{\hat{q}(t)+c}{1+\hat{q}(t)c}\left(-\hat{q}(t)u(0,t) + \int_0^1 u_t(y,t)dy\right)\,. \tag{14}$$

The main stability result under the adaptive controller (6), (7) is stated next.

Theorem 1 *Consider the closed-loop system consisting of the plant (1)–(3), the control law (6), and the parameter update law (7). Let Assumption 2.1 hold and pick any control gains $c_1 > 0$ and $c \in (0, c^*)$, where*

$$c^* = \frac{1}{\bar{q} - \underline{q}} \begin{cases} \underline{q}^2 - 1\,, & \underline{q} > 1 \\ 1 - \bar{q}^2\,, & \bar{q} < 1 \end{cases} \tag{15}$$

There exists $\gamma^* > 0$ such that for any $\gamma \in (0,\gamma^*)$ and any $\delta \in (0,1/2)$, the zero solution of the system $(u, v, \hat{q} - q)$ is globally stable in the sense that there exist positive constants R and ρ (independent of the initial conditions) such that for all initial conditions satisfying $(u_0, v_0, \hat{q}_0) \in H_1(0,1) \times L_2(0,1) \times [\underline{q}, \bar{q}]$, the following holds:

$$\Upsilon(t) \le R\left(\mathrm{e}^{\rho\Upsilon(0)} - 1\right)\,, \quad \forall t \ge 0\,, \tag{16}$$

where

$$\Upsilon(t) = \Omega(t) + (q - \hat{q}(t))^2 \tag{17}$$

and

$$\Omega(t) = \int_0^1 v^2(x,t)dx + \int_0^1 u_x^2(x,t)dx + u^2(1,t)\,. \tag{18}$$

Furthermore,

$$\int_0^\infty \Omega(t)dt \le \infty\,, \tag{19}$$

i.e., regulation is achieved in the sense that $\operatorname{ess\,lim}_{t\to\infty}\Omega(t) = 0$.

Though Lyapunov- and LaSalle-like stability theorems exist for infinite-dimensional systems [35], they are not nearly as useful as the theorems for finite-dimensional systems in [15] since satisfying the conditions of those theorems is immensely more challenging than proving such general theorems. For this reason, Lyapunov stability results for infinite-dimensional systems, particularly nonlinear ones, are best developed in a fashion customized to the specific PDE system that the designer is dealing with. We pursue such an approach here. Theorem 1 is proved through a series of lemmas established in Sections 4, 5, and 6.

We emphasize the global stability result in (16). This is a class $\mathscr{K}_\infty$ estimate in the spirit of stability definitions in [15]. Stability studies in adaptive control very rarely provide specific stability estimates as (16). Instead, due to the complexity of the stability analysis (for systems with unmatched parametric uncertainties, and particularly for controllers relying on the certainty equivalence approach), only boundedness is typically established, rather than Lyapunov stability. Here we work out an explicit bound on the norm of the overall system, in terms of its initial condition. The class $\mathscr{K}_\infty$ functional bound, $R\left(e^{\rho r}-1\right),\ r \geq 0$, is exponential and zero at zero. The norm of the overall infinite-dimensional system includes the kinetic and potential energies of the wave/string system and the parameter estimation error. Since no persistency of excitation assumption is being imposed in the present work, as it would not be a priori verifiable, the parameter estimate $\hat{q}(t)$ is not guaranteed to converge to the true parameter value q, however, the PDE state is guaranteed to be regulated to zero, in an appropriate weak sense.

4 System Transformation and Adaptive Target System

In this section we discuss the backstepping transformation (11), (12) and derive the target system based on this transformation. We start by rewriting the wave equation model (1)–(3) as a first-order-in-time evolution equation,

$$
\begin{aligned}
u_t(x,t) &= v(x,t) && (20)\\
v_t(x,t) &= u_{xx}(x,t) && (21)\\
u_x(0,t) &= -qv(0,t) && (22)\\
u_x(1,t) &= U(t)\,, && (23)
\end{aligned}
$$

where the variable $v(x,t)$ is the velocity. The state of this model is (u,v). Since (20), (21) is a second order PDE system, a backstepping transformation of the first-order form $u \mapsto w$, as (11) may appear to be, does not suffice. We need a transformation of the form $(u,v) \mapsto (w,\omega)$, namely one that involves both the displacement state and the velocity state. This transformation consists of (11) and of (12).

From (11) we obtain

$$
w_x(x,t) = u_x(x,t) + \frac{\hat{q}(t)+c}{1+\hat{q}(t)c} v(x,t) \tag{24}
$$

$$
w(1,t) = \frac{1-\hat{q}^2(t)}{1+\hat{q}(t)c} u(1,t) + \frac{\hat{q}(t)+c}{1+\hat{q}(t)c} \int_0^1 \left(v(x,t)+\hat{q}(t)u_x(x,t)\right) dx\,. \tag{25}
$$

The transformation $(u,v) \mapsto (w,\omega)$ can be viewed either in the form (11), (12) or in the form (12), (24), (25).

The transformation (12) follows from (11) and from the PDE model (1), (2) when $\hat{q}(t) \equiv q =$ const but not otherwise. Unlike the non-adaptive case [33], in introducing (12) we do not simply differentiate (11) with respect to t, as the time derivative of the transformed position state, $w_t(x,t)$, would include $\dot{\hat{q}}$. The exclusion of the $\dot{\hat{q}}$ term from the transformation (12) is crucial, because $\dot{\hat{q}}$ is a parameter update law that is yet to be designed, and its inclusion would result in the appearance of $\ddot{\hat{q}}$ in the expression for $w_{tt}(x,t)$, namely, in the wave equation for the transformed system. The appearance of $\ddot{\hat{q}}$ in the transformed wave equation would be fatal because we are not designing $\ddot{\hat{q}}$ but only $\dot{\hat{q}}$, so we would not be able to bound $\ddot{\hat{q}}$. In summary, as we shall see, the transformation (12) is the key to the feasibility of adaptive design.

It can be observed that the transformations (11), (12), (24), (25) are linear in the PDE state (u,v) and nonlinear only in $\hat{q}(t)$. Hence, the transformations can be viewed as a redefinition of the underlying inner product, as was done with parameter-dependent Lyapunov functions in [9].

The inverse of the transformation (12), (24), (25), namely $(w,\omega) \mapsto (u,v)$, is given by

$$v(x,t) = \frac{(1+\hat{q}(t)c)^2}{(1-\hat{q}^2(t))(1-c^2)} \left(\omega(x,t) - \frac{\hat{q}(t)+c}{1+\hat{q}(t)c} w_x(x,t) \right) \tag{26}$$

$$u_x(x,t) = \frac{(1+\hat{q}(t)c)^2}{(1-\hat{q}^2(t))(1-c^2)} \left(w_x(x,t) - \frac{\hat{q}(t)+c}{1+\hat{q}(t)c} \omega(x,t) \right) \tag{27}$$

$$\begin{aligned} u(1,t) =& \frac{1+\hat{q}(t)c}{1-\hat{q}^2(t)} w(1,t) \\ &+ \frac{(\hat{q}(t)+c)(1+\hat{q}(t)c)}{(1-\hat{q}^2(t))(1-c^2)} \int_0^1 (-\omega(x,t) + c w_x(x,t))\, dx. \end{aligned} \tag{28}$$

In the non-adaptive case, namely, when $\hat{q}(t) \equiv q$, the transformation (11) would convert the plant (1), (2) into the 'target system' $w_{tt} = w_{xx}, w_x(0,t) = c w_t(0,t)$, which has a damping boundary condition at $x = 0$. In the adaptive case, where the time-dependent $\hat{q}(t)$ appears in both (11) and (12), the derivation of the target system is rather more complicated. It is presented in the next lemma.

