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Delays are among the most common dynamic phenomena arising in control engineering practice. Interest in delay systems is driven by applications such as chemical process control, machining, combustion systems, teleoperation, and networked systems.

Delay systems belong to the class of distributed parameter systems but have a special structure, not possessed by partial differential equations, that can be exploited in analysis and design to arrive at compact or even explicit solutions.

An enormous wealth of results exists for controlling systems with state, input, and output delays. Control problems with input delays are among the earliest challenges to be tackled, for example, in Otto J.M. Smith's 1959 article [1], which presents the compensator known as the Smith predictor. The 50th anniversary of this popular engineering tool is a fitting occasion to consider new control problems with input delays and other infinite-dimensional input dynamics. Many problems have become tractable with the maturing of tools for controlling distributed parameter systems.

When the input delay is short relative to the plant's time scales, finite-dimensional feedback laws can be used to optimize the robustness margin to small input delays. However, when the input delay is long, the

Compensation of Infinite-Dimensional Actuator and Sensor Dynamics

NONLINEAR AND DELAY-ADAPTIVE SYSTEMS

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controller must account for the delay state. The resulting design is an infinite-dimensional controller, such as the Smith predictor. The Smith predictor approach can also be used to compensate a long output delay. Input and output delays are collectively referred to as dead time.

The Smith predictor, which is a modification of a nominal controller designed for a system without a delay, addresses dead time in setpoint regulation and constant disturbance rejection problems. However, when the plant is unstable, the Smith predictor may fail to achieve closed-loop stability, even though the nominal controller is designed to stabilize the system without delay.

A modification of the Smith predictor that removes its limitation to stable plants is given by the finite spectrum assignment (FSA) approach [2]–[6]. The basic idea of the Smith predictor, and its relation to FSA, is discussed in "Smith Predictor and Its Relation with Prediction-Based Feedback."

BASICS OF PREDICTOR FEEDBACK

FSA, also known as predictor feedback, addresses the state-space model

$$\dot{X}(t) = AX(t) + BU(t - D), \tag{1}$$

where X is the state vector, U is the control input (scalar in the present consideration), D is the delay, and (A, B) is a controllable pair. The primary problem being studied is

Smith Predictor and Its Relation with Predictor-Based Feedback

For the system

$$\dot{X}(t) = AX(t) + BU(t - D), \tag{S1}$$
$$Y(t) = CX(t), \tag{S2}$$

the Smith controller [1] is given by

$$U(s) = -\mathscr{C}(s)\{Y(s) - R(s) + P(s)(1 - e^{-sD})U(s)\}, \tag{S3}$$

where $P(s) = C(sI - A)^{-1}B$ is the plant transfer function, $R(s)$ is the reference input, and $\mathscr{C}(s)$ is an arbitrary compensator. In particular, the designer may choose an observer-based controller whose transfer function is

$$\mathscr{C}(s) = -K(sI - A - BK + LC)^{-1}L, \tag{S4}$$

where L is the observer gain vector.

The motivation for the Smith controller (S4) is to compensate for the input delay in tracking setpoints for stable plants. The reference-to-output closed-loop transfer function is obtained as

$$Y(s) = e^{-sD}\frac{\mathscr{C}(s)P(s)}{1 + \mathscr{C}(s)P(s)}R(s). \tag{S5}$$

Hence, the reference-following performance recovers the performance of the nominal, undelayed system, except for a delay of D. However, the input-output formula (S5) does not reveal the fundamental problem of the Smith predictor, namely, a potential closed-loop instability when the plant is unstable. This instability is unobservable in the representation (S5), but it can be observed when an input disturbance is introduced and when the closed-loop transfer function is computed from that disturbance to the output, as shown in [7]. Section 6.2.2 of [7] also presents a useful discussion regarding three "myths" associated with tuning, disturbance rejection, and sensitivity to model errors for the Smith predictor.

The FAS control laws [2]–[4] are often mentioned in the literature as advanced or generalized forms of the Smith predictor [1]. These control laws are quite different from the Smith predictor and the similarity is more in the use of the term predictor and in the employment of delayed feedback than in motivation or what is being achieved.

For comparison between [1] and [2]–[4], consider the observer-based version of the FSA predictor feedback, which is given by

$$\dot{\hat{X}}(t) = A\hat{X}(t) + BU(t - D) + L(Y(t) - C\hat{X}(t)), \tag{S6}$$
$$U(t) = K\left[e^{AD}\hat{X}(t) + \int_{t-D}^{t} e^{A(t-\theta)}BU(\theta)\,d\theta\right]. \tag{S7}$$

The transfer function of this compensator is

$$U(s) = -\mathscr{C}_D(s)\{Y(s) + (P_D(s) - P(s)e^{-sD})U(s)\}, \tag{S8}$$

where the transfer function $P_D(s)$ is given by

$$P_D(s) = C(sI - A)^{-1}B_D, \tag{S9}$$

the delay-adjusted compensator $\mathscr{C}_D(s)$ is given by

$$\mathscr{C}_D(s) = -K_D(sI - A - B_DK_D + LC)^{-1}L, \tag{S10}$$

and the vectors B_D and K_D are given by

$$B_D = e^{-AD}B, \tag{S11}$$
$$K_D = Ke^{AD}. \tag{S12}$$

For the sake of comparison between the original Smith controller (S3)–(S4) and the observer-based FSA feedback (S8)–(S10), we set $R(s) = 0$ in (S3). We then observe subtle but significant differences in how the input delay is being compensated, though the structure is the same, particularly, in how the infinite-dimensional element e^{-sD} appears in the control laws.

While the Smith controller (S3)–(S4) offers no stability guarantees when the plant is unstable, the closed-loop stability under the full-state FSA feedback (3) is established in Theorem 1 and under the observer-based feedback (S6), (S7) in [12].

stabilization when A is not Hurwitz. However, whether or not A is Hurwitz, the goal is to compensate for the effect of the input delay on a nominal control design. Assuming $D = 0$, a nominal design is a finite-dimensional full-state feedback controller of the form

$$U(t) = KX(t), \tag{2}$$

where K is chosen such that $A + BK$ is Hurwitz. In practice, K can be chosen either to place the nominal closed-loop poles at desired locations or to solve an optimal control problem. To compensate for the fact that $D > 0$, FSA [2]–[4] employs the infinite-dimensional predictor feedback

$$U(t) = K\left[e^{AD}X(t) + \int_{t-D}^{t} e^{A(t-\theta)}BU(\theta)d\theta\right], \tag{3}$$

where K is the nominal control gain. The feedback is infinite dimensional because the integral involves the control history over the interval $[t-D, t]$. The delay is compensated through both the integral term and the finite-dimensional term $Ke^{AD}X(t)$, where the compensation scales the nominal gain K by the predictor matrix e^{AD}. Unstable eigenvalues of A tend to increase the gains in K, whereas stable eigenvalues of A tend to decrease them, which is evident in the case of a scalar plant, where $|Ke^{AD}| > |K|$ if $A > 0$, and $|Ke^{AD}| < |K|$ if $A < 0$. Hence, the controller is more aggressive for unstable plants and more cautious for stable ones.

The idea of the predictor feedback (3) is to compensate for the delay by feeding back the future state $X(t+D)$. Applying the variation of constants formula to (1), we can express the future state as

$$X(t+D) = e^{AD}X(t) + \int_{t}^{t+D} e^{A(t+D-\eta)}BU(\eta - D)d\eta, \tag{4}$$

where the current state $X(t)$ is the initial condition. Shifting the time variable under the integral in (4), we obtain

$$X(t+D) = e^{AD}X(t) + \int_{t-D}^{t} e^{A(t-\theta)}BU(\theta)d\theta, \tag{5}$$

which gives the future state $X(t+D)$ in terms of the control signal $U(\theta)$ from the past time window $[t-D, t]$. Hence, with (5), the feedback law (3) is

$$U(t) = KX(t+D), \quad \text{for all } t \geq 0, \tag{6}$$

assuming no disturbances or modeling errors. The robustness of the feedback (3) to uncertainties, both parametric and dynamic, is of interest in practice. The developments in [8] make it possible to address these concerns due to the availability of Lyapunov functions for predictor feedback. Related issues in [9] include the robustness of the feedback law (3) to digital implementation of the distributed delay term given by the integral. This issue is resolved with appropriate discretization schemes [10], [11].

The feedback law (3) appears to be implicit since U is present on both sides. However, the input memory $U(\theta)$, where $\theta \in [t-D, t]$, is part of the state of an infinite-dimensional system, and thus the control law is effectively a full-state-feedback controller.

DELAYS MODELED AS PARTIAL DIFFERENTIAL EQUATIONS

Delay is a dynamic phenomenon that can be represented as a partial differential equation (PDE) of transport type, which evolves in one spatial dimension, with one derivative in space and one derivative in time. For example, in the case of an input delay, the signal $U(t-D)$ can be represented as the output of the transport PDE system $u_t(x,t) = u_x(x,t)$, where $x \in [0, D)$ is a time-like spatial coordinate along which the state variable u is convected at unity speed, the control $U(t)$ enters through the boundary condition $u(D,t) = U(t)$, and the output map is given by $Y(t) = u(0,t)$. In this example, the state of the dynamic system at time t is $u(\cdot, t)$, whereas the state equation, which is infinite dimensional, consists of the PDE $u_t(x,t) = u_x(x,t)$ and the boundary condition $u(D,t) = U(t)$.

In the PDE model of a delay, the input and output variables of the PDE system are both boundary values of the state $u(x,t)$. For this reason, the input operator and output operator of the state-space model of the PDE are both given by kernels that are δ-functions in x, namely, $Y(t) = \int_0^D \delta(x)u(x,t)dx$ and $\int_0^D \delta(x-D)u(x,t)dx = U(t)$. Input and output operators that include δ-function kernels are unbounded linear operators on the state space of square integrable functions of x. Because of this unbounded character, PDE systems with boundary inputs and outputs, including systems with delays, introduce greater challenges for control and observer design than PDE models whose input and output operators are bounded.

In PDE control systems, the control signal can enter the PDE model in two different ways. In *in-domain control*, the control signal enters the PDE, such as in a chemical tubular reactor with thermal actuation distributed along the entire length of the reactor, whereas in *boundary control*, the control signal enters through the boundary condition, such as in wall-actuated flow control, as well as in the case of a transport PDE modeling a delay.

Ordinary differential equations (ODEs) with delays, also known as delay differential equations (DDEs), are interconnected systems of ODEs and transport PDEs. A control system whose plant is an ODE and whose input is subject to a delay thus has a cascade PDE-ODE structure, where the control signal enters through a boundary condition of the PDE. Since a system with input delay is a PDE-ODE cascade with the control signal $U(t)$ entering through a boundary condition of the PDE, a system with an input delay is a boundary control system.

