Delay-adaptive feedback for linear feedforward systems

Nikolaos Bekiaris-Liberis *, Miroslav Krstic
Department of Mechanical and Aerospace Engineering, University of California at San Diego, La Jolla, CA 92093-0411, USA

A R T I C L E   I N F O
Article history:
Received 8 October 2009
Received in revised form 13 March 2010
Accepted 1 March 2010
Available online 13 April 2010

Keywords:
Delays systems
Adaptive control
Feedforward systems

A B S T R A C T
Predictor techniques are an indispensable part of the control design toolbox for plants with input and state delays of significant size. Yet, they suffer from sensitivity to the design values. Explicit feedback laws were recently introduced by Jankovic for a class of feedforward linear systems with simultaneous state and input delays. For the case where the delays are of unknown length, using the certainty equivalence principle, we design a Lyapunov-based adaptive controller, which achieves global stability and regulation, for arbitrary initial estimates for the delays. We consider a two-block sub-class of linear feedforward systems. A generalization to the n-block case involves a recursive application of the same techniques.

© 2010 Elsevier B.V. All rights reserved.

1. Introduction

The design of stabilizing controllers for systems with delays continues to be an active area of research. Controllers for both linear and nonlinear systems exist in the literature, many of which are based on predictor-like techniques [1–9,24–38]. However, systems with simultaneous input and state delay remain a challenge, even for linear systems [10,2–4,11–13].

An even more challenging problem is the adaptive control of systems with simultaneous input and state delays. From the practical point of view, controllers for delay systems should be robust to parametric uncertainties, including plant parameters and delays. The importance of designing robust controllers when the delays are the unknown parameters was highlighted in the control problems considered in [14] and [15]. On the other hand, since there already exists a rich literature for the control of time delay systems, adaptive control schemes that are based on existing control techniques are of interest. Since many control schemes are based on predictor-like techniques, which are known to be very sensitive to delay uncertainties [16], designing adaptive versions of these control schemes is crucial for making them usable in scenarios with uncertain delays.

Adaptive control schemes can be found in [17,18]. Yet, the adaptive control problem when the delays are the unknown parameters had not been solved until recently with the works in [19,20]. In [19] the problem of designing an adaptive control scheme for a linear system with unknown input delay is solved, and in [20] the result is extended to also incorporate unknown plant parameters. The aforementioned designs are based on predictor feedback together with tools that come from the adaptive control of parabolic PDEs [21].

In this paper we develop a delay-adaptive version of the design introduced by Jankovic in [4] for linear feedforward systems with simultaneous state and input delays. In [4], for the system

\[
\begin{align*}
\dot{X}_1(t) &= F_1X_1(t) + H_1X_2(t - D_1) + B_1U(t) \\
\dot{X}_2(t) &= F_2X_2(t) + B_2U(t),
\end{align*}
\]

a predictor-based controller is designed as

\[
U(t) = K_1D_1 \int_0^t e^{-\int_0^\theta D_1(\theta - 1)} d\theta H_1X_2(t + D_1(\theta - 1)) d\theta + K_1X_1(t) + K_2X_2(t).
\]

The above controller is based on a transformation that reduces the system to an equivalent system without state delay. This transformation is

\[
Z_1(t) = X_1(t) + D_1 \int_0^t e^{-\int_0^\theta D_1(\theta - 1)} H_1X_2(t + D_1(\theta - 1)) d\theta
\]

\[
Z_2(t) = X_2(t),
\]

and it transforms the system (1)–(2) to

\[
\begin{align*}
\dot{Z}_1(t) &= F_1Z_1(t) + e^{-\int_0^\theta D_1(\theta - 1)} H_1Z_2(t) + B_1U(t) \\
\dot{Z}_2(t) &= F_2Z_2(t) + B_2U(t).
\end{align*}
\]

The importance of the previous transformation, besides transforming the original system to an equivalent one without state delay, is that the system can be linearly parameterized in the state delay, which is the key for designing an adaptive control law.
Then, assuming that the pair \( \left[ \begin{array}{c} f_1 \\ 0 \\ f_2 \end{array} \right], \left[ \begin{array}{c} B_1 \\ B_2 \end{array} \right] \) is completely controllable, a state feedback controller \( U(t) = K \dot{Z}_1(t) + K_2 Z_2(t) \) is designed such that the transformed system is asymptotically stable. It can be shown that controllability of the original system is equivalent to controllability of the transformed system [22]. This design can be also applied in the case where there is a delay in the input, say \( D_2 \). In this case after employing the state transformation a predictor feedback is needed for the transformed system. In this case the controller that compensates for \( D_2 \) is given by

\[
U(t) = KE^{D_2}Z(t) + KD_2 \int_0^t e^{D_2(t-\theta)}B \left( u(t) - D_2 (\theta - 1) \right) d\theta,
\]

(8)

where

\[
A = \begin{bmatrix} f_1 & e^{-D_2 f_1} H_1 \\ 0 & f_2 \end{bmatrix},
\]

(9)

\[
B = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix},
\]

(10)

\[
K = \begin{bmatrix} K_1 \\ K_2 \end{bmatrix}.
\]

(11)

The controller (8) is the basis of our adaptive design.