Lemma 1 *Suppose that the functions $u,v,w,\omega : [0,1] \times \mathbb{R}_+ \to \mathbb{R}$ and $\hat{q} : \mathbb{R}_+ \to \mathbb{R}$ satisfy the transformation (12), (24), (25). Then they satisfy the system (20)–(22) if and only if they satisfy the system*

$$w_t(x,t) = \omega(x,t) + \tilde{q}(t) \frac{\hat{q}(t)+c}{1+\hat{q}(t)c} v(0,t) + \theta(x,t)\dot{\hat{q}}(t) \tag{29}$$

$$\omega_t(x,t) = w_{xx}(x,t) + \beta(x,t)\dot{\hat{q}}(t) \tag{30}$$

$$w_x(0,t) = c\omega(0,t) - \tilde{q}(t) \frac{1-c^2}{1+\hat{q}(t)c} v(0,t), \tag{31}$$

where

$$\tilde{q}(t) = q - \hat{q}(t) \tag{32}$$

is the parameter estimation error and

$$\theta(x,t) = \frac{1}{1-\hat{q}^2(t)}\left(\int_0^1 (\omega(y,t)+\hat{q}(t)w_x(y,t))\,dy - \frac{2\hat{q}(t)+c+c\hat{q}^2(t)}{1+\hat{q}(t)c}w(1,t)\right) - \int_x^1 \alpha(y,t)dy \tag{33}$$

$$\alpha(x,t) = \frac{1}{1-\hat{q}^2(t)}\left(\omega(x,t) - \frac{\hat{q}(t)+c}{1+\hat{q}(t)c}w_x(x,t)\right) \tag{34}$$

$$\beta(x,t) = \frac{1}{1-\hat{q}^2(t)}\left(w_x(x,t) - \frac{\hat{q}(t)+c}{1+\hat{q}(t)c}\omega(x,t)\right). \tag{35}$$

Proof First we derive (31) by setting $x=0$ in (12) and (24), obtaining

$$\omega(0,t) = v(0,t) + \frac{\hat{q}(t)+c}{1+\hat{q}(t)c}u_x(0,t) \tag{36}$$

$$w_x(0,t) = u_x(0,t) + \frac{\hat{q}(t)+c}{1+\hat{q}(t)c}v(0,t) \tag{37}$$

by substituting (22) into (36), (37), which yields

$$\omega(0,t) = v(0,t) - q\frac{\hat{q}(t)+c}{1+\hat{q}(t)c}v(0,t) \tag{38}$$

$$w_x(0,t) = -qv(0,t) + \frac{\hat{q}(t)+c}{1+\hat{q}(t)c}v(0,t), \tag{39}$$

by multiplying (38) with c, and by subtracting the result from (39). Next, we derive (30). We start by differentiating (12) with respect to t and (24) with respect to x, obtaining

$$\omega_t = u_{tt} + \frac{\hat{q}+c}{1+\hat{q}c}u_{xt} + \frac{1-c^2}{(1+\hat{q}c)^2}u_x\dot{\hat{q}} \tag{40}$$

$$w_{xx} = u_{xx} + \frac{\hat{q}+c}{1+\hat{q}c}u_{xt}. \tag{41}$$

Subtracting these two equations and substituting u_x from (27) we obtain (30) and (35). Finally, we derive (29). Differentiating (11) with respect to t we get

$$w_t = v - \frac{\hat{q}(\hat{q}+c)}{1+\hat{q}c}v_0 + \frac{\hat{q}+c}{1+\hat{q}c}\int_0^x v_t(\xi)d\xi + \mu\dot{\hat{q}}, \tag{42}$$

where $v_0(t) = u_t(0,t)$, $u_0(t) = u(0,t)$, and

$$\mu(x) = -\frac{2\hat{q}+c\hat{q}^2+c}{(1+\hat{q}c)^2}u_0 + \frac{1-c^2}{(1+\hat{q}c)^2}\int_0^x v(\xi)d\xi. \tag{43}$$

Substituting (22), integrating by parts, and then substituting (23), we get

$$w_t = v - \frac{\hat{q}(\hat{q}+c)}{1+\hat{q}c}v_0 + \frac{\hat{q}+c}{1+\hat{q}c}(u_x+qv_0) + \mu\dot{\hat{q}}, \tag{44}$$

Employing (12), we obtain

$$w_t = \omega + \tilde{q}\frac{\hat{q}+c}{1+\hat{q}c}v_0 + \mu\dot{\hat{q}}\,. \tag{45}$$

To establish (29), it remains to show that $\mu = \theta$. From (26) and (34) we note that

$$v = \frac{(1+\hat{q}c)^2}{1-c^2}\alpha\,. \tag{46}$$

Substituting (46) into (43), we get

$$\mu(x) = -\frac{2\hat{q}+c\hat{q}^2+c}{(1+\hat{q}c)^2}u_0 + \int_0^x \alpha(\xi)d\xi\,. \tag{47}$$

Setting $x = 0$ in (11) we get that

$$u_0 = \frac{1+\hat{q}c}{1-\hat{q}^2}w_0\,. \tag{48}$$

With the elementary observation that $w_0 = w_1 - \int_0^1 w_x(y)dy$. where $w_0(t) = w(0,t)$, we get

$$u_0 = \frac{1+\hat{q}c}{1-\hat{q}^2}\left(w_1 - \int_0^1 w_x(y)dy\right)\,. \tag{49}$$

Substituting (49) into (47) we get

$$\mu(x) = -\frac{2\hat{q}+c\hat{q}^2+c}{(1+\hat{q}c)(1-\hat{q}^2)}\left(w_1 - \int_0^1 w_x(y)dy\right) + \int_0^x \alpha(\xi)d\xi\,. \tag{50}$$

Noting that $\int_0^x \alpha = \int_0^1 \alpha - \int_x^1 \alpha$, we get

$$\begin{aligned}\mu(x) = &\int_0^1 \left(\frac{2\hat{q}+c\hat{q}^2+c}{(1+\hat{q}c)(1-\hat{q}^2)}w_x(y) + \alpha(y)\right)dy \\ &- \frac{2\hat{q}+c\hat{q}^2+c}{(1+\hat{q}c)(1-\hat{q}^2)}w_1 - \int_x^1 \alpha(\xi)d\xi\end{aligned} \tag{51}$$

With (46) and (26), a simple calculation shows that

$$\frac{2\hat{q}+c\hat{q}^2+c}{(1+\hat{q}c)(1-\hat{q}^2)}w_x + \alpha = \frac{1}{1-\hat{q}^2}(\omega + \hat{q}w_x) \tag{52}$$

Hence, from (51), (52), and (33), it follows that $\mu = \theta$. ∎

Note that only the first derivative $\dot{\hat{q}}$ appears in the target system (29), (30) and not the second derivative $\ddot{\hat{q}}$.

In the non-adaptive case, the 'target system' $w_{tt} = w_{xx}, w_x(0,t) = cw_t(0,t)$ combined with an appropriate boundary condition at $x = 1$, is exponentially stable. Our task is to prove stability of the target system (29)–(31) in the adaptive case, namely, when $\tilde{q}(t) \neq 0$ and $\dot{\hat{q}}(t) \neq 0$ are acting as perturbations to the system.