The challenges in boundary control and observer design manifest themselves differently depending on whether the design approach is based on optimality or spectrum assignment, though the root of the challenge is always that the input or output operator is unbounded. Among various methods that have been developed since the late 1960s for boundary control, methods based on backstepping for PDEs [12], [13] distinguish themselves by providing explicit formulas for the feedback laws. This quality makes these methods accessible to engineers and also usable within the context of adaptive control, where the parameters of the PDE model are unknown. The predictor feedback (3), which represents a particular form of boundary control, can be obtained by following the backstepping approach [12].

DESIGN TOOLS FOR STABILIZATION OF SYSTEMS WITH DELAY AND PDE DYNAMICS AT THE INPUT

In this article, we describe several extensions to predictor feedback design, including predictor feedback for nonlinear systems and PDEs with input delays, robustness and inverse optimality results, a delay-adaptive design, and a design for systems with time-varying delays. Proofs and solutions to additional control problems with delays and PDEs are given in [8].

In addition to systems with input delays, we also present predictor-like feedback for ODE systems with input dynamics given by PDEs. Finally, we complement the treatment of infinite-dimensional input dynamics with a treatment of infinite-dimensional sensor dynamics, presenting an observer design for ODE systems with sensor delays as well as for sensor dynamics given by a PDE. Actuator and sensor dynamics modeled by PDEs are motivated by physical applications. For example, combustion systems involve fuel transport (convection), diffusion, reaction, turbulent mixing, and acoustics, all of which are modeled by PDEs. This article is thus a tutorial introduction to design tools for PDE-ODE and PDE-PDE cascade systems.

The remainder of the article is organized into ten sections. The first seven sections are dedicated to delay-ODE systems, the following section considers delay-PDE systems, and the remaining two sections focus on PDE-ODE systems. We conclude with a brief review of some open problems and research opportunities.

This article deals with both delays and PDEs. The PDEs play various roles in the article, and these roles need to be clarified. We use PDEs to model delays, which allows us to develop delay-adaptive designs, as well as to model more general PDE actuator dynamics. In addition, in parts of the article a PDE assumes the role of the plant, and we consider delay-PDE cascades. From the PDE-ODE and PDE-PDE cascade problems for which design results are available [8], we present examples to highlight the key issues.

We proceed next with delay problems, where we consider four scenarios. In each scenario, the delay is allowed to be arbitrarily large. In the first scenario, the delay is constant and known, in which case we consider both linear and nonlinear plants. In the second scenario, the delay is constant and slightly uncertain; in this case we study robustness for linear plants. In the third scenario, the delay is time varying but known and the plant is linear. Finally, in the last scenario, the delay is constant and unknown. To handle uncertainty in the delay that is of the same order of magnitude as the delay itself, we employ adaptive control.

The article includes nonlinear systems with input delay. In the area of nonlinear control, several types of uncertainties are considered: unmeasurable disturbances, static nonlinear functional perturbations, dynamic perturbations on the state, and dynamic perturbations on the input. Among these uncertainties, unmodeled input dynamics present a significant challenge in robust nonlinear control, with long delays at the input of nonlinear systems remaining a difficult problem. Advances have been made in control of nonlinear systems with state delay [14]–[17] as well as on robustness to input delays of arbitrary length [18], however, systematic compensation of input delays of arbitrary length has not been considered. In this article, we present a control design for compensating input delays of arbitrary length in nonlinear control systems. We consider first the broad class of nonlinear control systems whose solutions remain bounded over any finite time interval, which are also referred to as the forward complete systems. For forward complete systems, the predictor feedback law is given implicitly but it is globally asymptotically stabilizing. Then we consider strict-feedforward systems, a subset of forward complete systems, for which the control law is given explicitly.

THE BACKSTEPPING TRANSFORMATION

While $U(t)$ is the value of the control signal at time t, the function $U(\cdot)$ on the sliding window $[t-D, t]$ is the infinite-dimensional actuator state. This state is infinite dimensional. The key to extending the predictor feedback (3) to systems more general than (1) is to construct an invertible infinite-dimensional backstepping transformation of the actuator state $U(\theta)$, $\theta \in [t-D, t]$, that converts the closed-loop system into a cascade form, called the *target system*. The target system facilitates Lyapunov analysis of the closed-loop system.

The idea of the backstepping transformation is to convert the infinite-dimensional system (1), (3), which is in feedback configuration, into a cascade configuration. In "Finite-Dimensional Backstepping," we review the idea of a backstepping transformation for a single integrator. While in the finite-dimensional case the backstepping transformation is a change of only one scalar variable, in the case of input delay the backstepping transformation is infinite dimensional. This transformation amounts to constructing a linear operator for the infinite-dimensional delay state. While in the finite-dimensional case the addition of more

Finite-Dimensional Backstepping

We use the finite-dimensional case to motivate the development of an infinite-dimensional backstepping transformation for systems with input delay. We consider the system

$$\dot{X}(t) = AX(t) + B\xi(t), \tag{S13}$$
$$\xi(t) = U(t), \tag{S14}$$

which has actuator dynamics modeled by a single integrator (S14) at the input. The backstepping transformation of the scalar actuator state ξ is given by

$$\zeta(t) \triangleq \xi(t) - KX(t), \tag{S15}$$

and the backstepping control law is given by

$$\begin{aligned} U(t) &= K\dot{X}(t) - c\zeta(t) \\ &= K[AX(t) + B\xi(t)] - c(\xi(t) - KX(t)) \\ &= K(A + cI)X(t) + (KB - c)\xi(t), \end{aligned} \tag{S16}$$

where $c > 0$. With (S15) and (S16) the system (S13), (S14) is converted into the target system

$$\dot{X}(t) = (A + BK)X(t) + B\zeta(t), \tag{S17}$$
$$\dot{\zeta}(t) = -c\zeta(t). \tag{S18}$$

The cascade system (S17), (S18) is exponentially stable since each of the component subsystems in the cascade is exponentially stable. If the actuator dynamics include more integrators than in (S13), (S14), the backstepping design proceeds one integrator at a time.

than one integrator leads to a recursive procedure for constructing a vector backstepping transformation, in the infinite-dimensional case the transformation is performed in one step, with a single infinite-dimensional transformation. The infinite-dimensional transformation employs an integral operator, which we construct in this section.

For the delay problem (1), (3) we transform the actuator state $U(\theta), \theta \in [t - D, t]$, by defining

$$W(\theta) \triangleq U(\theta) - KX(\theta + D), \tag{7}$$

where $\theta \in [t - D, t]$ and $t \geq 0$. From (7) with $\theta = t$ and (6), it follows that

$$W(t) = 0, \quad \text{for all } t \geq 0. \tag{8}$$

However, for $\theta \in [-D, 0]$, the function $W(\theta)$ may be nonzero. We write $U \mapsto W$.

Since the transformation (7) involves the noncausal component $X(\theta + D)$, we develop an alternative representation of $W(\theta)$. Toward that end, similar to (5), using the variation of constants formula with initial time $t - D$, initial state $X(t)$, and current time θ, from (1) we obtain

$$X(\theta + D) = e^{A(\theta + D - t)}X(t) + \int_{t-D}^{\theta} e^{A(\theta - \sigma)}BU(\sigma)d\sigma, \tag{9}$$

for all $\theta \in [t - D, t]$ and all $t \geq 0$. By substituting (9) into (7), we arrive at

$$W(\theta) = U(\theta) - K\left[\int_{t-D}^{\theta} e^{A(\theta - \sigma)}BU(\sigma)d\sigma + e^{A(\theta + D - t)}X(t)\right], \tag{10}$$

where $\theta \in [t - D, t]$ and $t \geq 0$. In the transformation (10), it is not helpful to view $W(\theta)$ as the value of a function but rather as a transformation of the function $U(\theta)$, where $\theta \in [t - D, t]$, into another function $W(\theta)$, where $\theta \in [t - D, t]$. A more detailed explanation for the construction of the transformation (10) is provided in the next section using a transport PDE representation of the actuator state.

Next we replace U by W in the closed-loop system. Setting $\theta = t - D$ in (10), solving the resulting equation as $U(t - D) = KX(t) + W(t - D)$, and substituting this expression into (1), we arrive at the target system

$$\dot{X}(t) = (A + BK)X(t) + BW(t - D), \tag{11}$$
$$W(t) = 0, \quad t \geq 0, \tag{12}$$

where

$$W(\theta) = U(\theta) - K\left[\int_{D}^{\theta} e^{A(\theta - \sigma)}BU(\sigma)d\sigma + e^{A(\theta + D)}X(0)\right], \quad \text{for } \theta \in [-D, 0]. \tag{13}$$

Using (12), it follows from (11) that $\dot{X}(t) = (A + BK)X(t)$ for all $t \geq D$, which means that the delay is perfectly compensated after $t = D$, namely, the system evolves as if the delay were absent after $t = D$. In the target system (11), (12), the signal $W(t)$ is neither a new control nor an exogenous disturbance but the output of a dynamical system. This dynamical system is a delay line of length D with zero input, the initial condition (13), and the state $W(\theta)$, where $\theta \in [t - D, t]$. The dynamical system that generates the signal $W(t)$ has a finite-time convergence property since the state W becomes zero at $t = D$ and, in the absence of disturbances, remaining zero for $t \geq D$.

By comparing the original system (1), (3) with the target system (11), (12), we see that the backstepping transformation has accomplished two things. First, while the open-loop system $\dot{X} = AX$ is possibly unstable, the closed-loop system (11) is stable. Second, while the original control signal $U(t)$ given by (3) depends on $X(t)$, the transformed control signal $W(t)$ given by (8) and (10) is the output of a delay line with zero input, namely, an autonomous system, which establishes a cascade configuration.

Having introduced the backstepping transformation and the target system, we now turn to examine the stability of the target system (11), (12). Since $W(\cdot)$ is nonzero only on

$[-D, 0]$ and the system $\dot{X} = AX$ is exponentially stable, the target system (11), (12) is exponentially stable. This fact is established in Theorem 1 but can be seen intuitively by noting that (11) is autonomous after $t = D$. For exponential stability to hold not only for the target system (11), (12) but also for the original feedback system (1), (3), whose state is $X(t), U(\theta), \theta \in [t-D, t]$, it is necessary that the transformation (10) be invertible. If the transformation were not invertible, then the closed-loop instability would be unobservable. In fact, the inverse of (10) is given explicitly by

$$U(\theta) = W(\theta) + K\left[\int_{t-D}^{\theta} e^{(A+BK)(\theta-\sigma)} BW(\sigma)d\sigma + e^{(A+BK)(\theta+D-t)}X(t)\right], \quad (14)$$

where $\theta \in [t-D, t]$ and $t \geq 0$. Hence, since the target system (11), (12) is exponentially stable, the actuator state (14) converges exponentially to zero, which means that the entire state of the closed-loop system in the original variables (X, U) converges exponentially to zero.