2. Problem formulation

In this paper we consider both the state and input delays to be unknown, that is, we consider the system

\[
\dot{X}_1(t) = f_1 X_1(t) + H_1 X_2(t - D_1) + B_1 U(t - D_2)
\]

(12)

\[
\dot{X}_2(t) = f_2 X_1(t) + B_2 U(t - D_2),
\]

(13)

with \( D_1 \) and \( D_2 \) unknown. Since \( D_1 \) and \( D_2 \) are unknown, in addition to the predictor based controller, we must design two estimators, one for each of the delays. We employ projector operators and assume a bound on the lengths of the delays to be known.

Assumption 1. There exist known constants \( D_1, \overline{D}_1 \) and \( D_2, \overline{D}_2 \) such that \( D_1 \in \overline{[D_1, \overline{D}_1]} \) and \( D_2 \in [0, \overline{D}_2] \).

Our controller is based on the transformed system (i.e., on the system without state delay). As indicated in Section 1, the pair \( \left[ \begin{array}{c} f_1 \\ 0 \\ f_2 \end{array} \right], \left[ \begin{array}{c} B_1 \\ B_2 \end{array} \right] \) must be completely controllable. Under this assumption we can find a stabilizing state feedback. In the case of unknown state delay \( D_1 \), we must assume that there exists a stabilizing state feedback for all values of the state delay in a given interval. We thus make the following assumption.

Assumption 2. The pair \( \left[ \begin{array}{c} f_1 \\ 0 \\ f_2 \end{array} \right], \left[ \begin{array}{c} B_1 \\ B_2 \end{array} \right] \) is completely controllable \( \forall D_1 \in \overline{[D_1, \overline{D}_1]} \). Furthermore, we assume that there exists a triple of vector/valued functions \( K(D_1), P(D_1), Q(D_1) \) such that \( K(D_2) \in C^1([D_2, \overline{D}_2]), P(D_2) \in C^1([D_2, \overline{D}_2]), Q(D_2) \in \mathbb{C}^0([D_2, \overline{D}_2]), \) the matrices \( P(D_2) \) and \( Q(D_2) \) are positive definite and symmetric, and the following Lyapunov equation is satisfied \( \forall D_1 \in \overline{[D_1, \overline{D}_1]} \):

\[
(A(D_1) + BK(D_1))^T P(D_1) + P(D_1) (A(D_1) + BK(D_1)) = -Q(D_1), \forall D_1 \in \overline{[D_1, \overline{D}_1]}.
\]

(14)

Our final assumption is needed in the choice of the normalization coefficients in the adaptation law for delay estimates.

Assumption 3. The quantities \( \lambda = \inf_{D_1 \in \overline{[D_1, \overline{D}_1]}} \min(\lambda_{\min}(Q(D_1)), \lambda_{\max}(P(D_1))) \) and \( \overline{\lambda} = \sup_{D_1 \in \overline{[D_1, \overline{D}_1]}} \lambda_{\max}(P(D_1)) \) exist and are known.

3. Controller design

We first rewrite (12)-(13) using a PDE representation of the delayed states and control as

\[
\dot{X}_1(t) = f_1 X_1(t) + H_1 \dot{\xi}(0, t) + B_1 u(0, t)
\]

(15)

\[
D_1 \dot{\xi}_1(t, x) = \xi_1(t, x)
\]

(16)

\[
\xi_1(1, t) = X_2(t)
\]

(17)

\[
\dot{X}_2(t) = f_2 X_1(t) + B_2 u(0, t)
\]

(18)

\[
D_1 u_1(t, x) = u_k(t, x)
\]

(19)

\[
u_1(1, t) = U(t),
\]

(20)

where \( x \in [0, 1] \). We assume that the infinite-dimensional states \( \xi_1(t, x), u_1(t, x), x \in [0, 1] \) are available for measurement. This assumption is not in contradiction with the assumption that the convection speeds \( 1/D_1 \) and \( 1/D_2 \) are unknown. As restrictive as the requirement for measurement of \( \xi_1(t, x), u_1(t, x), x \in [0, 1] \) may appear, we do not believe that the delay-adaptive problem without such measurements is solvable globally because it cannot be formulated as linearly parameterized in the unknown delays \( D_1 \) and \( D_2 \).