Before we proceed with further analysis, in the remainder of this section we give several more relations that are satisfied by the transformed state variables. They will come in handy in the design and in the stability proof. The first of these relations is

$$w_{xt}(x,t) = \omega_x(x,t) + \alpha(x,t)\dot{\hat{q}}(t) \tag{53}$$

and the second one is

$$u_t(0,t) = \frac{1+\hat{q}(t)c}{1+\hat{q}(t)c - q(\hat{q}(t)+c)}\omega(0,t), \tag{54}$$

which, along with (31), gives a damper-like boundary condition at $x = 0$, but with time-varying damping:

$$\begin{aligned} w_x(0,t) =& \frac{c - \tilde{q}(t)\frac{1+\hat{q}(t)c}{1-\hat{q}^2(t)}}{1 - \tilde{q}(t)\frac{\hat{q}(t)+c}{1-\hat{q}^2(t)}}\omega(0,t) & (55)\\ =& \frac{c + \frac{\tilde{q}(t)}{q\hat{q}(t)-1}}{1 + c\frac{\tilde{q}(t)}{q\hat{q}(t)-1}}\omega(0,t) & (56)\\ =& \frac{\hat{q}(t) + c - q(1+\hat{q}(t)c)}{1+\hat{q}(t)c - q(\hat{q}(t)+c)}\omega(0,t). & (57) \end{aligned}$$

Each of the three forms of the damping coefficient (multiplying $\omega(0,t)$ on the right-hand side) will be useful in the subsequent analysis. We will have to restrict the size of the gain c to prevent the denominator of this time-varying damping coefficient from going through zero as our estimate $\hat{q}(t)$ undergoes possibly broad transients. Note however that none of the three forms (55)–(57) of the damping boundary condition at $x = 0$, which are not linear in $\tilde{q}(t)$, will be used for update law design. The form (31), which is linear in $\tilde{q}(t)$, will be used for update law design.

The target system (17)–(19) can be represented completely in terms of the target system variables, by substituting (25) and (27) into (17) and (19), respectively. This representation is

$$\begin{aligned} w_t(x,t) =& \omega(x,t) + \tilde{q}(t)\frac{\hat{q}(t)+c}{1+\hat{q}(t)c - q(\hat{q}(t)+c)}\omega(0,t) + \theta(x,t)\dot{\hat{q}}(t) & (58)\\ \omega_t(x,t) =& w_{xx}(x,t) + \beta(x,t)\dot{\hat{q}}(t) & (59)\\ w_x(0,t) =& \frac{c + \dfrac{\tilde{q}(t)}{q\hat{q}(t)-1}}{1 + c\dfrac{\tilde{q}(t)}{q\hat{q}(t)-1}}\omega(0,t). & (60) \end{aligned}$$

This is an equally valid representation as (29)–(31), however (60) cannot be used for update law design because it is not linearly parameterized in $\tilde{q}(t)$.

$\Omega(t)$	the system norm of the nonadaptive system	(18)
$\Upsilon(t)$	the system norm of the adaptive system	(17)
$\mathscr{E}(t)$	the total energy of the nonadaptive target system	(62)
$E(t)$	the Lyapunov function of the nonadaptive target system	(68)
$V(t)$	the Lyapunov function of the adaptive target system	(75)
$\Sigma(t)$	the system norm of the nonadaptive target system	(61)

Table 1: System norms and Lyapunov-like functions used in the paper.

5 Proof of Theorem 1—Step I: Lyapunov Calculations

The proof of Theorem 1 is rather complicated, since the closed-loop adaptive system is a nonlinear infinite-dimensional system. This system is globally stable and achieves global regulation of the PDE state, however, it is not globally asymptotically stable, since, as is normally the case in adaptive control, the parameter estimation error does not converge to zero.

Our analysis will employ several Lyapunov functions and system norms of increasing complexity. We list them in Table 1 for the reader's convenience. The norms $\Omega(t)$ and

$$\Sigma(t) = \int_0^1 \omega^2(x,t)dx + \int_0^1 w_x^2(x,t)dx + w^2(1,t) \tag{61}$$

are on the space $H^1(0,1) \times L^2(0,1)$, for the states, respectively, (u,v) and (w,ω), whereas the norm $\Upsilon(t)$ is on the space $H^1(0,1) \times L^2(0,1) \times \mathbb{R}$ for the state $(u,v,\tilde{q})$.

We prove Theorem 1 through a series of lemmas, starting with a lemma that employs a simple "total energy" functional as the Lyapunov functional for the system and provides the justification for the selection of the control law (65).

Lemma 2 *Consider the Lyapunov function candidate*

$$\mathscr{E}(t) = \frac{1}{2}\left(\int_0^1 \omega^2(x,t)dx + \int_0^1 w_x^2(x,t)dx + c_1 w^2(1,t)\right), \tag{62}$$

where $c_1 > 0$. The control law (65) guarantees that

$$\begin{aligned}\dot{\mathscr{E}}(t) = &\int_0^1 \left(w_x(x,t)\alpha(x,t) + \omega(x,t)\beta(x,t)\right) dx\dot{\hat{q}}(t) \\ &+ c_1 w(1,t)\theta(1,t)\dot{\hat{q}}(t) - c\omega^2(0,t) \\ &+ \tilde{q}(t)\frac{(1-c^2)\omega(0,t) + c_1(\hat{q}(t)+c)w(1,t)}{1+\hat{q}(t)c}u_t(0,t).\end{aligned} \tag{63}$$

Proof The derivative of the energy function (62), after an integration by parts and substitution of (30) and (53), is obtained in the form

$$\begin{aligned}\dot{\mathscr{E}}(t) &= \int_0^1 (\omega(x,t)w_{xx}(x,t) + w_x(x,t)\omega_x(x,t))\,dx \\ &\quad + \int_0^1 (w_x(x,t)\alpha(x,t) + \omega(x,t)\beta(x,t))\,dx\dot{\hat{q}}(t) \\ &\quad + c_1 w(1,t)w_t(1,t) \\ &= \int_0^1 (w_x(x,t)\alpha(x,t) + \omega(x,t)\beta(x,t))\,dx\dot{\hat{q}}(t) \\ &\quad + \omega(1,t)w_x(1,t) - \omega(0,t)w_x(0,t) + c_1 w(1,t)w_t(1,t)\,, \end{aligned} \tag{64}$$

and, after the substitution of (31) and of (29) for $x = 1$, it becomes

$$\begin{aligned}\dot{\mathscr{E}}(t) &= \int_0^1 (w_x(x,t)\alpha(x,t) + \omega(x,t)\beta(x,t))\,dx\dot{\hat{q}}(t) \\ &\quad + c_1 w(1,t)\theta(1,t)\dot{\hat{q}}(t) - c\omega^2(0,t) \\ &\quad + \tilde{q}(t)\frac{(1-c^2)\omega(0,t) + c_1(\hat{q}(t)+c)w(1,t)}{1+\hat{q}(t)c}u_t(0,t) \\ &\quad + \omega(1,t)w_x(1,t) + c_1 w(1,t)\omega(1,t)\,. \end{aligned} \tag{65}$$

The basis of the controller selection (6) is dealing with the cross-terms in the last line of (65). The control law (6) yields a simple Robin boundary condition

$$w_x(1,t) = -c_1 w(1,t)\,, \tag{66}$$

which, by substitution into (65), gives (63). ∎

Remark 1 The boundary condition (66), which introduces a stiffness/spring-like term in the closed-loop PDE system, helps explain the control law (6). When c_1 is small, the control law (6) uses mainly velocity feedback, whereas larger values of c_1 allow the introduction of position feedback, which is essential for the regulation of the entire (position and velocity) state to zero. Figures 4 and 5 show graphically the distribution of the closed-loop eigenvalues of the PDE system $w_{tt} = w_{xx}, w_x(0,t) = cw_t(0,t), w_x(1,t) = -c_1 w(1,t)$ for $c_1 \gg 1$ and the dependence of the eigenvalues on c and n. The eigenvalues are given by

$$\lambda_n = \varepsilon - \frac{1}{2}\ln\left|\frac{1+c}{1-c}\right| + j\pi\begin{cases} n+1/2\,, & 0 \le c \le 1 \\ n\,, & c > 1 \end{cases} \tag{67}$$

for $n \in \mathbb{Z}$, where $\varepsilon \in \mathbb{C}$, $|\varepsilon| = O(1/c_1)$, and $\mathrm{Re}\varepsilon \le 0$. As gain c grows from 0 to $+1$, the eigenvalues move leftward, all the way to $-\infty$. As c further grows from $+1$ to $+\infty$, the eigenvalues move rightward from $-\infty$ towards the imaginary axis, whereas their imaginary parts drop down by $\pi/2$. For a fixed c, the eigenvalues are always distributed on a vertical line. They depend linearly on n, namely, they are equi-distant along the vertical line.