To prove that the equilibrium $X(t) = 0, W(\theta) \equiv 0, \theta \in [t-D, t]$ of (11), (12), and, hence, the equilibrium $X(t) = 0, U(\theta) \equiv 0, \theta \in [t-D, t]$ of the original system (1), (3), is exponentially stable, we use a Lyapunov-Krasovskii functional. In particular, the Lyapunov functional for the target system (11), (12) has the form

$$V(X(t), W(\cdot)_{[t-D,t]}) = X^T(t)PX(t) + a\int_{t-D}^{t}(1+\theta+D-t)W^2(\theta)d\theta, \quad (15)$$

where a is a positive constant. The Lyapunov functional (15) depends on the state variables (X, W), where X is a vector and $W(\cdot)$ is a function defined on the infinite-dimensional space of square-integrable functions whose argument belongs to the interval $[t-D, t]$.

The stability analysis for the closed-loop system (1), (3) consists of two steps. The first step is the construction of the backstepping transformation (10), while the second step is the construction of the Lyapunov functional (15). To obtain the Lyapunov functional for the original system (1), (3) involving $X(t), U(\theta), \theta \in [t-D, t]$, we substitute (10) into (15), which yields

$$\begin{aligned}
&\hat{V}(X(t), U(\cdot)_{[t-D,t]}) \\
&= X(t)^T PX(t) + a\int_{t-D}^{t}(1+\theta+D-t)U^2(\theta)d\theta \\
&+ aX^T(t)\left(\int_{t-D}^{t}(1+\theta+D-t)e^{A^T(\theta+D-t)} \times K^T K e^{A(\theta+D-t)}d\theta\right)X(t) \\
&+ a\int_{t-D}^{t}(1+\theta+D-t)\left(K\int_{t-D}^{\theta} e^{A(\theta-\sigma)}BU(\sigma)d\sigma\right)^2 d\theta \\
&- 2a\int_{t-D}^{t}(1+\theta+D-t)U(\theta)K\int_{t-D}^{\theta} e^{A(\theta-\sigma)}BU(\sigma)d\sigma d\theta \\
&- 2a\left(\int_{t-D}^{t}(1+\theta+D-t)U(\theta)Ke^{A(\theta+D-t)}d\theta\right)X(t) \\
&+ 2a\left(\int_{t-D}^{t}(1+\theta+D-t)K\int_{t-D}^{\theta} e^{A(\theta-\sigma)} \times BU(\sigma)d\sigma Ke^{A(\theta+D-t)}d\theta\right)X(t),
\end{aligned} \quad (16)$$

where $\hat{V}(X(t), U(\cdot)_{[t-D,t]}) = V(X(t), W(\cdot)_{[t-D,t]})$. The Lyapunov functional (16) contains cross terms involving X and several nested integrals of U and thus is substantially more complex than (15). The simpler form (15) is the benefit of using backstepping. With the Lyapunov functional (15) it is shown in [8] that there exist positive constants $\alpha_1, \alpha_2, \alpha_3$ such that

$$\dot{V}(t) \leq -\alpha_1 V(t) \quad (17)$$

and

$$\alpha_2\left[\|X(t)\|^2 + \int_{t-D}^{D}U^2(\theta)d\theta\right] \leq V(t) \leq \alpha_3\left[\|X(t)\|^2 + \int_{t-D}^{D}U^2(\theta)d\theta\right], \quad (18)$$

which implies exponential stability of the equilibrium $X(t) = 0, U(\theta) \equiv 0, \theta \in [t-D, t]$.

BACKSTEPPING IN THE TRANSPORT PDE SETTING AND CLOSED-LOOP STABILITY

As discussed above, we model the input delay in the system (1) as the transport PDE

$$u_t(x, t) = u_x(x, t), \quad x \in [0, D], \quad (19)$$

$$u(D, t) = U(t). \quad (20)$$

Both t and x have dimension of time. The PDE (19) has transport speed u_t/u_x of unity in nondimensional units. In the transport PDE model (19), (20) of the input delay, the undelayed control signal $U(t)$ enters the boundary condition (20). The initial condition of the system (19), (20) is $u(x, 0) \triangleq u_0(x) \triangleq U(x-D)$, namely, the control history over the time interval $[-D, 0]$. The input signal $U(\cdot)$ thus acts both as an input and as an initial condition. The transport PDE (19), (20) has the explicit solution

$$u(x, t) = U(t + x - D). \quad (21)$$

The output

$$u(0, t) = U(t - D) \quad (22)$$

is the D-second delayed input. For this reason, with (22) system (1) is written as

$$\dot{X}(t) = AX(t) + Bu(0, t). \quad (23)$$

Equations (23), (20), and (19) form a PDE-ODE cascade, which, as illustrated in Figure 1, is driven by the input U from the boundary of the PDE.

We now introduce the backstepping transformation of the actuator state and the inverse of this transformation. These transformations are given, respectively, by

$$w(x,t) = u(x,t) - \int_0^x Ke^{A(x-y)}Bu(y,t)dy - Ke^{Ax}X(t), \tag{24}$$

$$u(x,t) = w(x,t) + \int_0^x Ke^{(A+BK)(x-y)}Bw(y,t)\,dy + Ke^{(A+BK)x}X(t), \tag{25}$$

where $w(x,t)$ is the transformed actuator state. Similar to (21), the transformed actuator state $w(x,t)$ can be written as $w(x,t) = W(t+x-D)$. The transformations (10) and (24) can be obtained from each other by using $\theta = x+t-D$ and $\sigma = y+t-D$. The inverse transformations (14) and (25) can be related in a similar manner.

With the transport PDE representation of the input delay, the closed-loop system (1), (3) is alternatively represented as

$$\dot{X}(t) = AX(t) + Bu(0,t), \tag{26}$$

$$u_t(x,t) = u_x(x,t), \quad x \in [0,D), \tag{27}$$

$$u(D,t) = \int_0^D Ke^{A(D-y)}Bu(y,t)dy + Ke^{AD}X(t). \tag{28}$$

The direct backstepping transformation (24) converts the system (26)–(28) into the target system

$$\dot{X}(t) = (A+BK)X(t) + Bw(0,t), \tag{29}$$

$$w_t(x,t) = w_x(x,t), \quad x \in [0,D), \tag{30}$$

$$w(D,t) = 0, \tag{31}$$

which is the cascade of the undriven transport PDE w-subsystem (30), (31) and the exponentially stable system $\dot{X} = (A+BK)X$ in (29). The motivation for the construction (24) and its derivation are given in "Predictor Feedback from Infinite-Dimensional Backstepping Design."

Since the undriven transport PDE (30), (31) is exponentially stable [8], the overall cascade (29), (31) is exponentially stable. This fact is established with the Lyapunov functional

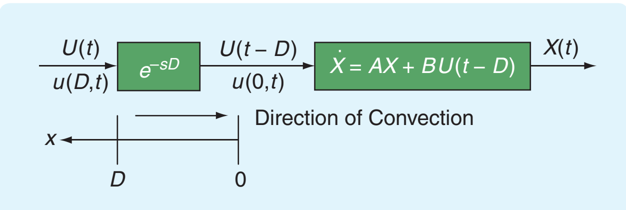

FIGURE 1 The linear system $\dot{X}(t) = AX(t) + BU(t-D)$ with the actuator delay D. The actuator delay is represented by the transport partial differential equation (19)–(20) with the spatial coordinate $x \in [0, D)$.

$$V(X(t), w(\cdot,t)_{[0,D]}) = X(t)^{\mathrm{T}}PX(t) + 2\frac{\|PB\|^2}{\lambda_{\min}(Q)} \times \int_0^D (1+x)w^2(x,t)dx, \tag{32}$$

where $\|\cdot\|$ represents the Euclidean norm and P is the solution of the Lyapunov equation

$$(A+BK)^{\mathrm{T}}P + P(A+BK) = -Q. \tag{33}$$

An alternative representation of the Lyapunov functional (32) is given by (15), with $a = 2\|PB\|^2/\lambda_{\min}(Q)$. The exponential stability of the closed-loop infinite-dimensional system is summarized in the following theorem.

Theorem 1

There exist positive constants G and g such that all solutions of the closed-loop system (26)–(28) satisfy

$$\Gamma(t) \le Ge^{-gt}\Gamma(0) \tag{34}$$

for all $t \ge 0$, where

$$\Gamma(t) \triangleq \|X(t)\|^2 + \int_0^D u^2(x,t)dx. \tag{35}$$

INVERSE OPTIMALITY, DISTURBANCE ATTENUATION, AND ROBUSTNESS TO ACTUATOR BANDWIDTH LIMITATIONS

We now use the Lyapunov functional (32) to derive disturbance attenuation estimates when the system (1) is subject to an additive disturbance. We then prove robustness to bandwidth limitations in the actuator dynamics and conduct an inverse optimal redesign of the predictor feedback.

We consider the system

$$\dot{X}(t) = AX(t) + BU(t-D) + B_1 d(t), \tag{36}$$

where $d(t)$ is an unknown, scalar-valued bounded disturbance whose bounds are unknown, along with the dynamic controller

$$\dot{U}(t) = -cU(t) + cU_{\mathrm{nom}}(t), \tag{37}$$

where $c > 0$ and

$$U_{\mathrm{nom}}(t) = K\left[e^{AD}X(t) + \int_{t-D}^{t} e^{A(t-\theta)}BU(\theta)d\theta\right] \tag{38}$$

is the nominal predictor feedback. Alternatively, the dynamic controller can be represented as

$$\check{U}(s) = L(s)\left(\check{U}_{\mathrm{nom}}(s) + \frac{U(0)}{c}\right), \tag{39}$$

where $\check{U}(s), \check{U}_{\mathrm{nom}}(s)$ are the Laplace transforms of $U(t), U_{\mathrm{nom}}(t)$, respectively, and where

Predictor Feedback from Infinite-Dimensional Backstepping Design

Consider the system (1). Suppose the static state-feedback control (2) is designed for the case without delay (that is, with $D=0$) such that $A+BK$ is Hurwitz. We want to map the original system (19), (20), (23), which may be unstable (when A is not Hurwitz), into the desirable target system (29)–(31), which is a cascade of two exponentially stable subsystems, one PDE and one ODE.