The transport PDE states can be expressed in terms of the past values of \( X_2 \) and \( U \) as

\[
\dot{\xi}_1(t, x) = X_2(t + D_1 (x - 1))
\]

(21)

\[
u_1(t, x) = U(t + D_2 (x - 1)).
\]

(22)

Using the certainty equivalence principle the controller (8) is taken as

\[
U(t) = K \hat{D}_1 e^{\hat{D}_1 \hat{D}_2(t)} \left[ X_1(t) + \hat{D}_1(t) \int_0^1 e^{-\hat{D}_1 \hat{D}_2(t)} H_1 \xi(y, t) dy \right]
\]

(23)

\[
+ K \hat{D}_2(t) \int_0^1 e^{\hat{D}_1 \hat{D}_2(t)} \left[ \dot{\xi}_1(t, y) - \xi(t, y) \right] dy.
\]

The update laws for the estimations of the unknown delays \( D_1 \) and \( D_2 \) are given by

\[
\hat{\dot{D}}_1(t) = \gamma_1 \text{Proj}_{[\overline{D}_1, \overline{D}_1]} [\tau_{D_1}]
\]

(24)

\[
\hat{\dot{D}}_2(t) = \gamma_2 \text{Proj}_{[0, \overline{D}_2]} [\tau_{D_2}],
\]

(25)

where the projector operators are defined as

\[
\text{Proj}_{[\overline{D}_1, \overline{D}_1]} [\tau_{D_1}] = \begin{cases} 0 & \text{if } \hat{D}_1 = \overline{D}_1 \text{ and } \tau_{D_1} < 0 \\ 0 & \text{if } \hat{D}_1 = \overline{D}_1 \text{ and } \tau_{D_1} > 0 \\ 1 & \text{else} \end{cases}
\]

(26)

and where

\[
\tau_{D_1} = \frac{\int_0^1 (1 + x) u(x, t) K(\hat{D}_1) e^{\hat{D}_1 \hat{D}_2(t)} dx - \frac{k}{\alpha} Z(t) P(\hat{D}_1)}{\Gamma(t)} \times R_2(t)
\]

(27)

\[
\tau_{D_2} = \frac{-\int_0^1 (1 + x) u(x, t) K(\hat{D}_1) e^{\hat{D}_1 \hat{D}_2(t)} dx}{\Gamma(t)} \times (Bu(0, t) + A(\hat{D}_1) Z(t))
\]

(28)

\[
\Gamma(t) = 1 + Z(t) P(\hat{D}_1) Z(t) + \alpha_2 \int_0^1 (1 + x) u^2(x, t) dx
\]

+ \int_0^1 (1 + x) \dot{\xi}_1^2(t, x) \dot{\xi}_1(t, x) dx,
\]

(29)
with

\[ k \leq \frac{\lambda D_1}{\delta} \]  

\[ D_2 \sup_{\tilde{b}_1 \in [\tilde{b}_1, \tilde{b}_1]} |P(\tilde{D}_1)B|^2 \]

\[ a_2 \geq \frac{\lambda}{\delta} \]  

In the above relation we use the following signals which are derived in the stability analysis of the closed-loop system

\[ Z_1(t) = x_1(t) + \tilde{b}_1(t) \int_0^t e^{-\tilde{b}_1(t\theta)}H_1\xi(\theta, t)d\theta \]  

(32)

\[ Z_2(t) = x_2(t) \]  

(33)

\[ R_2(t) = \left[ e^{-\tilde{b}_1(t\theta)}H_1Z_2(t) - H_1\xi(0, t) + \tilde{b}_1(t)F_1 \int_0^t e^{-\tilde{b}_1(t\theta)}H_1\xi(\theta, t)d\theta \right] \]  

(34)

and the transformed infinite dimensional state of the actuator

\[ u(x, t) = u(x, t) - K(\tilde{D}_1) \left( e^{\tilde{b}_1(t\theta)}H_2Z_2(t) + \tilde{D}_2(t) \right) \]  

\[ \times \int_0^t e^{\tilde{b}_1(t\theta)}H_2Z_2(t)(\theta - y)Bu(y, t)d\theta \].  

(35)

### 4. Stability analysis

This section is devoted to the proof of the main result. We start by giving the main theorem and in the rest of the section we prove it using a series of technical lemmas.

**Theorem 1.** Let Assumptions 1–3 hold. Then system (12)–(13) with the controller (23) and the update laws (24)–(25) is stable in the sense that there exist constants \( R \) and \( \rho \) such that

\[ \Omega(t) \leq R (e^{\rho \Omega(0)} - 1) \],

where

\[ \Omega(t) = \|X(t)\|^2 + \|\xi(t)\|^2 + \|u(t)\|^2 + \tilde{D}_1^2(t) + \tilde{D}_2^2(t), \]

(37)

\[ \|\xi(t)\|^2 = \int_0^t \|\xi(\theta, t)\|^2 d\theta \]  

(38)

\[ \|u(t)\|^2 = \int_0^t \|u(\theta, t)\|^2 d\theta . \]

(39)

Furthermore

\[ \lim_{t \to \infty} X(t) = 0 \]

(40)

\[ \lim_{t \to \infty} \Omega(t) = 0 \].

(41)

We start proving the above theorem by first transforming the system (15)–(20) using the transformations (32)–(33) and (35). By differentiating with respect to time (32) and (33) and by using (15) and (18) we get

\[ \dot{Z}_1(t) = F_1X_1(t) + H_2\xi(0, t) + B_1u(0, t) \]

\[ + \tilde{D}_1(t) \int_0^t e^{-\tilde{b}_1(t\theta)}H_1\xi(\theta, t)d\theta \]  

(42)

\[ - \tilde{D}_1(t)\tilde{D}_2(t) \int_0^t F_1e^{-\tilde{b}_1(t\theta)}H_1\xi(\theta, t)d\theta \]

\[ + \tilde{D}_1(t) \int_0^t e^{-\tilde{b}_1(t\theta)}H_1\xi(\theta, t)d\theta \]

\[ \dot{Z}_2(t) = F_2X_2(t) + B_2u(0, t) \].