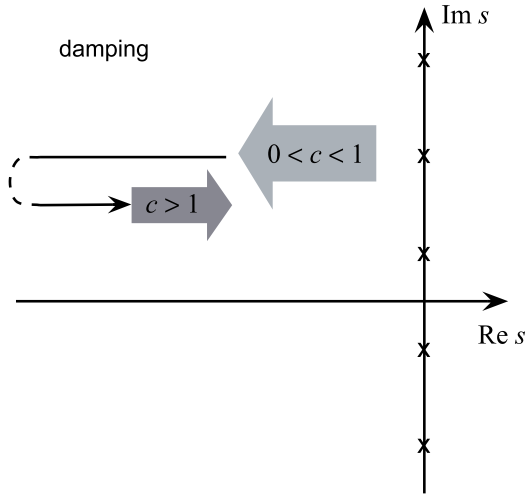

Figure 4: Eigenvalues of the wave PDE with boundary damping for $c_1 \gg 1$. As c grows from 0 to $+1$, the eigenvalues move leftward, all the way to $-\infty$. As c further grows from $+1$ to $+\infty$, the eigenvalues move rightward from $-\infty$ towards the imaginary axis, whereas their imaginary parts drop down by $\pi/2$.

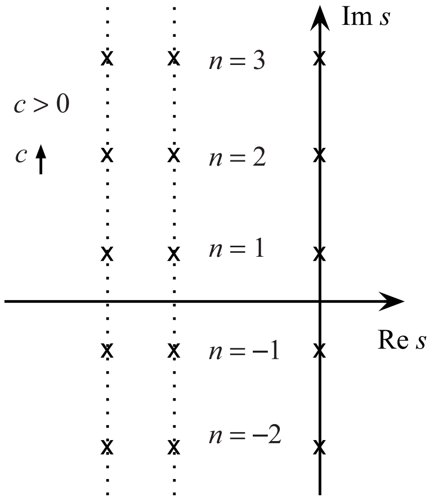

Figure 5: Eigenvalues of the wave PDE with boundary damping for $c_1 \gg 1$, continued from Figure 4. For a fixed c, the eigenvalues are always distributed on a vertical line. They depend linearly on n, namely, they are equidistant along the vertical line.

Even in the nonadaptive case, the result of Lemma 2 is insufficient for proving stability because, when $\tilde{q}(t) \equiv \dot{\hat{q}}(t) \equiv 0$, we only have $\dot{\mathscr{E}}(t) = -c\omega^2(0,t)$, which is only negative semidefinite relative to the Lyapunov function (62).

Lemma 3 *Consider the Lyapunov function candidate*

$$E(t) = \mathscr{E}(t) + \delta \int_0^1 (-2+x)\omega(x,t)w_x(x,t)dx\,. \tag{68}$$

For all $\delta \in (0,1/2)$ *the following holds*

$$(1-2\delta)\mathscr{E}(t) \le E(t) \le (1+2\delta)\mathscr{E}(t)\,. \tag{69}$$

Furthermore,

$$\begin{aligned}\dot{E}(t) = &- \delta \int_0^1 \left(\omega^2(x,t) + w_x^2(x,t)\right) dx \\ &- \frac{\delta}{2}\left(\omega^2(1,t) + c_1^2 w^2(1,t)\right) \\ &- \left(c - \delta\left(1+n(t)\right)\right)\omega^2(0,t) + \eta(t)\dot{\hat{q}}(t) \\ &+ \tilde{q}(t)\frac{(1-c^2)\omega(0,t) + c_1(\hat{q}(t)+c)w(1,t)}{1+\hat{q}(t)c}u_t(0,t)\,,\end{aligned} \tag{70}$$

where

$$n(t) = \left(\frac{c + \frac{\tilde{q}(t)}{q\hat{q}(t)-1}}{1 + c\frac{\tilde{q}(t)}{q\hat{q}(t)-1}}\right)^2 = \left(\frac{\hat{q}(t) + c - q(1+\hat{q}(t)c)}{1+\hat{q}(t)c - q(\hat{q}(t)+c)}\right)^2 \tag{71}$$

and where

$$\begin{aligned}\eta(t) = &\int_0^1 \left(w_x(x,t)\alpha(x,t) + \omega(x,t)\beta(x,t)\right) dx \\ &+ \delta \int_0^1 (-2+x)\left(w_x(x,t)\beta(x,t) + \omega(x,t)\alpha(x,t)\right) dx \\ &+ c_1 w(1,t)\theta(1,t)\,.\end{aligned} \tag{72}$$

Proof Such augmentation is common in the stability theory for wave equations. To our knowledge, it has been used for the first time in the context of adaptive estimation in [9]. It is easy to see, with the help of Young's inequality, that

$$-2\delta\mathscr{E}(t) \le \delta \int_0^1 (-2+x)\omega(x,t)w_x(x,t)dx \le 2\delta\mathscr{E}(t)\,, \tag{73}$$

which implies (69). With a lengthy calculation we now obtain

$$\begin{aligned}\dot{E}(t) = &- \delta \int_0^1 \left(\omega^2(x,t) + w_x^2(x,t)\right) dx \\ &- \frac{\delta}{2}\left(\omega^2(1,t) + w_x^2(1,t)\right) + \delta\left(\omega^2(0,t) + w_x^2(0,t)\right) \\ &- c\omega^2(0,t) + \eta(t)\dot{\hat{q}}(t) \\ &+ \tilde{q}(t)\frac{(1-c^2)\omega(0,t) + c_1(\hat{q}(t)+c)w(1,t)}{1+\hat{q}(t)c}u_t(0,t)\,.\end{aligned} \tag{74}$$

Substituting (56) and (66), we obtain (70). ∎

In the next lemma, we further augment the Lyapunov function, bringing it into its final, complete form. This lemma provides a justification for the choice of the update law.

Lemma 4 *Consider the Lyapunov function candidate*

$$V(t) = \ln(1+E(t)) + \frac{1}{2\gamma}\tilde{q}^2(t)\,, \tag{75}$$

where $\gamma > 0$. The derivative of the Lyapunov function is bounded by

$$\begin{aligned} \dot{V}(t) \leq & \frac{1}{1+E(t)} \left\{ -\delta \int_0^1 \left(\omega^2(x,t) + w_x^2(x,t)\right) dx \right. \\ & - \frac{\delta}{2}\left(\omega^2(1,t) + c_1^2 w^2(1,t)\right) \\ & \left. - \left(c - \delta\left(1+n(t)\right)\right)\omega^2(0,t) + \eta(t)\dot{\hat{q}}(t) \right\}. \end{aligned} \tag{76}$$

Proof Taking a derivative of (75), with the help of (70), we obtain

$$\begin{aligned} \dot{V}(t) = & \frac{1}{1+E(t)} \left\{ -\delta \int_0^1 \left(\omega^2(x,t) + w_x^2(x,t)\right) dx \right. \\ & - \frac{\delta}{2}\left(\omega^2(1,t) + c_1^2 w^2(1,t)\right) \\ & \left. - \left(c - \delta\left(1+n(t)\right)\right)\omega^2(0,t) + \eta(t)\dot{\hat{q}}(t) \right\} \\ & + \tilde{q}(t)\frac{(1-c^2)\omega(0,t) + c_1(\hat{q}(t)+c)w(1,t)}{(1+\hat{q}(t)c)(1+E(t))} u_t(0,t) \\ & - \frac{1}{\gamma}\tilde{q}(t)\dot{\hat{q}}(t)\,, \end{aligned} \tag{77}$$