The PDE-ODE system (19), (20), (23) has a block-lower-triangular structure, where the lower left off-diagonal block is the potentially unstable ODE plant $\dot{X}=AX(t)$. For this reason, we seek a 2×2 transformation $(X,u)\mapsto(X,w)$, that has the lower triangular form

$$\begin{bmatrix} X \\ w \end{bmatrix} = \begin{bmatrix} I_{n\times n} & 0_{n\times[0,D]} \\ \Gamma & \mathcal{Q}+I_{[0,D]\times[0,D]} \end{bmatrix}\begin{bmatrix} X \\ u \end{bmatrix}, \tag{S19}$$

where $I_{n\times n}$ denotes the identity matrix, $I_{[0,D]\times[0,D]}$ denotes the identity operator on a function $u(x,t)$ of the argument $x\in[0,D]$, the symbol Γ denotes the operator $\Gamma\colon X(t)\mapsto -\gamma(x)^T X(t)$, and the symbol $\mathcal{Q}$ denotes the Volterra integral operator $\mathcal{Q}\colon u(x,t)\mapsto -\int_0^x q(x,y)u(y,t)\,dy$, where the kernel functions $\gamma(x)$ and $q(x,y)$ are to be determined. Due to the lower triangularity of $\mathcal{Q}$, the overall transformation $(X,u)\mapsto(X,w)$ is lower triangular. Furthermore, the diagonal of this transformation is the identity operator $\mathrm{Id}=\mathrm{diag}\{I_{n\times n}, I_{[0,D]\times[0,D]}\}$. Due to the triangular structure, the transformation (S19) is not only suitable for converting the system (19), (20), (23) into the target form (29)–(31), but is also invertible.

Hence, following (S19), we seek a backstepping transformation of the form

$$w(x,t)=u(x,t)-\int_0^x q(x,y)u(y,t)\,dy-\gamma(x)^T X(t). \tag{S20}$$

To determine the kernel functions $\gamma(x)$ and $q(x,y)$, we calculate the time and spatial derivatives of the transformation (S20), obtaining

$$\begin{aligned} w_x(x,t) &= u_x(x,t)-q(x,x)u(x,t)-\int_0^x q_x(x,y)u(y,t)\,dy \\ &\quad -\gamma'(x)^T X(t) \end{aligned} \tag{S21}$$

$$\begin{aligned} w_t(x,t) &= u_t(x,t)-\int_0^x q(x,y)u_t(y,t)\,dy \\ &\quad -\gamma(x)^T[AX+Bu(0)] \end{aligned} \tag{S22}$$

$$\begin{aligned} &= u_x(x,t)-q(x,x)u(x,t)+q(x,0)u(0,t) \\ &\quad +\int_0^x q_y(x,y)u(y,t)\,dy-\gamma(x)^T[AX+Bu(0,t)]. \end{aligned} \tag{S23}$$

Subtracting (S21) from (S23) we obtain

$$\begin{aligned} &\int_0^x (q_x(x,y)+q_y(x,y))u(y,t)\,dy + \\ &\quad [q(x,0)-\gamma(x)^T B]u(0,t) + \\ &\quad\quad [\gamma'(x)^T-\gamma(x)^T A]X(t)=0. \end{aligned} \tag{S24}$$

Since (S24) must be valid for all functions $u(x,t)$ and $X(t)$, as well as for all the values of their arguments (x,t), we thus have the conditions

$$q_x(x,y)+q_y(x,y)=0, \tag{S25}$$

$$q(x,0)=\gamma(x)^T B, \tag{S26}$$

$$\gamma'(x)=A^T\gamma(x). \tag{S27}$$

The first two conditions form a first-order hyperbolic PDE, while the third condition is an ODE. To find the initial condition for this ODE, we set $x=0$ in (S20), which gives

$$w(0,t)=u(0,t)-\gamma(0)^T X(t). \tag{S28}$$

Substituting this expression into (29), we obtain

$$\dot{X}(t)=AX(t)+Bu(0,t)+B(K-\gamma(0)^T)X(t). \tag{S29}$$

Comparing this equation with (23), we have

$$\gamma(0)=K^T. \tag{S30}$$

Therefore the solution to the ODE (S27) is $\gamma(x)=e^{A^T x}K^T$, which gives

$$\gamma(x)^T=Ke^{Ax}. \tag{S31}$$

The general solution to (S25) is

$$q(x,y)=\omega(x-y), \tag{S32}$$

where the function ω is determined from (S26). We obtain

$$q(x,y)=Ke^{A(x-y)}B. \tag{S33}$$

We now substitute the gains $\gamma(x)$ and $q(x,y)$ into the transformation (S20) and set $x=D$, obtaining the control law

$$u(D,t)=\int_0^D Ke^{A(D-y)}Bu(y,t)\,dy+Ke^{AD}X(t). \tag{S34}$$

With (20) and (21), we observe that the backstepping control law (S34) is the same as the FSA/predictor feedback law (3).

$$L(s)=\frac{c}{s+c}. \tag{40}$$

The transfer function $L(s)$ can arise from either lowpass unmodeled dynamics of the actuator, in which case $L(s)$ is a neglected part of the plant, or it can be intentionally introduced by the designer as part of the control law for the sake of achieving inverse optimality.

The following result is established with the Lyapunov functional

$$\begin{aligned} V(X(t),w(\cdot,t)_{[0,D]}) &= X(t)^T PX(t)+2\frac{\|PB\|^2}{\lambda_{\min}(Q)} \\ &\quad \times\int_0^D (1+x)w^2(x,t)\,dx+\frac{1}{2}w^2(D,t). \end{aligned} \tag{41}$$

The PDE backstepping approach is a potentially powerful tool for advancing design techniques for systems with input and output delays.

Theorem 2

There exists a positive constant c^*, such that for all $c > c^*$, the feedback system (36)–(38) is L_∞-stable, that is, there exist positive constants β_1, β_2, γ_1 such that

$$N(t) \le \beta_1 e^{-\beta_2 t} N(0) + \gamma_1 \sup_{\tau \in [0,t]} |d(\tau)| \tag{42}$$

for all $t \ge 0$, where

$$N(t) \triangleq \left(\|X(t)\|^2 + \int_{t-D}^{t} U^2(\theta) d\theta + U^2(t) \right)^{1/2}. \tag{43}$$

Furthermore, there exist a constant $c^\circ > c^*$, a function $\mu:(0,\infty)^2 \to (0,\infty)$ with the property that, for all $\gamma_2 > 0$, $\mu(c,\gamma_2) \to \infty$ as $c \to \infty$, and a functional Ω with the property that

$$\Omega(X(t), U(\cdot)_{[t-D,t]}) \ge \mu(c,\gamma_2) N^2(t), \quad \text{for all } t \ge 0, \tag{44}$$

such that, for all $c \ge c^\circ$ and for all

$$\gamma_2 \ge \gamma_2^\circ \triangleq 8 \frac{\|PB\|^2}{\lambda_{\min}(Q)}, \tag{45}$$

the feedback (39) minimizes the cost functional

$$J = \sup_{d \in \mathcal{D}} \lim_{t \to \infty} \left[2cV(t) + \int_0^t (\Omega(\tau) + \dot{U}^2(\tau) - c\gamma_2 d^2(\tau)) d\tau \right], \tag{46}$$

where $\mathcal{D}:(X,U) \mapsto d$ is the set of linear scalar-valued mappings whose arguments are vectors $X(t)$ and square-integrable functions $U(\theta), \theta \in [t-D,t]$.

We emphasize that the bounds $c^*, c^\circ, \gamma_2^\circ$ are independent of $\sup_{\tau\in[0,t]}|d(\tau)|$ and of $\int_0^\infty d^2(\tau)d\tau$.

The crucial property of the functional Ω is its positive definiteness in (X,U), as stated in (44). While, in direct optimal control, the designer is given a positive-definite functional Ω as a cost on the plant state and is tasked with synthesizing an optimal feedback law, in the inverse optimal approach, the designer is allowed to choose the feedback law and the cost functional Ω simultaneously. This additional freedom may be challenging since the functional Ω must be chosen to be positive definite. A tradeoff exists between the direct and inverse optimal control frameworks. In the direct optimal control framework the designer has to solve a Riccati equation that, in the case of infinite-dimensional systems such as systems with delays, is not a matrix equation but an operator equation. Nonlinear and infinite-dimensional equations are often difficult to numerically approximate and solve. In the inverse optimal approach, optimality is achieved, albeit not for an a priori prescribed cost, but with a feedback law (37), (38) that does not require the solution of an operator Riccati equation. The benefits of optimality include guarantees of an infinite gain margin and a 60° phase margin [20], [21].

Theorem 2 shows that, for sufficiently high c, the predictor feedback (3) is robust to unmodeled actuator bandwidth limitations, as depicted in Figure 2, modeled by the first-order transfer function $c/(s+c)$. Alternatively, this transfer function can be a part of the control law, as in (39).

Theorem 2 also shows that the system under predictor feedback (3), as well as under feedback (39) with sufficiently high c, has a finite L_∞ gain relative to an additive disturbance.

Furthermore, Theorem 2 shows that the feedback (39) is an inverse optimal stabilizer for sufficiently high c, in the absence of the disturbance d. This result is obtained by writing the feedback law in terms of $\dot{U}(t)$ as the control input, in which case the feedback law is of the $-L_gV$ form [20], namely, in the form where the control is the negative of the inner product of the input operator and of the gradient of the Lyapunov functional with respect to the system's state.

Finally, Theorem 2 shows that, in the presence of the disturbance, the feedback (39) with sufficiently high c is an inverse optimal solution to a differential game problem [21] with a positive-definite penalty on the state and control, and a negative-definite penalty on the disturbance.

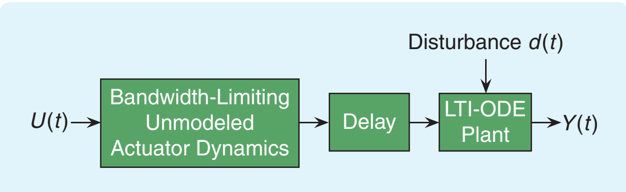

FIGURE 2 An ordinary differential equation with input delay, perturbed by unmodeled bandwidth-limiting actuator dynamics and additive disturbance. Theorem 2 establishes robustness with respect to both perturbations under predictor feedback (3).

ROBUSTNESS TO DELAY ERROR

In this section we study the problem of robustness to delay error, depicted in Figure 3. We construct Lyapunov functional that allows us to provide a positive answer to the question of whether predictor feedback has robustness to sufficiently small delay error.

This article is a tutorial introduction to design tools for PDE-ODE and PDE-PDE cascade systems.

We consider the feedback system

$$\dot{X}(t) = AX(t) + BU(t - D_0 - \Delta D), \tag{47}$$

$$U(t) = K\left[e^{AD_0}X(t) + \int_{t-D_0}^{t} e^{A(t-\theta)}BU(\theta)d\theta\right]. \tag{48}$$

The actuator delay error ΔD can be either positive or negative relative to the assumed actuator delay $D_0 > 0$. However, the actual delay must be nonnegative, that is, $D_0 + \Delta D \geq 0$. For the study of robustness to a small ΔD, we use two Lyapunov functionals, namely, one for $\Delta D > 0$, which is the easier of the two cases, and another for $\Delta D < 0$, in which case we employ

$$\begin{aligned} V(X(t), w(\cdot, t)_{[\Delta D, D_0+\Delta D]}) &= X(t)^T PX(t) \\ &\quad + \frac{a}{2}\int_0^{D_0+\Delta D} (1 + x)w^2(x,t)dx \\ &\quad + \frac{1}{2}\int_{\Delta D}^{0} (D_0 + x)w^2(x,t)dx \end{aligned} \tag{49}$$

with a sufficiently large a.