(43)

Using relations (16) and (17), the fact that \( \frac{\dot{b}_1}{\delta} = 1 - \frac{\dot{b}_1}{\delta} \) and integrating by parts the last integral in (42) we obtain

\[ \dot{Z}_1(t) = F_1X_1(t) + H_2\xi(0, t) + B_1u(0, t) \]

\[ + \tilde{D}_1(t) \int_0^t e^{-\tilde{b}_1(t\theta)}H_1\xi(\theta, t)d\theta \]  

\[ - \tilde{D}_1(t)\tilde{D}_2(t) \int_0^t F_1e^{-\tilde{b}_1(t\theta)}H_1\xi(\theta, t)d\theta \]

\[ + \tilde{D}_1(t) \int_0^t e^{-\tilde{b}_1(t\theta)}H_1\xi(\theta, t)d\theta \]

\[ + 1 - \frac{\tilde{D}_1}{\tilde{D}_1} \int_0^t F_1e^{-\tilde{b}_1(t\theta)}H_1\xi(\theta, t)d\theta \]

\[ + 1 - \frac{\tilde{D}_1}{\tilde{D}_1} \int_0^t F_1e^{-\tilde{b}_1(t\theta)}H_1\xi(\theta, t)d\theta \].

(44)

\[ \dot{Z}_2(t) = F_2X_2(t) + B_2u(0, t) \].

(45)

Using (32)–(33) and after some algebra we arrive at

\[ \dot{Z}(t) = \left[ \frac{\dot{Z}_1(t)}{\dot{Z}_2(t)} \right] \]

\[ = A(\tilde{D}_1)Z(t) + Bu(0, t) + \tilde{D}_1(t)R_1(t) - \frac{\tilde{D}_1}{\tilde{D}_1}R_2(t) \].

(46)

where \( R_2(t) \) is defined in (34) and

\[ A(\tilde{D}_1) = \begin{bmatrix} F_1 & -e^{-\tilde{b}_1(t\theta)}H_1 \\ 0 & F_2 \end{bmatrix} \]

(47)

\[ B = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} \]

(48)

\[ R_1(t) = \left[ \int_0^t \left( I - \frac{\tilde{D}_1}{\tilde{D}_1}(1 - \tilde{D}_1) \right)e^{-\tilde{b}_1(t\theta)}H_1\xi(\theta, t)d\theta \right] \]

(49)

Using relation (35) for \( x = 0 \) we get that

\[ \dot{Z}(t) = A(\tilde{D}_1)Z(t) + Bu(0, t) \]

\[ + \tilde{D}_1(t)R_1(t) - \frac{\tilde{D}_1}{\tilde{D}_1}R_2(t) \].

(50)

Moreover the transformation of the actuator state \( w \) satisfies

\[ D_2u_1(x, t) = w_1(x, t) + \frac{\tilde{D}_1}{\tilde{D}_1}D_2P_1(t, x, t) - \tilde{D}_2P_2(x, t) \]

\[ - D_2\frac{x_1(t)}{\dot{x}_1(t)}(x_1(t) - D_2\frac{x_2(t)}{\dot{x}_2(t)}(x_2(t), t) \]

(51)

where

\[ p_1(x, t) = K(\tilde{D}_1)e^{\tilde{b}_1(t\theta)}h_1\tilde{D}_2(t) \]

(52)

\[ q_1(x, t) = \int_0^x \left( \frac{\partial K(\tilde{D}_1)}{\partial \tilde{D}_1} + K(\tilde{D}_1)\frac{\partial A(\tilde{D}_1)}{\partial \tilde{D}_1} \right)(x - y) \]

\[ \times \tilde{D}_2(t)e^{\tilde{b}_1(t\theta)}h_1\tilde{D}_2(t) \]

\[ + \left( \frac{\partial K(\tilde{D}_1)}{\partial \tilde{D}_1} + K(\tilde{D}_1)\frac{\partial A(\tilde{D}_1)}{\partial \tilde{D}_1} \right) \tilde{D}_2(t) \]

\[ \times e^{\tilde{b}_1(t\theta)}h_1\tilde{D}_2(t) \]

\[ \times \tilde{D}_2(t)e^{\tilde{b}_1(t\theta)}h_1\tilde{D}_2(t) \]

\[ + K(\tilde{D}_1)A(\tilde{D}_1) \]

\[ \times e^{\tilde{b}_1(t\theta)}h_1\tilde{D}_2(t) \]