The update law (7) is chosen to achieve cancellation of the last two terms in (77). To see how the cancellation of the last two terms in (77) occurs, even in the presence of parameter projection, we first introduce a more compact notation for the update law by substituting (54) into (7), thus getting

$$\dot{\hat{q}}(t) = \gamma \frac{\mathrm{Proj}\left\{\xi(t)\right\}}{1+E(t)} \tag{78}$$

$$\xi(t) = \frac{(1-c^2)\omega(0,t) + c_1(\hat{q}(t)+c)w(1,t)}{1+\hat{q}(t)c - q(\hat{q}(t)+c)}\omega(0,t)\,. \tag{79}$$

Hence, the last two terms in (77) are

$$\tilde{q}(t)\frac{\xi(t) - \mathrm{Proj}\left\{\xi(t)\right\}}{1+E(t)}\,. \tag{80}$$

By (8), $\mathrm{Proj}\left\{\xi(t)\right\} = 0$ whenever $(\xi,\hat{q}) \in \mathscr{G} = \left\{\hat{q} \leq \underline{q} \text{ and } \xi < 0\right\} \cup \left\{\hat{q} \geq \bar{q} \text{ and } \xi > 0\right\}$ and $\mathrm{Proj}\left\{\xi(t)\right\} = \xi(t)$ otherwise. Since $\mathscr{G} \in \{\tilde{q}(t)\xi(t) \leq 0\}$, we get that

$$\tilde{q}(t)\frac{\xi(t) - \mathrm{Proj}\left\{\xi(t)\right\}}{1+E(t)} \leq 0\,, \tag{81}$$

thus obtaining (76). ∎

In the next lemma we deal with the possibly detrimental effect of $n(t)$, defined in (71), in the Lyapunov inequality (76).

Lemma 5 *For all $c \in (0, c^*)$, there exists a finite $n^* \geq 0$ such that*

$$\begin{aligned} \dot{V}(t) \leq & \frac{1}{1+E(t)} \left\{ -\delta \int_0^1 \left(\omega^2(x,t) + w_x^2(x,t) \right) dx \right. \\ & -\frac{\delta}{2} \left(\omega^2(1,t) + c_1^2 w^2(1,t) \right) - \left(c - \delta \left(1 + n^* \right) \right) \omega^2(0,t) \\ & \left. +\gamma \frac{\eta(t) \mathrm{Proj}\{\xi(t)\}}{1+E(t)} \right\}. \end{aligned} \tag{82}$$

Proof Substituting (78) and (79) into (76) we get

$$\begin{aligned} \dot{V}(t) \leq & \frac{1}{1+E(t)} \left\{ -\delta \int_0^1 \left(\omega^2(x,t) + w_x^2(x,t) \right) dx \right. \\ & -\frac{\delta}{2} \left(\omega^2(1,t) + c_1^2 w^2(1,t) \right) - \left(c - \delta \left(1 + n(t) \right) \right) \omega^2(0,t) \\ & \left. +\gamma \frac{\eta(t) \mathrm{Proj}\{\xi(t)\}}{1+E(t)} \right\}. \end{aligned} \tag{83}$$

Due to standard properties of the projection operator, we obtain that, with the update law (7), the following is satisfied

$$\hat{q}(t) \in \left[\underline{q}, \bar{q} \right], \quad \forall t \geq 0. \tag{84}$$

Then, due to (84), choosing c such that $0 < c < c^*$, guarantees that

$$1 + \hat{q}(t)c - q(\hat{q}(t) + c) \neq 0 \tag{85}$$

for all time. For a given $c \in (0, c^*)$, let us denote

$$n^* = \max_{\hat{q} \in [\underline{q}, \bar{q}]} \left(\frac{\hat{q} + c - q(1 + \hat{q}c)}{1 + \hat{q}c - q(\hat{q} + c)} \right)^2 < \infty. \tag{86}$$

Hence, we get (87). ∎

With the standard property of the projection operator that $|\mathrm{Proj}\{\xi\}| \leq |\xi|$, we get

$$\begin{aligned} \dot{V}(t) \leq & \frac{1}{1+E(t)} \left\{ -\delta \int_0^1 \left(\omega^2(x,t) + w_x^2(x,t) \right) dx \right. \\ & -\frac{\delta}{2} \left(\omega^2(1,t) + c_1^2 w^2(1,t) \right) - \left(c - \delta \left(1 + n^* \right) \right) \omega^2(0,t) \\ & \left. +\gamma \frac{|\eta(t)||\xi(t)|}{1+E(t)} \right\}. \end{aligned} \tag{87}$$

We need to bound the non-negative term $|\eta(t)||\xi(t)|$, which we do in the next two lemmas.

Lemma 6 *There exists a positive constant* m_1 *such that*

$$|\eta(t)| \leq m_1 \mathscr{E}(t)\,. \tag{88}$$

Proof From (72), with the help of the Cauchy-Schwartz and Young's inequalities we get

$$\begin{aligned} |\eta| \leq & \|w_x\|\|\alpha\| + \|\omega\|\|\beta\| + 2\delta\left(\|w_x\|\|\beta\| + \|\omega\|\|\alpha\|\right) \\ & + \frac{c_1}{2}\left(w_1^2 + \theta_1^2\right) \\ \leq & \left(\frac{1}{2} + \delta\right)\left(\|w_x\|^2 + \|\omega\|^2 + \|\alpha\|^2 + \|\beta\|^2\right) \\ & + \frac{c_1}{2}\left(w_1^2 + \theta_1^2\right), \end{aligned} \tag{89}$$

where $\theta_1(t) \triangleq \theta(1,t)$. Then, from (33)–(35) we get

$$\|\alpha\|^2 \leq 2\left(\frac{\|\omega\|^2}{(1-\hat{q}^2)^2} + \left(\frac{\hat{q}+c}{(1-\hat{q}^2)(1+\hat{q}c)}\right)^2 \|w_x\|^2\right) \tag{90}$$

$$\|\beta\|^2 \leq 2\left(\frac{\|w_x\|^2}{(1-\hat{q}^2)^2} + \left(\frac{\hat{q}+c}{(1-\hat{q}^2)(1+\hat{q}c)}\right)^2 \|\omega\|^2\right) \tag{91}$$

$$\theta_1^2 \leq \frac{3}{(1-\hat{q}^2)^2}\left[\|\omega\|^2 + \hat{q}^2\|w_x\|^2 + \left(\frac{2\hat{q}+c+c\hat{q}^2}{1+\hat{q}c}\right)^2 w_1^2\right] \tag{92}$$

Substituting (90)–(92) into (89), noting from (62) that $\mathscr{E} = \left(\|w_x\|^2 + \|\omega\|^2 + w_1^2\right)/2$, with the help of (84) and a few simple majorizations we get

$$\begin{aligned} m_1 = \max_{0 \leq \underline{q} \leq \hat{q} \leq \bar{q}} \max \Bigg\{ & 1 + 3\left(\frac{2\hat{q}+c+c\hat{q}^2}{(1-\hat{q}^2)(1+\hat{q}c)}\right)^2, \\ & (1+2\delta)\left[1 + 2\left(\frac{1}{(1-\hat{q}^2)^2} \right.\right. \\ & \left.\left. + \left(\frac{\hat{q}+c}{(1-\hat{q}^2)(1+\hat{q}c)}\right)^2\right)\right] + 3c_1 \frac{1+\hat{q}^2}{(1-\hat{q}^2)^2} \Bigg\}\,. \end{aligned} \tag{93}$$

∎

With the next lemma, we complete our Lyapunov calculation, showing that $\dot{V}$ is negative semidefinite.