Theorem 3

There exists a positive constant δ such that, for all $\Delta D \in (-\delta, \delta)$, there exist positive constants G and g such that all solutions of the closed-loop system (47), (48) satisfy

$$\Gamma(t) \leq Ge^{-gt}\Gamma(0) \tag{50}$$

for all $t \geq 0$, where

$$\Gamma(t) \triangleq \|X(t)\|^2 + \int_{t-\overline{D}}^{t} U^2(\theta)d\theta \tag{51}$$

and

$$\overline{D} \triangleq D_0 + \max\{0, \Delta D\}. \tag{52}$$

Although finite-dimensional feedback laws for finite-dimensional plants remain stabilizing in the presence of small delays [22], this result does not apply to systems considered here, which are infinite-dimensional even before a delay perturbation is introduced. The delay perturbation to predictor feedback incorporates the possibility of two different classes of perturbations, depending on whether ΔD is positive or negative, so off-the-shelf results cannot be used.

Theorem 3 may be surprising in light of results of [23], which show that boundary controllers for hyperbolic PDEs of second order, such as wave equations, have a zero delay margin. Even though the input-delay problem involves a transport PDE, which is hyperbolic, and the predictor feedback is a boundary controller, the results on the lack of delay-robustness do not hold for predictor feedback because of a distinction between first-order and second-order hyperbolic PDEs. The second-order hyperbolic PDEs in [23] have infinitely many eigenvalues on the imaginary axis, which causes the appearance of closed-loop eigenvalues in the right-half plane even for infinitesimal delays in the feedback loop. On the other hand, the spectrum of an ODE with input delay consists of the spectrum of the ODE and the spectrum of the transport PDE, which consists of infinitely many eigenvalues with real parts equal to negative infinity. Upon closing the loop with the stabilizing predictor feedback law, stability is not lost in the presence of a small delay error since the closed-loop spectrum, which consist of finitely many eigenvalues of $A + BK$ in the left-half plane, and of the infinitely many eigenvalues of the transport PDE $w_t = w_x$ at negative infinity, is shifted by only a small amount by the delay perturbation.

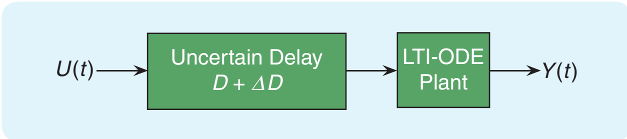

FIGURE 3 An ordinary differential equation with input delay D, which is known up to a small error ΔD, which may be either positive or negative. Theorem 3 shows that stability is preserved under predictor feedback (48) for sufficiently small $|\Delta D|$ but arbitrarily large D.

DELAY-ADAPTIVE CONTROL

We now turn our attention from robustness to small delay error to adaptivity for large delay uncertainty. Relevant results on adaptive control of systems with input delays include [24] and [25]. However, these results deal with parametric uncertainties in the ODE plant, whereas the key challenge is uncertainty in the delay.

We now design an adaptive predictor feedback, where an estimate $\hat{D}(t)$ is employed instead of the unknown delay D. Let us consider the plant (1) but with a transport PDE representation of the input delay given as

$$\dot{X}(t) = AX(t) + Bu(0, t), \tag{53}$$

$$Du_t(x, t) = u_x(x, t), \tag{54}$$

$$u(1, t) = U(t). \tag{55}$$

We take the predictor feedback in the certainty equivalence form

$$U(t) = K\left[e^{A\hat{D}(t)}X(t) + \hat{D}(t)\int_0^1 e^{A\hat{D}(t)(1-y)}Bu(y,t)dy \right], \quad (56)$$

with the update law for the estimate $\hat{D}(t)$ given by

$$\dot{\hat{D}}(t) = \gamma \mathrm{Proj}_{[0,\bar{D}]}\{\tau(t)\}, \quad (57)$$

$$\tau(t) = -\frac{\int_0^1 (1+x)W(x,t)Ke^{A\hat{D}(t)x}dx\,(AX(t)+Bu(0,t))}{1+X(t)^T PX(t) + b\int_0^1 (1+x)W^2(x,t)dx}, \quad (58)$$

$$w(x,t)=u(x,t)-\hat{D}(t)\int_0^x Ke^{A\hat{D}(t)(x-y)}Bu(y,t)dy-Ke^{A\hat{D}(t)x}X(t), \quad (59)$$

where

$$\mathrm{Proj}_{[0,\bar{D}]}\{\tau\} = \tau\begin{cases} 0, & \hat{D} \le 0 \text{ and } \tau < 0, \\ 0, & \hat{D} \ge \bar{D} \text{ and } \tau > 0, \\ 1, & \text{else}, \end{cases} \quad (60)$$

is the standard projection operator, the initial condition for the parameter estimate is restricted to

$$\hat{D}(0) \in [0, \bar{D}], \quad (61)$$

where $\bar{D}$ is a known upper bound on D, and the normalization coefficient b is chosen such that

$$b \ge \frac{4\bar{D}\|PB\|^2}{\lambda_{\min}(Q)}.$$

The structure of the adaptive control system is shown in Figure 4. The choice of the update law (57)–(59) is motivated by a Lyapunov analysis, resulting in a normalization of the update law, without the use of any filters or overparametrization.

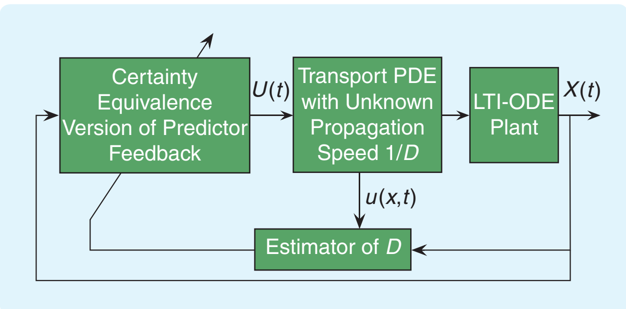

FIGURE 4 Delay-adaptive predictor feedback. The actual delay D may be between zero and a known upper bound $\bar{D}$, which is finite but arbitrarily large. The certainty-equivalence controller (56) is combined with the update law (57)–(59). Theorem 4 guarantees global stability and regulation of the state and control.

Theorem 4

Consider the closed-loop adaptive system (53)–(59). There exists $\gamma^* > 0$ such that, for all $\gamma \in (0, \gamma^*)$, there exist positive constants R and ρ such that, for all initial conditions satisfying $(X_0, u_0, \hat{D}_0) \in \mathbb{R}^n \times L_2[0,1] \times [0, \bar{D}]$,

$$\Upsilon(t) \le R(e^{\rho\Upsilon(0)} - 1), \quad \text{for all } t \ge 0, \quad (62)$$

where

$$\Upsilon(t) \triangleq \|X(t)\|^2 + \int_0^1 u^2(x,t)dx + (D-\hat{D}(t))^2. \quad (63)$$

Furthermore,

$$\lim_{t\to\infty} X(t) = 0, \quad \lim_{t\to\infty} U(t) = 0. \quad (64)$$

Example 1

To illustrate the delay-adaptive design (53)–(59), consider plant

$$X(s) = \frac{e^{-s}}{s - 0.75}U(s). \quad (65)$$

As shown in Figure 5, the time period from 0 s to $D = 1$ s is dead time, in which the control signal is propagating through the actuator delay, the parameter estimation (57) is active until about 3 s, the control evolution is exponential (corresponding to a predominantly LTI system) after 3 s, and the state evolves exponentially after 4 s. The adaptive controller achieves regulation of $X(t)$ to zero, with an acceptable adaptation transient, both with $\hat{D}(0) = 0$, which corresponds to a controller that initially overlooks the presence of the delay, and with $\hat{D}(0) = 2$ s, which corresponds to a controller that overcompensates the delay. In both cases the parameter error is 100% relative to the actual delay value of $D = 1$. ■

TIME-VARYING INPUT DELAY

We now consider the problem of time-varying known input delays, which is depicted in Figure 6. We consider the system

$$\dot{X}(t) = AX(t) + BU(t - \delta(t)), \quad (66)$$

where $\delta(t)$ is the time-varying delay. For convenience, let

$$\phi(t) \triangleq t - \delta(t) \quad (67)$$

represent the time transformation affecting the input U. In the case of a constant delay D, the time transformation (67) is a shift of time by D, that is, $\phi(t) = t - D$.

The predictor feedback for (66) is

$$U(t)=K\left[e^{A(\phi^{-1}(t)-t)}X(t) + \int_{\phi(t)}^t e^{A(\phi^{-1}(t)-\phi^{-1}(\theta))}B\frac{U(\theta)}{\phi'(\phi^{-1}(\theta))}d\theta \right]. \quad (68)$$

We now employ a transport PDE representation with

$$u(x,t) = U(\phi(t + x(\phi^{-1}(t) - t))) \quad (69)$$

and apply the time-varying backstepping transformation

$$w(x,t) = u(x,t) - Ke^{Ax(\phi^{-1}(t)-t)}X(t) - K\int_0^x e^{A(x-y)(\phi^{-1}(t)-t)}Bu(y,t)(\phi^{-1}(t)-t)dy \quad (70)$$

of (66), (68) into the target system

$$\dot{X}(t) = (A+BK)X(t) + Bw(0,t), \quad (71)$$

$$w_t(x,t) = \pi(x,t)w_x(x,t), \quad (72)$$

$$w(1,t) = 0, \quad (73)$$

where the variable speed of propagation $\pi(x,t)$ of the transport equation w is given by

$$\pi(x,t) = \frac{1+x\left(\frac{d(\phi^{-1}(t))}{dt} - 1\right)}{\phi^{-1}(t)-t}. \quad (74)$$

This procedure yields the following stabilization result.