(53)

\[ p_2(x, t) = K(\tilde{D}_1)e^{\tilde{b}_1(t\theta)}h_1\tilde{D}_2(t)Bu(0, t) + K(\tilde{D}_1)A(\tilde{D}_1) \]

\[ \times e^{\tilde{b}_1(t\theta)}h_1\tilde{D}_2(t) \]

\[ \times Bu(0, t) \]

(54)

\[ q_2(x, t) = K(\tilde{D}_1) \]

\[ \int_0^x \left( I + \tilde{D}_2(t)(x - y)A(\tilde{D}_1) \right) e^{\tilde{b}_1(t\theta)}h_1\tilde{D}_2(t) \]

\[ \times Bu(0, t) \]

(55)
Thus now, system (15)–(20) is mapped to the target system that is comprised of (50) and (51). Moreover, the inverse transformation of the state $X(t)$ is easily obtained from Eqs. (32)–(33) and the inverse transformation of (35) is given by

$$u(x, t) = w(x, t) + K(\tilde{D}_1) \left( e^{(R(\tilde{D}_1)+B(\tilde{D}_1))}\tilde{D}_2(t)Z(t) + \tilde{D}_2(t) \int_0^t e^{(R(\tilde{D}_1)+B(\tilde{D}_1))}(\xi-y)Bu(y, t) dy \right).$$  

We first prove that the signals in (50) and (51) that multiply the “disturbances” $\tilde{D}_1$ and $\tilde{D}_2$, are bounded with respect to the system's transformed states $Z(t)$, $\xi(x, t)$, and $w(x, t)$. Before doing that, we point out that boundedness of the transformed states is equivalent to boundedness of the original states.

**Lemma 1.** There exist constants $M_w, M_u, M_\xi$ and $M_2$ such that

$$\|u(t)\|^2 \leq M_w (\|w(t)\|^2 + |Z(t)|^2)$$  

$$|X(t)|^2 \leq M_x (|Z(t)|^2 + \|\xi(t)\|^2)$$  

$$\|w(t)\|^2 \leq M_w (\|u(t)\|^2 + |Z(t)|^2)$$  

$$|Z(t)|^2 \leq M_z (|X(t)|^2 + \|\xi(t)\|^2).$$  

**Proof.** First observe that the signals $K(\tilde{D}_1)$, $P(\tilde{D}_1)$ and $A(\tilde{D}_1)$ are continuously differentiable with respect to $\tilde{D}_1$. Moreover, since $\tilde{D}_1$ and $\tilde{D}_2$ are uniformly bounded, the signals $K(\tilde{D}_1)$, $P(\tilde{D}_1)$ and $A(\tilde{D}_1)$ and their derivatives are also uniformly bounded. Denote by $M_k$, $M_p$, and $M_a$ the bounds of $K(\tilde{D}_1)$, $P(\tilde{D}_1)$ and $A(\tilde{D}_1)$ respectively, and with $M'_k$, $M'_p$, and $M'_a$ the bounds of their derivatives. From relations (32)–(33) and (35), (36) and using Young’s and Cauchy–Schwarz’s inequalities it easy to show that the above bounds hold with

$$M_u = 3 \max \left\{ 1 + M_k^2 \tilde{D}_2 e^{2\tilde{D}_2(M_k + |B|)|B|}, M_k^2 e^{2\tilde{D}_2(M_k + |B|)|B|} \right\}$$  

$$M_x = 3 \max \left\{ 1 + M_k^2 \tilde{D}_2 e^{2\tilde{D}_2(M_k + |B|)|B|} |B|, M_k^2 e^{2\tilde{D}_2(M_k + |B|)|B|} \right\}$$  

$$M_\xi = 3 \max \left\{ 1 + M_k^2 \tilde{D}_2 e^{2\tilde{D}_2(M_k + |B|)|B|} |B|, M_k^2 e^{2\tilde{D}_2(M_k + |B|)|B|} \right\}.$$  

We are now ready to state the following lemma.

**Lemma 2.** There exist constants $M_{R_1}$, $M_{R_2}$, $M_{P_1}$, $M_{P_2}$, $M_{Q_1}$ and $M_{Q_2}$ such that the following bounds hold

$$|R_1(t)|^2 \leq M_{R_1} \|\xi(t)\|^2$$  

$$|R_2(t)|^2 \leq M_{R_2} (|Z(t)|^2 + \|\xi(0, t)\|^2 + \|\xi(t)\|^2)$$  

$$p_1(t, x) \leq M_{P_1} (|Z(t)|^2 + \|\xi(0, t)\|^2 + \|\xi(t)\|^2)$$  

$$p_2(t, x) \leq M_{P_2} (|Z(t)|^2 + u^2(0, t))$$  

$$q_1(t, x) \leq M_{Q_1} (|Z(t)|^2 + \|u(t)\|^2 + \|\xi(t)\|^2)$$  

$$q_2(t, x) \leq M_{Q_2} \|u(t)\|^2,$$

for all $x \in \mathbb{R}$.  