Lemma 7 *There exist positive constants* δ^* *and* γ^* *such that, for any* $\delta \in (0, \delta^*)$ *and any* $\gamma \in (0, \gamma^*)$*, there exists a positive constant* μ *such that*

$$\begin{aligned} \dot{V}(t) \leq -\frac{\mu}{1+E(t)} \Bigg\{ & \int_0^1 \left(\omega^2(x,t) + w_x^2(x,t)\right) dx \\ & + \omega^2(1,t) + w^2(1,t) + \omega^2(0,t) \Bigg\}\,. \end{aligned} \tag{94}$$

Proof With the help of Lemma 6 and (69), we get

$$\begin{aligned}\dot V(t) \le &\frac{1}{1+E(t)}\left\{-\delta\int_0^1\left(\omega^2(x,t)+w_x^2(x,t)\right)dx\right.\\&-\frac{\delta}{2}\left(\omega^2(1,t)+c_1^2w^2(1,t)\right)\\&-\left(c-\delta\left(1+n^*\right)\right)\omega^2(0,t)\\&\left.+\gamma\frac{m_1}{1-2\delta}|\xi(t)|\right\}. \end{aligned} \tag{95}$$

Finally, with $c\in(0,c^*)$, (84), and Young's inequality, it follows that there exist positive constants m_2 and m_3 such that

$$\frac{m_1}{1-2\delta}|\xi(t)| \le m_2\omega(0,t)^2+m_3w(1,t)^2\,. \tag{96}$$

Consequently,

$$\begin{aligned}\dot V(t) \le &\frac{1}{1+E(t)}\left\{-\delta\int_0^1\left(\omega^2(x,t)+w_x^2(x,t)\right)dx\right.\\&-\frac{\delta}{2}\omega^2(1,t)-\left(\frac{\delta c_1^2}{2}-\gamma m_3\right)w^2(1,t)\\&\left.-\left(c-\gamma m_2-\delta\left(1+n^*\right)\right)\omega^2(0,t)\right\}. \end{aligned} \tag{97}$$

To finish the Lyapunov analysis, we first chose γ sufficiently small (as a function of δ) to make

$$\frac{\delta c_1^2}{2}-\gamma m_3>0 \tag{98}$$

and then δ sufficiently small to make

$$c-\gamma m_2-\delta\left(1+n^*\right)>0\,. \tag{99}$$

Indeed, there exist constants

$$\delta^*(\underline{q},\bar q,c,c_1) = \frac{2c}{\frac{c_1^2m_2}{m_3}+4(1+n^*)} \tag{100}$$

$$\gamma^*(\underline{q},\bar q,c,c_1) = \frac{c_1^2\delta^*}{4m_3}\,, \tag{101}$$

which are found (conservatively) as explicit functions of their arguments, so that (94) holds. ∎

6 Proof of Theorem 1—Step II: Global Stability Bound

From the negative semi-definiteness of (94) it is clear that global stability follows since

$$V(t)\le V(0)\,,\quad \forall t\ge 0\,, \tag{102}$$

however, we want to derive a specific stability estimate in terms of the system norm $\Upsilon(t)$. In the next two lemmas we give some norm estimates that allow us to derive a stability bound.

Lemma 8 *There exist positive constants s_1 and s_2 such that*

$$s_1\Sigma(t) \le \Omega(t) \le s_2\Sigma(t)\,. \tag{103}$$

Proof From (12), (24), (25), (26)–(28), (18), and (61), with the help of (84) and the Cauchy-Schwartz inequality, one obtains

$$\begin{aligned} s_1 = \Bigg(\max_{0\le\underline{q}\le\hat{q}\le\bar{q}} \max\Bigg\{ & 3\left(\frac{1-\hat{q}^2}{1+\hat{q}c}\right)^2, 3\left(\frac{\hat{q}+c}{1+\hat{q}c}\right)^2, \\ & 3\left(\hat{q}\frac{\hat{q}+c}{1+\hat{q}c}\right)^2, 2+2\left(\frac{\hat{q}+c}{1+\hat{q}c}\right)^2\Bigg\}\Bigg)^{-1} \end{aligned} \tag{104}$$

$$\begin{aligned} s_2 = \max_{0\le\underline{q}\le\hat{q}\le\bar{q}} \max\Bigg\{ & 3\left(\frac{1+\hat{q}c}{1-\hat{q}^2}\right)^2, \\ & 3\left(\frac{(1+\hat{q}c)(\hat{q}+c)}{(1-\hat{q}^2)(1-c^2)}\right)^2, 3c^2\left(\frac{(1+\hat{q}c)(\hat{q}+c)}{(1-\hat{q}^2)(1-c^2)}\right)^2, \\ & 2\left(\frac{(1+\hat{q}c)^2}{(1-\hat{q}^2)(1-c^2)}\right)^2 + 2\left(\frac{(1+\hat{q}c)(\hat{q}+c)}{(1-\hat{q}^2)(1-c^2)}\right)^2\Bigg\}\,. \end{aligned} \tag{105}$$

∎

Lemma 9

$$\frac{2s_1}{(1+2\delta)\max\{1,c_1\}}E(t) \le \Omega(t) \le \frac{2s_2}{(1-2\delta)\min\{1,c_1\}}E(t)\,. \tag{106}$$

Proof Consider the norm (61) and note that

$$\frac{\min\{1,c_1\}}{2}\Sigma(t) \le \mathscr{E}(t) \le \frac{\max\{1,c_1\}}{2}\Sigma(t)\,. \tag{107}$$

The result of the lemma follows from Lemma 8, (69), and (107). ∎

Now we are ready to establish a stability bound in terms of the system norm $\Upsilon(t)$.

Lemma 10 *The global stability estimate (16) holds with*

$$R = 2\left(\frac{s_2}{(1-2\delta)\min\{1,c_1\}} + \gamma\right) \tag{108}$$

$$\rho = \frac{1}{2}\left(\frac{(1+2\delta)\max\{1,c_1\}}{s_1} + \frac{1}{\gamma}\right)\,. \tag{109}$$

Proof From (75) it follows that

$$\begin{aligned}\tilde{q}^2(t) &\leq 2\gamma V(t) \leq 2\gamma\left(\mathrm{e}^{V(t)}-1\right) &(110)\\ E(t) &\leq \mathrm{e}^{V(t)}-1\,. &(111)\end{aligned}$$

With Lemma 9, (110), (111), and (102), we obtain

$$\Upsilon(t) \leq 2\left(\frac{s_2}{(1-2\delta)\min\{1,c_1\}}+\gamma\right)\left(\mathrm{e}^{V(0)}-1\right). \tag{112}$$

Finally, with (75) and Lemma 9, we get

$$\begin{aligned}V(0) &\leq E(0)+\frac{1}{2\gamma}\tilde{q}^2(0)\\ &\leq \frac{1}{2}\left(\frac{(1+2\delta)\max\{1,c_1\}}{s_1}+\frac{1}{\gamma}\right)\Upsilon(0)\,. &(113)\end{aligned}$$

With the constants (108) and (109), we obtain a global stability estimate (16). ∎

It still remains to establish the regulation result. This result is proved in the following lemma.

Lemma 11 $\int_0^\infty \Omega(t)dt < \infty$.