Theorem 5

Consider the closed-loop system (66), (68). Assume that the delay function $\delta(t)$ is positive and bounded from above, and $\delta'(t)$ is less than one and bounded from below. Then there exist positive constants G and g such that, for all $t \geq 0$,

$$\|X(t)\|^2 + \int_{\phi(t)}^t U^2(\theta)d\theta \leq Ge^{-gt}\left(|X_0|^2 + \int_{\phi(0)}^0 U^2(\theta)d\theta\right),$$
$$\text{for all } t \geq 0 \quad (75)$$

Example 2

We consider the case of a bounded delay function without a limit. Let

$$\delta(t) = t - \rho^{-1}(t), \quad (76)$$

where

$$\rho(t) \triangleq t + 1 + \frac{1}{2}\cos t. \quad (77)$$

Consider the first-order system

$$\dot{X}(t) = X(t) + U(\rho^{-1}(t)), \quad (78)$$

so that $A = B = 1$ in (66). The control law (68) in this case becomes

$$U(t) = -(1+c)\left[e^{1+\frac{1}{2}\cos t}X(t) + \int_{\rho^{-1}(t)}^t \left(1 - \frac{1}{2}\sin\theta\right)e^{t+\frac{1}{2}\cos t - \theta - \frac{1}{2}\cos\theta}U(\theta)d\theta\right]. \quad (79)$$

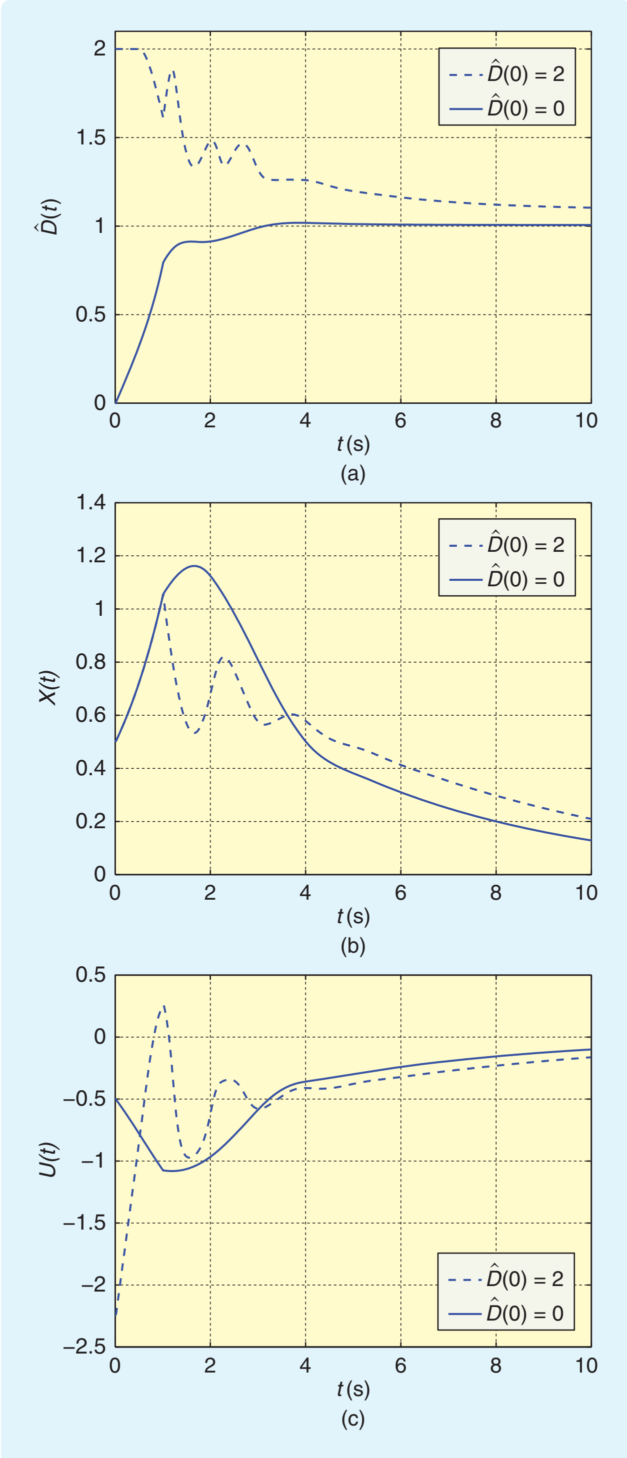

FIGURE 5 Time responses of (a) $\hat{D}(t)$, (b) $X(t)$, and (c) $U(t)$ under delay-adaptive predictor feedback for an unstable first-order plant. Stabilization is achieved both with $\hat{D}(0) = 0$ and with $\hat{D}(0) > D$.

Figure 7 shows the closed-loop state and control signals for $c = 0.13$. The oscillating $U(t)$ compensates the oscillation in the delay function.

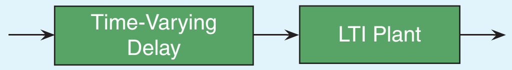

FIGURE 6 The linear system $\dot{X}(t) = AX(t) + BU(\phi(t))$ with the time-varying actuator delay $\delta(t) = t - \phi(t)$. Theorem 5 guarantees exponential stability under predictor feedback (68) with compensation of the time-varying delay.

PREDICTOR FEEDBACK FOR NONLINEAR SYSTEMS

We now compensate input delays in nonlinear control systems, shown in Figure 8, through an extension of predictor feedback. Consider the nonlinear system

$$\dot{X}(t) = f(X(t), U(t-D)), \quad f(0,0) = 0, \tag{80}$$

and assume that we know a feedback law $U = \kappa(X)$ with $\kappa(0) = 0$ that globally asymptotically stabilizes the zero solution of the system (80) when $D = 0$. Denote the initial conditions as $X_0 = X(0)$ and $U_0(\theta) = U(\theta)$, $\theta \in [-D, 0]$. A predictor feedback is given by

$$U(t) = \kappa(P(t)), \tag{81}$$

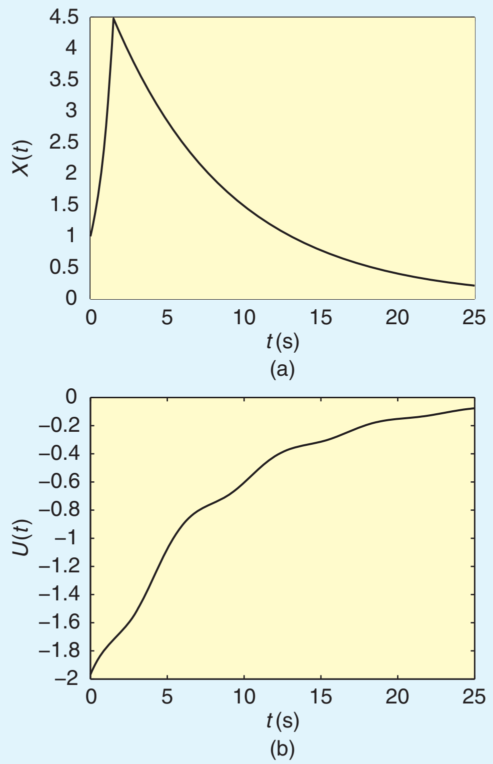

FIGURE 7 (a) State $X(t)$ and (b) control $U(t)$ evolution for Example 2. The control starts having an effect on the plant state at $t = \rho(0) = 3/2$ s. The waviness in the control signal compensates the delay, which varies sinusoidally.

where the predictor $P(t)$ is defined as

$$P(t) = \int_{t-D}^{t} f(P(\theta), U(\theta))\, d\theta + X(t), \quad t \geq 0, \tag{82}$$

$$P(\theta) = \int_{D}^{\theta} f(P(\sigma), U_0(\sigma))\, d\sigma + X_0, \quad \theta \in [-D, 0]. \tag{83}$$

The predictor $P(t)$ is defined implicitly through a nonlinear integral equation, rather than explicitly, through matrix exponentials and the variation of constants formula when the plant is linear. The lack of an explicit formula for $P(t)$ is not an obstacle numerically, since $P(t)$ is defined in terms of its past values.

The nonlinear integral equation (82) is equivalent to the plant model (80) since $P(t) = X(t+D)$. A solution $P(t)$ to (82) does not always exist since the control applied after $t = 0$ has no effect on the plant over the time interval $[0, D]$, and, as a consequence, system (80) can exhibit finite escape over that interval, resulting also in a finite escape for $P(t)$. Hence, a natural way to ensure global existence of the predictor state is to assume that the plant has the property that, for all initial conditions and all locally bounded input signals, its solutions exist for all time. This property is called *forward completeness*.

Forward completeness does not require the solutions of (80) to be uniformly bounded. For example, solutions can be growing to infinity as time approaches infinity. All linear time invariant (LTI) systems, stable or unstable, driven by inputs with exponential growth, are forward complete. The same is true of nonlinear systems with globally Lipschitz right-hand sides, as well as many systems that are neither globally Lipschitz nor have stable equilibria, including systems that contain superlinear nonlinearities that induce limit cycles.

We now develop a nonlinear predictor design for two classes of systems. The first class consists of forward-complete systems, that is, systems that do not exhibit a finite escape time for all initial conditions and all input signals that are bounded over finite time intervals, which includes all piecewise-continuous inputs. For this class, we develop predictor feedback, which achieves global asymptotic stability, as long as the system without delay is globally asymptotically stabilizable. However, the predictor requires the solution of a nonlinear integral equation, or a nonlinear DDE, in real time. The stability result formulated in the following theorem employs the standard class $\mathcal{KL}$ functions, which are functions of two variables,

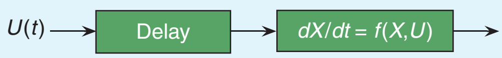

FIGURE 8 Nonlinear control in the presence of an arbitrarily long input delay. Theorem 6 guarantees global asymptotic stabilization with the predictor feedback (81)–(83) if the plant is forward-complete and globally asymptotically stabilizable in the absence of delay.

continuous in both variables, monotonically increasing and zero at zero in the first variable, and decaying to zero in the second variable.

Theorem 6

Let $\dot{X} = f(X, U)$ be forward complete, and let $\dot{X} = f(X, \kappa(X))$ be globally asymptotically stable at $X = 0$. Consider the closed-loop system (80)–(83). Then there exists a function $\hat{\beta}$ in class $\mathcal{KL}$ such that

$$\|X(t)\| + \|U\|_{L_\infty[t-D,t]} \leq \hat{\beta}(|X(0)| + \|U_0\|_{L_\infty[-D,0]}, t) \qquad (84)$$

for all $(X_0, U_0) \in \mathbb{R}^n \times L_\infty[-D, 0]$ and for all $t \geq 0$.

Although $P(t)$ may not be explicitly computable, many nonlinear system are not only forward complete and globally stabilizable but also have the property that $P(t)$ is explicitly computable. These nonlinear systems comprise the class of strict-feedforward systems [20]. The relation between strict-feedforward systems and strict-feedback systems [19], as well as with the forward-complete systems, is described in Figure 9. For more information on strict-feedforward systems, see "Strict-Feedforward Systems and Integrator Forwarding."