**Proof.** From relations (49) and (34) and by using Young's and Cauchy–Schwarz's inequalities we get the bounds for $R_1(t)$ and $R_2(t)$ with

$$M_{R_1} = (1 + \tilde{D}_1 |F_1|^2) e^{\tilde{D}_2 |F_1|^2} |H_1|^2$$  

$$M_{R_2} = 3 |H_1|^2 \max \left\{ e^{\tilde{D}_2 |F_1|^2}, 1, \tilde{D}_1 |F_1|^2 e^{\tilde{D}_2 |F_1|^2} \right\},$$

Using relations (52)–(55) together with Young's and Cauchy–Schwarz's inequalities, relations (65)–(66) and (57) we get the bounds of the lemma with

$$M_{P_1} = M_k^2 e^{2M_2 \tilde{D}_2} M_{P_2}$$  

$$M_{P_2} = 3 \max \left\{ M_k^2 e^{2M_2 \tilde{D}_2} |B|^2, M_k^2 M_\xi e^{2M_2 \tilde{D}_2} \right\}$$  

$$M_{Q_1} = 3 \max \left\{ (M_k^2 \tilde{D}_2 + \tilde{D}_2 M_k^2 M_\xi)^2 |B|^2 e^{2M_2 \tilde{D}_2}, M_k^2 e^{2M_2 \tilde{D}_2} M_{Q_2}, \left( M_k + M_\xi M_k^2 \tilde{D}_2 \right)^2 e^{2M_2 \tilde{D}_2} \right\}$$  

$$M_{Q_2} = M_k^2 (1 + \tilde{D}_2 M_k)^2 e^{2M_2 \tilde{D}_2}. \quad \square$$

**Lemma 3.** There exist constants $k$, $a_2$, $\gamma_1$ and $\gamma_2$ such that for the Lyapunov function

$$V(t) = D_2 \log(1 + \mathcal{S}(t)) + a_2 \tilde{D}_1(t) \tilde{D}_1(t) + a_2 \tilde{D}_2(t) \gamma_2,$$

where

$$\mathcal{S}(t) = \xi(t)^T P \left( \tilde{D}_1 \right) Z(t) + k \int_0^t (1 + x)\xi^T(x, t)\xi(x, t) dx$$

$$- a_2 \int_0^t (1 + x)w^2(x, t) dx,$$

the following holds

$$V(t) \leq V(0).$$

**Proof.** Taking the time derivative of the above function we obtain

$$\dot{V}(t) = -2a_2 \tilde{D}_2 \left( \tilde{D}_1(t) - \gamma_1 \tau_{D_1} \right) - 2a_2 \tilde{D}_2 \left( \tilde{D}_2(t) - \gamma_2 \tau_{D_2} \right)$$

$$+ \frac{D_2}{1 + \mathcal{S}(t)} \left( -Z^T(t)Q(\tilde{D}_1)Z(t) + 2Z^T(t)P(\tilde{D}_1)Bu(0, t) \right)$$

$$+ 2k \frac{\mathcal{Z}_1(t)\mathcal{Z}_2(t) - k}{D_1} \xi(0, t)^2$$

$$- a_2 \int_0^t w^2(x, t) dx - k \frac{\tilde{D}_1}{D_2} \int_1^t \xi^T(x, t)\xi(x, t) dx$$

$$- a_2 \frac{\int_0^t w^2(x, t) dx + \tilde{D}_1(t) \left( Z^T(t) \frac{\partial P(\tilde{D}_1)}{\partial D_1} Z(t) \right) + 2Z^T(t)P(\tilde{D}_1)R_1(t) - 2a_2 \int_0^t (1 + x)w(x, t)q_1(x, t) dx}{\int_0^1 (1 + x)w(x, t)q_2(x, t) dx}.$$  

Using the properties of the projector operators and relations (24)–(25) we get

$$\dot{V}(t) \leq \frac{D_2}{1 + \mathcal{S}(t)} \left( -Z^T(t)Q(\tilde{D}_1)Z(t) + 2k \mathcal{Z}_1(t)\mathcal{Z}_2(t) \right)$$

$$- k \frac{\mathcal{Z}_1(t)\mathcal{Z}_2(t) - k}{D_2} \xi(0, t)^2 - a_2 w^2(0, t) + Z^T(t)P(\tilde{D}_1)Bu(0, t)$$

$$- k \int_0^1 \xi^T(x, t)\xi(x, t) dx - a_2 \int_0^1 w^2(x, t) dx$$

$$+ \tilde{D}_1(t) \left( Z^T(t) \frac{\partial P(\tilde{D}_1)}{\partial D_1} Z(t) + 2Z^T(t)P(\tilde{D}_1)R_1(t) \right).$$
\[-2a_2 \int_0^1 (1 + x) w(x, t) q_1(x, t) dx \]
\[-2a_2 D(t) \int_0^1 (1 + x) w(x, t) q_2(x, t) dx \].