Proof By integrating (94) in time, we get that

$$\int_0^\infty \frac{\Sigma(t)+\omega^2(0,t)+\omega^2(1,t)}{1+E(t)}dt \leq \frac{V(0)}{\mu}\,. \tag{114}$$

Next, we note that

$$\begin{aligned}&\int_0^\infty \left(\Sigma(t)+\omega^2(0,t)+\omega^2(1,t)\right)dt\\ &= \int_0^\infty \frac{\Sigma(t)+\omega^2(0,t)+\omega^2(1,t)}{1+E(t)}\left(1+E(t)\right)dt\\ &\leq \left(1+\sup_{\tau\geq 0}E(\tau)\right)\int_0^\infty \frac{\Sigma(t)+\omega^2(0,t)+\omega^2(1,t)}{1+E(t)}dt\,. &(115)\end{aligned}$$

From (102) and (111) we get that $E(t) \leq \mathrm{e}^{V(0)}-1$, and hence

$$\int_0^\infty \left(\Sigma(t)+\omega^2(0,t)\right)dt \leq \frac{V(0)}{\mu}\left(\mathrm{e}^{V(0)}-1\right). \tag{116}$$

By (103), from the integrability of $\Sigma(t)$, it follows that

$$\int_0^\infty \Omega(t)dt \leq \frac{s_2 V(0)}{\mu}\left(\mathrm{e}^{V(0)}-1\right) < \infty\,, \tag{117}$$

which completes the proof. ∎

Remark 2 Though $\int_0^\infty \Omega(t)dt < \infty$ guarantees that $\Omega(t)$ is regulated to below any arbitrarily small positive constant for all times except possibly for a measure zero subset of the time axis, it is also desirable to have that $\lim_{t\to\infty}\Omega(t) = 0$, namely regulation in a strong sense. One would pursue this by proving that $\lim_{t\to\infty}\mathscr{E}(t) = 0$ and infer the same result for $\Omega(t)$ from the fact that $\Omega(t) \leq \frac{2s_2}{\min\{1,c_1\}}\mathscr{E}(t)$. Since $\mathscr{E}(t)$ is integrable, to satisfy the conditions of Barbalat's lemma one would have to show that $\dot{\mathscr{E}}(t)$ in (63) is bounded. For this, one has to show that $\omega(0,t)$ is bounded, using the already established boundedness of the signals $\Sigma(t)$ and $w_x(1,t)$, as well as the square integrability of $\omega(0,t)$ and absolute integrability of $\dot{\hat{q}}(t)$. The additional analysis would have to be performed in higher order norms, with the aid of non-standard changes of variables for achieving homogenization of boundary conditions. Such analysis is beyond the scope of this paper since the challenges come exclusively from the fact that the parametric uncertainty in (2) arises in a boundary condition and multiplies a trace term $u_t(0,t)$ which is not a part of the natural norm of the system $\Omega(t)$. Note that the issue of strong convergence is not that of using more suitable PDE-oriented Barbalat's type lemmas, which have been provided in [3, 27] but of deriving an estimate on the regressor (in terms or a higher order norm) such that the conditions of such lemmas are satisfied.

7 A Lipschitz Alternative to Standard Projection

Parameter projection has been in common use in adaptive control of PDEs since the paper of Demetriou and Rosen [11]. However, a perennial criticism of the standard projection operator is that the discontinuity in the projection operator creates a challenge for establishing uniqueness of solutions. This issue is further aggravated in the area of adaptive control of PDEs because a general theory of Filippov solutions does not exist for PDEs. For this reason, we present in this section a version of the projection operator which is locally Lipschitz and removes the concerns regarding uniqueness of solutions created by the discontinuity of the standard projection operator. However, we point out that this concern is purely theoretical—in simulations, despite the discontinuity in the standard projection operator, due to the uniform boundedness of the right-hand side in (7), the evolution of $\hat{q}(t)$ would remain continuous, as is always the case in simulations of adaptive systems with projection, including those in adaptive control of PDEs [6, 26, 31, 32].

To make the projection operator Lipschitz, we introduce a boundary layer around the boundary of the parameter set, in which the projection ramps up linearly. Outside of the parameter set the projection operator's value is zero, whereas inside the parameter set, or on the boundary of the parameter set and when the update law points inward, the projection operator's value is equal to the nominal update law. This idea was introduced in [29] and specialized to scalar parameter problems in [24].

We replace the discontinuous update law (78) by the update law

$$\dot{\hat{q}}(t) = \gamma \frac{\mathrm{Proj}^{\mathrm{L}}\{\xi(t)\}}{1+E(t)}, \qquad \hat{q}(0) \in \left[\underline{q}, \bar{q}\right], \tag{118}$$

where ξ is defined in (79) and where $\mathrm{Proj}^{\mathrm{L}}$ is the Lipschitz (in ξ and $\hat{q}$) projection operator

$$\mathrm{Proj}^{\mathrm{L}}_{[\underline{q},\bar{q}]}\{\xi\} = \xi \begin{cases} \max\left\{0, 1+\frac{1}{\varepsilon}\left(\hat{q}-\underline{q}\right)\right\}, & \hat{q} \leq \underline{q} \text{ and } \xi < 0 \\ \max\left\{0, 1-\frac{1}{\varepsilon}\left(\hat{q}-\bar{q}\right)\right\}, & \hat{q} \geq \bar{q} \text{ and } \xi > 0 \\ 1, & \text{else} \end{cases} \tag{119}$$

which has a linear (in $\hat{q}$) boundary layer of thickness ε around the projection set $[\underline{q},\bar{q}]$. The parameter ε is chosen as $\varepsilon \in (0,\varepsilon^*)$, where

$$\varepsilon^* = \min\left\{\left|1-\underline{q}\right|, \left|1-\bar{q}\right|\right\}. \tag{120}$$

The addition of the boundary layer is trivial. What is not trivial however, is that all the desirable properties of the standard projection operator still hold in the presence of the boundary layer, which is established in the following lemma.

Lemma 12 *The following properties hold for the projection operator (119):*

1. *The projection operator (119) is locally Lipschitz in* $(\xi,\hat{q})$.

2. *For all* $(\xi,\hat{q}) \in \mathbb{R}^2$, *the following holds:*

$$\left|\mathrm{Proj}^{\mathrm{L}}_{[\underline{q},\bar{q}]}\{\xi\}\right| \leq |\xi|. \tag{121}$$

3. *With* $\hat{q}(0) \in \left[\underline{q},\bar{q}\right]$, *the update law (118) with the projection operator (119) guarantees that the parameter estimate is maintained in the expanded projection set, namely,*

$$\hat{q}(t) \in \left[\underline{q}-\varepsilon, \bar{q}+\varepsilon\right], \qquad \forall t \geq 0. \tag{122}$$

4. *For all* $(\xi,\hat{q}) \in \mathbb{R}^2$, *the following holds:*

$$(q-\hat{q})\left(\xi - \mathrm{Proj}^{\mathrm{L}}\{\xi\}\right) \leq 0. \tag{123}$$

Proof Points 1, 2, and 3 are immediate. To establish Point 4, first we note from (119) that

$$\xi - \mathrm{Proj}^{\mathrm{L}}\{\xi\} = \xi \begin{cases} \min\left\{1, -\frac{1}{\varepsilon}\left(\hat{q}-\underline{q}\right)\right\}, & \hat{q} \leq \underline{q} \text{ and } \xi < 0 \\ \min\left\{1, \frac{1}{\varepsilon}\left(\hat{q}-\bar{q}\right)\right\}, & \hat{q} \geq \bar{q} \text{ and } \xi > 0 \\ 0, & \text{else} \end{cases} \tag{124}$$