Example 3

We illustrate the explicit computability of the predictor, and thus of the feedback law, for a strict-feedforward system. Consider the third-order system

$$\dot{X}_1(t) = X_2(t) + X_3^2(t), \qquad (85)$$

$$\dot{X}_2(t) = X_3(t) + X_3(t)U(t-D), \qquad (86)$$

$$\dot{X}_3(t) = U(t-D), \qquad (87)$$

which is not feedback linearizable but is a strict-feedforward system. The globally asymptotically stabilizing predictor feedback for this system is given by

$$\begin{aligned} U(t) = &-P_1(t) - 3P_2(t) - 3P_3(t) - \frac{3}{8}P_2^2(t) + \frac{3}{4}P_3(t) \\ &\times \left(-P_1(t) - 2P_2(t) + \frac{1}{2}P_3(t) + \frac{P_2(t)P_3(t)}{2}\right. \\ &\left. + \frac{5}{8}P_3^2(t) - \frac{1}{4}P_3^3(t) - \frac{3}{8}\left(P_2(t) - \frac{P_3^2(t)}{2}\right)^2\right), \end{aligned} \qquad (88)$$

where the D-second-ahead predictor of $(X_1(t), X_2(t), X_3(t))$ is given explicitly by

$$\begin{aligned} P_1(t) = &X_1(t) + DX_2(t) + \frac{1}{2}D^2X_3(t) + DX_3^2(t) \\ &+ 3X_3(t)\int_{t-D}^{t}(t-\theta)U(\theta)d\theta + \frac{1}{2}\int_{t-D}^{t}(t-\theta)^2U(\theta)d\theta \\ &+ \frac{3}{2}\int_{t-D}^{t}\left(\int_{t-D}^{\theta}U(\sigma)d\sigma\right)^2 d\theta, \end{aligned} \qquad (89)$$

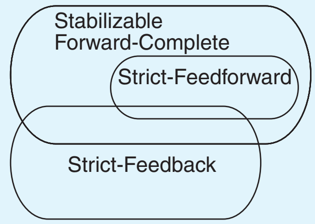

FIGURE 9 Relations among system classes studied in nonlinear control. The class of strict-feedback systems contains both systems that are forward complete and those that are not. As a result, some of the strict-feedback (and thus feedback-linearizable) systems are not globally stabilizable by predictor feedback in the presence of an input delay. In contrast, all strict-feedforward systems are forward complete. Hence, strict-feedforward systems are always globally stabilizable in the presence of an input delay of arbitrary length. In addition, predictor feedback is obtained explicitly for strict-feedforward systems. Strict-feedback and strict-feedforward systems have a small intersection, namely, the chain of integrators, which is both upper triangular and lower triangular.

$$\begin{aligned} P_2(t) = &X_2(t) + DX_3(t) + X_3(t)\int_{t-D}^{t}U(\theta)d\theta \\ &+ \int_{t-D}^{t}(t-\theta)U(\theta)d\theta + \frac{1}{2}\left(\int_{t-D}^{t}U(\theta)d\theta\right)^2, \end{aligned} \qquad (90)$$

$$P_3(t) = X_3(t) + \int_{t-D}^{t}U(\theta)d\theta. \qquad (91)$$

Note that the nonlinear infinite-dimensional feedback operator employs a finite Volterra series in $U(\theta)$. ∎

Example 4

Consider the second-order system

$$\dot{X}_1(t) = X_2(t) - X_2^2(t)U(t-D), \qquad (92)$$

$$\dot{X}_2(t) = U(t-D). \qquad (93)$$

The control law is derived as

$$\begin{aligned} U(t) = &-X_1(t) - (2+D)X_2(t) - \frac{1}{3}X_2^3(t) \\ &- \int_{t-D}^{t}(2+t-\theta)U(\theta)d\theta. \end{aligned} \qquad (94)$$

For $D = 3$ and initial conditions $X_1(0) = 0$ and $X_2(0) = 1$, we obtain closed-loop solutions shown in Figure 10. ∎

DELAY-PDE CASCADES

When a plant with an input delay is a PDE, as in Figure 11, special challenges arise in the design of predictor feedback, particularly if the PDE is actuated through boundary control, which makes the B operator unbounded. In [8] we consider two benchmark delay-PDE cascades, one where the plant is a parabolic PDE and the other where the plant

Strict-Feedforward Systems and Integrator Forwarding

Strict-feedforward systems of the form

$$\dot{X}_1 = X_2 + \psi_1(X_2, X_3, \ldots, X_n) + \phi_1(X_2, X_3, \ldots, X_n)U, \tag{S35}$$

$$\dot{X}_2 = X_3 + \psi_2(X_3, \ldots, X_n) + \phi_2(X_3, \ldots, X_n)U, \tag{S36}$$

$$\vdots$$

$$\dot{X}_{n-2} = X_{n-1} + \psi_{n-2}(X_{n-1}, X_n) + \phi_{n-2}(X_{n-1}, X_n)U, \tag{S37}$$

$$\dot{X}_{n-1} = X_n + \phi_{n-1}(X_n)U, \tag{S38}$$

$$\dot{X}_n = U, \tag{S39}$$

where $\phi_i(0) = 0, \psi_i(X_{i+1}, 0, \ldots, 0) \equiv 0, \partial\psi_i(0)/\partial x_j = 0$ for $i = 1, 2, \ldots, n-1, j = i+1, \ldots, n$, are globally asymptotically stabilizable [26], even though they are generically not feedback linearizable. A small subset of this class is feedback linearizable [27]. In many ways, strict-feedforward systems are the exact opposite of strict-feedback systems [19]. While strict-feedback systems can exhibit finite-escape instability, strict-feedforward systems can be unstable but not exponentially unstable. By inspection of their structure, from the bottom toward the top, it can be seen that the system consists of n cascaded "nonlinear integrators," without any feedback loops that could generate exponential instability, much less finite escape time. This is not to say that strict-feedforward systems are easy to stabilize. On the contrary, due to the potential presence of control in many of the nonlinear state equations, nonlinear nonminimum-phase-like behavior is possible, which creates challenges for stabilization. While the strict-feedback systems, which allow exponential and finite-escape instabilities, call for aggressive forms of feedback laws to overcome the instabilities, strict-feedforward systems require a more careful approach to prevent inducing such behaviors in some of the state variables while controlling other state variables.

Strict-feedforward systems can be stabilized using either the method of nested saturations [26] or the method of integrator forwarding [20]. We illustrate these methods on the third-order system (85)–(87) of Example 3 without the input delay, that is,

$$\dot{X}_1 = X_2 + X_3^2, \tag{S40}$$

$$\dot{X}_2 = X_3 + X_3 U, \tag{S41}$$

$$\dot{X}_3 = U, \tag{S42}$$

The nested saturation approach employs the feedback law

$$U = -b_3\text{sat}\{a_{33}X_3 + b_2\text{sat}[a_{22}X_2 + a_{23}X_3 + b_1\text{sat}(a_{11}X_1 + a_{12}X_2 + a_{13}X_3)]\}, \tag{S43}$$

where b_i and a_{ij} are positive constants and $\text{sat}(\cdot)$ is the standard unit saturation function. The nested saturation design consists of deriving sufficient conditions for the coefficients b_i and a_{ij} so that stability is ensured. Such conditions are typically conservative.

The integrator forwarding design, on the other hand, employs Lyapunov functions and a suitable triangular nonlinear change of variables to achieve stabilization. For the system (85)–(87) a globally stabilizing controller is derived as [27]

$$\begin{aligned} U = &-X_1 - 3X_2 - 3X_3 - \frac{3}{8}X_2^2 \\ &+ \frac{3}{4}X_3\left(-X_1 - 2X_2 + \frac{1}{2}X_3 + \frac{X_2X_3}{2}\right. \\ &\left. + \frac{5}{8}X_3^2 - \frac{1}{4}X_3^3 - \frac{3}{8}\left(X_2 - \frac{X_3^2}{2}\right)^2\right). \end{aligned} \tag{S44}$$

The Lyapunov candidate

$$V = Z_1^2 + Z_2^2 + Z_3^2 \tag{S45}$$

has the derivative

$$\dot{V} = -\left(1 + \frac{3}{4}Z_3\right)^2 Z_1^2 - Z_2^2 - Z_3^2 - \left(\left(1 + \frac{3}{4}Z_3\right)Z_1 + Z_2 + Z_3\right)^2, \tag{S46}$$

where the triangular change of variables is

$$Z_3 = X_3, \tag{S47}$$

$$Z_2 = X_2 + X_3 - \frac{X_3^2}{2}, \tag{S48}$$

$$Z_1 = X_1 + 2X_2 + X_3 - \frac{5}{8}X_3^2 + \frac{3}{8}\left(X_2 - \frac{X_3^2}{2}\right)^2. \tag{S49}$$

Comparing the integrator forwarding controller (88) with $P = X$, namely, the controller (S44) with the nested saturation controller (S43), we observe that the nested saturation controller (S43) is bounded, while the integrator forwarding controller (S44) has considerable polynomial growth. Not surprisingly, by investing more control effort when the state is larger, the integrator forwarding controllers can achieve better performance [20]. However, the integrator forwarding procedure employs integrations and cannot always be carried out explicitly.

is a second-order hyperbolic PDE. We review here the parabolic case, where the plant is an unstable reaction-diffusion equation, which has an arbitrarily large number of open-loop unstable eigenvalues.

Consider the PDE system

$$u_t(x,t) = u_{xx}(x,t) + \lambda u(x,t), \tag{95}$$

$$u(0,t) = 0, \tag{96}$$

$$u(1,t) = U(t-D), \tag{97}$$

where λ is a constant. Since the open-loop eigenvalues of this system are $\sigma_n = \lambda - n^2\pi^2$, the system can have arbitrarily many unstable eigenvalues for large positive λ. We derive the stabilizing feedback law as

$$U(t) = 2\sum_{n=1}^{\infty}\int_0^1 \sin(\pi n\xi)\lambda\xi \frac{I_1\left(\sqrt{\lambda(1-\xi^2)}\right)}{\sqrt{\lambda(1-\xi^2)}}d\xi \times\left(-e^{(\lambda-\pi^2n^2)D}\int_0^1 \sin(\pi ny)u(y,t)dy\right.$$

$$+ \pi n(-1)^n \int_{t-D}^{t} e^{(\lambda - \pi^2 n^2)(t-\theta)} U(\theta)\, d\theta \Bigg), \qquad (98)$$

where $I_1(\cdot)$ is the modified Bessel function of the first kind.