(81)

Then using Young’s inequality and (30)–(31) we get

\[ \dot{V}(t) \leq \frac{D_2}{1 + \mathcal{S}(t)} \left( -\frac{\lambda}{2} |Z(t)|^2 - \frac{k}{D_1} |\xi(0, t)|^2 - \frac{k}{D_1} |\xi(t)|^2 \right) \]
\[ - \frac{a_2}{2D_2} w^2(0, t) - \frac{a_2}{D_2} \|w(t)\|^2 + \dot{D}_1(t) \left( Z^T(t) \frac{\partial P(\hat{D}_1)}{\partial D_1} Z(t) \right) \]
\[ + 2Z^T(t) P(\hat{D}_1) R(t) - 2a_2 \int_0^1 (1 + x) w(x, t) q_1(x, t) dx \]
\[ - 2a_2 \dot{D}_2(t) \int_0^1 (1 + x) w(x, t) q_2(x, t) dx \].

(82)

Using bounds (65)–(70) together with relations (26) and (24)–(25), and by employing Young’s inequality one more time we get

\[ |\dot{D}_1(t)| \leq \gamma_1 |r_0| \]
\[ \leq \gamma_1 M_1 \left( \frac{|Z(t)|^2 + |\xi(0, t)|^2 + |\xi(t)|^2 + \|w(t)\|^2}{1 + \mathcal{S}(t)} \right) \]
\[ \dot{|D}_2(t)| \leq \gamma_2 |r_0| \]
\[ \leq \gamma_2 M_2 \left( \frac{|Z(t)|^2 + w^2(0, t) + \|w(t)\|^2}{1 + \mathcal{S}(t)} \right). \]

(83) (84)

where

\[ M_1 = \max \left\{ M_{p_1} + \frac{1}{a_2} M_p + \frac{1}{a_2} M_2 M_{k_2}, 1 \right\} \]
\[ M_2 = \max \left\{ 1, 2M_{p_2}, 2M_{p_2} M_{k_2} + M_{p_2} \right\}. \]

(85) (86)

Plugging in the above bound to (82) (and applying once more Young’s and Cauchy–Schwarz’s inequalities) we get

\[ \dot{V}(t) \leq \frac{D_2}{1 + \mathcal{S}(t)} \left( -\frac{\lambda}{2} |Z(t)|^2 - \frac{k}{D_1} |\xi(0, t)|^2 - \frac{k}{D_1} |\xi(t)|^2 \right) \]
\[ - \frac{a_2}{2D_2} w^2(0, t) - \frac{a_2}{D_2} \|w(t)\|^2 \]
\[ + B_1 \gamma_1 \left( \frac{|Z(t)|^2 + |\xi(0, t)|^2 + |\xi(t)|^2 + \|w(t)\|^2}{1 + \mathcal{S}(t)} \right) \]
\[ \times \left( |Z(t)|^2 + \|\xi(t)\|^2 + \|w(t)\|^2 \right) \]
\[ + B_2 \gamma_2 \left( \frac{|Z(t)|^2 + w^2(0, t) + \|w(t)\|^2}{1 + \mathcal{S}(t)} \right) \]
\[ \times \left( |Z(t)|^2 + \|w(t)\|^2 \right). \]

(87)

where

\[ B_1 = M_1 \max \left\{ M_{p_1} + M_p + a_2 M_{q_1} + 2a_2 M_{q_2} M_p + 2a_2 M_{q_1}, \right. \]
\[ M_p M_{k_1} + 2a_2 M_{q_2} M_p + 2a_2 + 2a_2 M_{q_1} M_p \}
\[ B_2 = 2M a_2 \left( 1 + M_{q_2} M_p \right). \]

(88) (89)

Now by defining the constants

\[ m_2 = \max \left\{ B_1, B_2 \right\} \]
\[ \min \left\{ \frac{\lambda}{2}, \frac{k}{D_1}, \frac{a_2}{2D_2} \right\}. \]

(90)

we get

\[ \dot{V}(t) \leq - \frac{D_2}{1 + \mathcal{S}(t)} \left( m_1 - m_2 (\gamma_1 + \gamma_2) \right) \left( |Z(t)|^2 + w^2(0, t) + \|w(t)\|^2 + \|\xi(0, t)\|^2 + \|\xi(t)\|^2 \right). \]

(92)

Thus when \( \gamma_1 + \gamma_2 \leq \frac{m_1}{m_2}, \) \( \dot{V}(t) \) is negative definite and thus

\[ V(t) \leq V(0). \]

(93)

To prove the stability bound of Theorem 1 we use the following lemma.

**Lemma 4.** There exist constants \( M \) and \( \mathcal{M} \) such that

\[ M \mathcal{S}(t) \leq |P(t)| \leq \mathcal{M} \mathcal{S}(t), \]

(94)

where

\[ |P(t)| = |X(t)|^2 + \|\xi(t)\|^2 + \|w(t)\|^2. \]

(95)

**Proof.** Immediate, using (57)–(60) with

\[ \mathcal{M} = \max \left\{ M_n + M_{k_n}, M_{k_1} + 1 \right\}, \]
\[ \bar{M} = \max \left\{ M_n + M_{k_1} - 2 + M_{k_2} + 1 \right\}. \]

(96) (97)