and then that

$$
\begin{aligned}
\tilde{q}\left(\xi - \mathrm{Proj}^{\mathrm{L}}\{\xi\}\right) &= (q-\hat{q})\,\xi \begin{cases} \min\left\{1, +\frac{1}{\varepsilon}\left(\underline{q}-\hat{q}\right)\right\}, & \hat{q} \le \underline{q} \text{ and } \xi < 0 \\ \min\left\{1, \frac{1}{\varepsilon}\left(\hat{q}-\bar{q}\right)\right\}, & \hat{q} \ge \bar{q} \text{ and } \xi > 0 \\ 0, & \text{else} \end{cases} \\
&= \begin{cases} (q-\hat{q})\,\xi \min\left\{1, +\frac{1}{\varepsilon}\left(\underline{q}-\hat{q}\right)\right\}, & \hat{q} \le \underline{q} \text{ and } \xi < 0 \\ (q-\hat{q})\,\xi \min\left\{1, \frac{1}{\varepsilon}\left(\hat{q}-\bar{q}\right)\right\}, & \hat{q} \ge \bar{q} \text{ and } \xi > 0 \\ 0, & \text{else} \end{cases} \\
&= \begin{cases} |q-\hat{q}|\,\xi \min\left\{1, +\frac{1}{\varepsilon}\left|\underline{q}-\hat{q}\right|\right\}, & \hat{q} \le \underline{q} \text{ and } \xi < 0 \\ -|q-\hat{q}|\,\xi \min\left\{1, \frac{1}{\varepsilon}\left|\hat{q}-\bar{q}\right|\right\}, & \hat{q} \ge \bar{q} \text{ and } \xi > 0 \\ 0, & \text{else} \end{cases} \\
&\le 0,
\end{aligned}
\tag{125}
$$

which completes the proof of Point 4. ∎

Though the projection operator is locally Lipschitz, it is not globally Lipschitz, due to the bilinear dependence $\frac{1}{\varepsilon}\xi\hat{q}$ in the set $\left\{\left\{\underline{q}-\varepsilon \le \hat{q} \le \underline{q}\right\} \cap \{\xi < 0\}\right\} \cup \left\{\left\{\bar{q} \le \hat{q} \le \bar{q}+\varepsilon\right\} \cap \{\xi > 0\}\right\}$.

To see that the update law with projection $\mathrm{Proj}^{\mathrm{L}}\{\cdot\}$ with a boundary layer ε has the same desirable effect on $\dot{V}$ as the update law with standard discontinuous projection, we return to (77),

$$
\begin{aligned}
\dot{V}(t) = &\frac{1}{1+E(t)}\left\{-\delta\int_0^1\left(\omega^2(x,t)+w_x^2(x,t)\right)dx - \frac{\delta}{2}\left(\omega^2(1,t)+c_1^2 w^2(1,t)\right)\right. \\
&\left. -\left(c-\delta\left(1+n(t)\right)\right)\omega^2(0,t)+\eta(t)\dot{\hat{q}}(t)\right\} \\
&+\tilde{q}(t)\frac{\xi(t)-\mathrm{Proj}^{\mathrm{L}}\{\xi(t)\}}{1+E(t)},
\end{aligned}
\tag{126}
$$

and focus on the last term. From (123) it follows that

$$
\begin{aligned}
\dot{V}(t) \le &\frac{1}{1+E(t)}\left\{-\delta\int_0^1\left(\omega^2(x,t)+w_x^2(x,t)\right)dx - \frac{\delta}{2}\left(\omega^2(1,t)+c_1^2 w^2(1,t)\right)\right. \\
&\left. -\left(c-\delta\left(1+n(t)\right)\right)\omega^2(0,t)+\eta(t)\dot{\hat{q}}(t)\right\},
\end{aligned}
\tag{127}
$$

as was the case in (76) with projection without the boundary layer ε.

With (127) we obtain the following stability result.

Theorem 2 *Consider the closed-loop system consisting of the plant (1)–(3), the control law (6), and the parameter update law (118) with the Lipschitz projection operator (119). Let Assumption 2.1 hold and pick any control gain* $c_1 > 0$ *and any boundary layer thickness* $\varepsilon \in (0, \varepsilon^*)$ *in the projection operator. There exist* $c^{**}(\varepsilon) \in (0, c^*)$ *and* $\gamma^* > 0$ *such that for all* $c \in (0, c^{**})$*, all* $\gamma \in (0, \gamma^*)$*, and all* $\delta \in (0, 1/2)$*, the zero solution of the system* $(u, v, \hat{q} - q)$ *is globally stable in the sense that there exist positive constants R and* ρ *(independent of the initial conditions)* *such that for all initial conditions satisfying* $(u_0, v_0, \hat{q}_0) \in H_1(0,1) \times L_2(0,1) \times [\underline{q}, \bar{q}]$, *the following holds:*

$$\Upsilon(t) \leq R\left(e^{\rho\Upsilon(0)} - 1\right), \quad \forall t \geq 0, \tag{128}$$

where

$$\Upsilon(t) = \Omega(t) + (q - \hat{q}(t))^2 \tag{129}$$

and

$$\Omega(t) = \int_0^1 v^2(x,t)dx + \int_0^1 u_x^2(x,t)dx + u^2(1,t). \tag{130}$$

Furthermore,

$$\int_0^\infty \Omega(t)dt \leq \infty, \tag{131}$$

i.e., regulation is achieved in the sense that $\mathrm{ess}\lim_{t\to\infty}\Omega(t) = 0$.

The only difference between the results in Theorems 1 and 2 is that the gain c in Theorem 2 has to be restricted to lower values, $0 < c < c^{**}(\varepsilon) < c^*$, to accommodate the fact that the Lipschitz projection operator no longer keeps the parameter estimate $\hat{q}(t)$ restricted to the set $\left[\underline{q}, \bar{q}\right]$ but to the larger set $\left[\underline{q} - \varepsilon, \bar{q} + \varepsilon\right]$. The only difference in the proof of Theorem 2 is in a slightly different proof of Lemma 5, notably in a slightly different expression in (86).

It is also worth noting that, not only can the projection operator be made Lipschitz, and still preserve all of the desirable properties of the discontinuous projection operator, but it can be made arbitrarily many times differentiable. This extension of the results in [24, 29] is provided in [7]. This approach can be used in the present paper to further smoothen the right-hand side of the closed-loop system, although it is not necessary to go beyond the Lipschitz projection operator in our case. The smooth projector in [7] was proposed for a usage in recursive overparametrization-based backstepping designs for ODEs, where the update law needs to be differentiated a certain number of times relative to the plant state and the parameter estimate.

8 Conclusions

We presented an adaptive feedback law for boundary control of an unstable wave equation with an unmatched parametric uncertainty. The basic idea introduced here for how to approach second-order-in-time PDE problems is potentially usable in other similar PDEs, from variations on the wave equation to beam equations.

For example, if we consider the wave equation, but with the anti-damping boundary condition (2) replaced by an anti-stiffness boundary condition given as $u_x(0,t) = -qu(0,t)$, such as studied in [23], we would introduce the transformation

$$w(x,t) = u(x,t) + (c + \hat{q}(t))\int_0^x e^{\hat{q}(t)(x-y)}u(y,t)dy \tag{132}$$

$$\omega(x,t) = u_t(x,t) + (c + \hat{q}(t))\int_0^x e^{\hat{q}(t)(x-y)}u_t(y,t)dy, \tag{133}$$

and then proceed with adaptive control design as in this paper. The update law would be obtained as

$$\dot{\hat{q}}(t) = \gamma \mathrm{Proj}\left\{ \frac{\zeta(t)u(0,t)}{1+E(t)} \right\} \tag{134}$$

$$\zeta(t) = \omega(0,t) + (c+\hat{q}(t)) \int_0^1 \mathrm{e}^{\hat{q}(t)x} \omega(x,t)dx \tag{135}$$

$$E(t) = \frac{1}{2}\left(\int_0^1 \omega^2(x,t)dx + \int_0^1 w_x^2(x,t)dx + cw^2(0,t) \right) + \delta \int_0^1 (1+x)\omega(x,t)w_x(x,t)dx. \tag{136}$$

For the PDE system considered in this paper, an output-feedback adaptive design can be pursued using a combination of tools from [33] and [32].

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