We are now ready to derive the stability estimate of Theorem 1. Using (77) it follows that

\[ \mathcal{S}(t) \leq \left( e^{y_0 t} - 1 \right) \]
\[ \tilde{D}_1 + \tilde{D}_2 \leq C_2 \frac{V(t)}{D_2} \leq C_2 \left( e^{y_0 t} - 1 \right), \]

(98) (99)

where

\[ C_2 = \left( \frac{\gamma_0 \tilde{D}_2 + \gamma_1 \tilde{D}_1}{a_2} \right). \]

(100)

Consequently

\[ \Omega(t) \leq \bar{M} + C_2 \left( e^{y_0 t} - 1 \right). \]

(101)

Moreover, from (77) we take

\[ V(0) \leq \max \left\{ \lambda, k, a_2 \right\} \left( |Z(0)|^2 + \|\xi(0)\|^2 + \|w(0)\|^2 \right) \]
\[ + \max \left\{ \frac{a_2}{\gamma_2}, \frac{a_2 \tilde{D}_2}{\gamma_1 \tilde{D}_1} \right\} \left( \tilde{D}_1^2(0) + \tilde{D}_2^2(0) \right), \]

(102)

and using Lemma 4 we have

\[ V(0) \leq C_3 \Omega(0), \]

(103)

where

\[ C_3 = \max \left\{ \frac{1}{\gamma_2}, \frac{\tilde{D}_2}{2 \gamma_1 \tilde{D}_1}, \frac{\lambda, k, a_2}{\bar{M}} \right\}. \]

(104)

Thus, by setting

\[ R = \bar{M} + C_2 \]
\[ \rho = C_3, \]

(105) (106)

we get the stability result in Theorem 1.
We now turn our attention to proving the convergence of $X(t)$ and $U(t)$ to zero. We use here an alternative to Barbalat’s Lemma from [23]. We first point out that from (93) it follows that $|Z(t)|$, $\|w(t)\|$, $\|\xi(t)\|$, $\hat{D}_1$ and $\hat{D}_2$ are uniformly bounded. Moreover, using (57) and (58) we get the uniform boundness of $|X(t)|$ and $\|u(t)\|$. Using (23) it follows that $U(t)$ is uniformly bounded. From (24)–(25), (83)–(84) and (50) we conclude that $\frac{d\hat{w}^2(t)}{dt}$ is uniformly bounded. Finally, since from (92) it turns out that $\|Z(t)\|$ and $\|\xi(t)\|$ are square integrable, using (58) and the alternative to Barbalat’s Lemma from [23], we conclude that $\lim_{t \to \infty} X(t) = 0$. We now turn our attention to proving the convergence of $U(t)$. Using (92) we also get that $\|w(t)\|$ is square integrable in time. Thus, with the help of (57) and by the square integrability of $|Z(t)|$ we conclude using (23) that $U(t)$ is square integrable. It only remains to show that $\frac{d\hat{w}^2(t)}{dt}$ is uniformly bounded. Hence, it is sufficient to show that $\hat{U}(t)$ is uniformly bounded. From (23) one can observe that since $\hat{D}_1$ and $\hat{D}_2$ are uniformly bounded, with the help of (16) and (19) we conclude the uniform boundness of $\frac{d\hat{w}^2(t)}{dt}$.

5. Simulations

We give here a simulation example to illustrate the effectiveness of the proposed adaptive scheme. We choose a second order feedforward system with parameters $F_1 = F_2 = 0.25$, $H_1 = 1$, $D_1 = 0.4$ and $D_2 = 0.8$. This is an unstable system with two poles at 0.25. The lower bounds for the unknown delays are $D_1 = 0.1$ for $D_1$ and 0 for $D_2$. Analogously the upper bounds are chosen as twice the real values of the delays i.e., $D_1 = 0.8$ and $D_2 = 1.6$. The initial conditions are chosen as $X_1(0) = 0.5, X_2(0) = 0.5$ and $X_2(\theta) = 0.5, \forall \theta \in [-D_1, 0]$, and finally $D_1(0) = D_1 = 0.1$. The controller parameters are chosen as $q_0 = 200, k = 0.005, \gamma_1 = 25, \gamma_2 = 25$ and $K(\hat{D}_1) = \left[-10.0625e^{0.25\hat{D}_1} -8.5 \right]$. 

Figs. 1–3 show two distinct simulations, starting from two extreme initial values for the input delay estimate, one at zero, and the other at twice the true delay value. In Figs. 1 and 3 we observe that, as Theorem 1 predicts, convergence to zero is achieved for the states and the input, despite starting with initial estimate for the input delay at the two extreme values and for the state delay at the lower bound. In Fig. 2 one can see the evolution of the estimates for the two distinct simulation cases. The estimates for the two delays, after a transient response, converge to stabilizing for the system values.

6. Conclusions

In this paper we presented an adaptive control design for a system in feedforward form with simultaneous unknown input and state delays. The design of the controller is based on predictor feedback. The update laws for the estimation of the unknown delays are based on the construction of a Lyapunov function with
normalization. Convergence to zero is then proved using the linear boundness of the relative signals.

References