Theorem 7

Consider the closed-loop system (95)–(98). There exists a continuous function $\rho : \mathbb{R}^2 \to (0,\infty)$ such that, for all initial conditions $(u_0, U_0) \in L_2[0,1] \times H_1[0,D]$ and for all $c > 0$, all solutions satisfy the bound

$$\Upsilon(t) \leq \rho(D,\lambda) e^{cD} \Upsilon(0) e^{-\min\{2,c\}t}, \quad \text{for all } t \geq 0, \qquad (99)$$

where

$$\Upsilon(t) \triangleq \int_0^1 u^2(x,t)\,dx + \int_{t-D}^{t} (U^2(\theta) + \dot{U}^2(\theta))\,d\theta. \qquad (100)$$

Two elements of this result are relevant. First, the feedback law (98) is derived explicitly. The explicit determination of the control gains is made possible by first deriving the control gain for $D = 0$ explicitly as in [28] and then by solving the undriven version of the PDE system (95)–(97) with an initial condition given by the control gain for $D = 0$. In more specific terms, we solve the PDE systems

$$k_{xx}(x,y) = k_{yy}(x,y) + \lambda k(x,y), \quad 0 \leq y \leq x \leq 1, \qquad (101)$$

$$k(x,0) = 0, \qquad (102)$$

$$k(x,x) = -\frac{\lambda}{2}x, \qquad (103)$$

and

$$\gamma_x(x,y) = \gamma_{yy}(x,y) + \lambda\gamma(x,y), \quad (x,y) \in [1, 1+D] \times (0,1), \qquad (104)$$

$$\gamma(x,0) = 0, \qquad (105)$$

$$\gamma(x,1) = 0, \qquad (106)$$

$$\gamma(1,y) = k(1,y). \qquad (107)$$

Note that the k-system is hyperbolic and defined on a triangular domain, whereas the γ-system is parabolic and defined on a rectangular semi-infinite domain. Furthermore, the solution to the k-system acts as an initial condition to the γ-system, as given by (107). The process of explicitly solving for $\gamma(x,y)$ is the PDE equivalent of analytically finding the vector Ke^{AD} in (3).

Second, when dealing with boundary control of a PDE with input delay, we face the problem of controlling two PDEs from different classes, such as, in the case covered here, a parabolic PDE and a first-order hyperbolic PDE, where the PDEs are interconnected through a boundary. While for each one of the two PDEs individually a natural system norm may be the standard L_2 norm, for the interconnected system this may not be the case, and a higher order Sobolev norm may have to be used for one of the subsystems, such as the L_2 norm of $\dot{U}$ in (100).

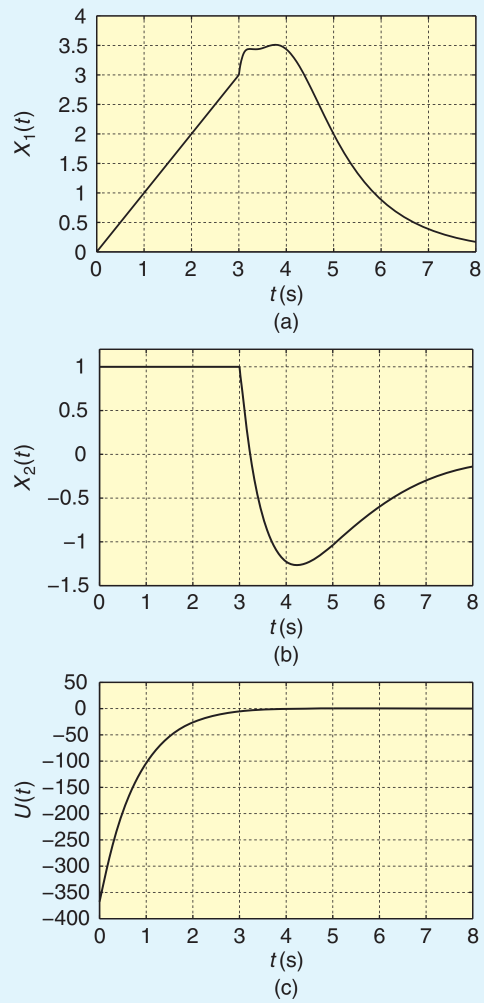

FIGURE 10 Time responses from Example 4 for $D = 3$ s. Note the transient of $X_1(t)$ in (a) after $t = 3$ s, which highlights the nonlinear character of the closed-loop system. The size of the control input $U(t)$ shown in (c) is due to the need to compensate for the long input delay of $D = 3$ s.

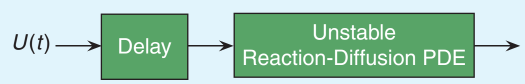

FIGURE 11 Control of an unstable parabolic partial differential equation (PDE) with input delay, that is, of a boundary controlled cascade of a transport PDE and a reaction-diffusion PDE. Explicit gains are derived for the predictor feedback (98). As stated in Theorem 7, stability is achieved in a Sobolev norm, rather than in the L_2 norm of the state of the PDE cascade.

PDE-ODE CASCADES

For the linear ODE (1) with input delay, the predictor feedback law (3) provides a boundary controller for a PDE-ODE cascade, where the input delay is a PDE of transport type. The same boundary control tools can be extended to stabilization of other PDE-ODE cascades, where the transport PDE is replaced by heat, wave, and other PDEs. Here we consider an ODE controlled through the undamped wave (string) PDE, depicted in Figure 12 and modeled by

$$\dot{X}(t) = AX(t) + Bu(0,t), \tag{108}$$
$$u_{tt}(x,t) = u_{xx}(x,t), \tag{109}$$
$$u_x(0,t) = 0, \tag{110}$$
$$u(D,t) = U(t). \tag{111}$$

The transfer function of the actuator dynamics, which is not rational, can be written explicitly as

$$\check{Y}(s) = \frac{1}{\cosh(Ds)}\check{U}(s), \tag{112}$$

where $Y(t) = u(0,t)$ and $\check{Y}(s), \check{U}(s)$ are the Laplace transforms of $Y(t), U(t)$, respectively.

The explicit feedback law for stabilization of (108)–(111) is designed as

$$U(t) = K\Sigma(D,c)X(t) + \int_0^D \varphi(D-y)u(y,t)dy + \int_0^D \psi(D-y)u_t(y,t)dy, \tag{113}$$

where $c > 0$,

$$\Sigma(D,c) = M(D) + c\int_0^D M(y)Ady, \tag{114}$$

$$M(x) = [I \quad 0]\, e^{\begin{bmatrix} 0 & A^2 \\ I & 0 \end{bmatrix}x} \begin{bmatrix} I \\ 0 \end{bmatrix}, \tag{115}$$

$$\varphi(x) = \mu(x) + cK(I+M(x))B, \tag{116}$$

$$\psi(x) = \int_0^x KM(\xi)Bd\xi - c + c\int_0^x \mu(\eta)d\eta, \tag{117}$$

$$\mu(x) = \int_0^x KM(\xi)ABd\xi. \tag{118}$$

The following stability result is expressed in terms of the Euclidean norm of $X(t)$ as well as the total (potential and kinetic) energy norm of the wave PDE state. We use the symbol j for $\sqrt{-1}$.

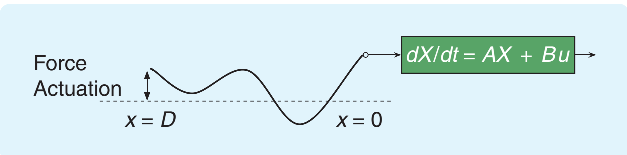

FIGURE 12 An arbitrary ordinary differential equation controlled through wave/string partial differential equation (PDE) dynamics at its input, using the control law (113). Theorem 8 guarantees exponential stability in a norm that includes the total energy of the wave PDE.

Theorem 8

Consider the closed-loop system (108)–(111), (113), where $c \in (0,1)\cup(1,\infty)$. Then there exist positive constants G, g such that

$$\Gamma(t) \le Ge^{-gt}\Gamma(0) \tag{119}$$

for all $t \ge 0$, where

$$\Gamma(t) \triangleq \|X(t)\|^2 + \int_0^D u_x^2(x,t)dx + \int_0^D u_t^2(x,t)dx. \tag{120}$$

Furthermore, the spectrum of the system (108)–(111), (113) is given by

$$\text{eig}\{A+BK\} \cup \left\{ -\frac{1}{2}\ln\left|\frac{1+c}{1-c}\right| + j\frac{\pi}{D}(n+\eta(c)),\ n \in \mathbb{Z} \right\}, \tag{121}$$

where $\eta(c) = 1/2$ if $c \in (0,1)$ and $\eta(c) = 0$ if $c > 1$.

While the controller (68) for the case of the input delay is easy to guess based on the predictor idea, the controller (113) for the wave PDE case is more complicated. The target dynamics governing the backstepping design of the controller (113) are given by

$$\dot{X}(t) = (A+BK)X(t) + Bw(0,t), \tag{122}$$
$$w_{tt}(x,t) = w_{xx}(x,t), \tag{123}$$
$$w_x(0,t) = cw_t(0,t), \quad c > 0, \tag{124}$$
$$w(D,t) = 0, \tag{125}$$

which is a cascade of a wave equation with the stable ODE $\dot{X} = (A+BK)X$. The backstepping transformation from (109)–(111) into (122)–(125) is given by

$$w(x,t) = u(x,t) - \int_0^x (\mu(x-y) + cK(I+M(x-y))B)u(y,t)dy - \int_0^x \left(m(x-y) - c + c\int_0^{x-y}\mu(s)ds \right)u_t(y,t)dy - \left(\gamma(x) + c\int_0^x \gamma(y)Ady \right)X(t), \tag{126}$$

where $m(x) = \int_0^x KM(\xi)Bd\xi$. The construction of this transformation, given in [8], involves a detailed study of the backstepping approach for wave equations. For a wave PDE-ODE cascade, the designer's task is not only to compensate the wave PDE dynamics to stabilize the ODE but also to stabilize the wave PDE, which has all of its eigenvalues on the imaginary axis. To stabilize the wave equation, damping must be added at the boundary $x = 0$ since the position actuation at $x = D$ does not allow direct addition of a force like damping. The transformation (126) allows the addition of damping at the boundary $x = 0$ using control at the boundary $x = D$.

We use PDEs to model delays, which allows us to develop delay-adaptive designs, as well as to model more general PDE actuator dynamics.

Example 5

Consider the undamped oscillator

$$A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}. \tag{127}$$

We need to add damping both in the plant and in the actuator dynamics governed by the undamped wave PDE (109). We take the nominal feedback as a pure damping (velocity-based) feedback, with the gain vector

$$K = [0 \;\; -h], \quad h > 0. \tag{128}$$

The feedback law (113) is obtained as

$$\begin{aligned} U(t) &= hc\sin(D)X_1(t) - h\cos(D)X_2(t) \\ &\quad - ch\int_0^D (1+\cos(D-y))u(y,t)dy \\ &\quad - \int_0^D (h\sin(D-y)+c)u_t(y,t)dy. \end{aligned} \tag{129}$$

For $D = 0$ this controller reduces to the nominal controller

$$U(t) = -hX_2(t) = -h\dot{X}_1(t). \tag{130}$$

For $h = 0$ the controller (129) is a stabilizing controller for the wave equation (109) alone. The controller's formula is

$$U(t) = -c\int_0^D u_t(y,t)dy. \tag{131}$$

This full-state feedback adds damping to the wave equation. According to (121), in each of the intervals $(0, 1)$ and $(1, \infty)$, where the gain c is stabilizing, the imaginary parts of all the eigenvalues of the wave equation (109), (110), (111) with feedback (131) are independent of c, but the real parts decrease as $c \to 1$, achieving $-\infty$ for $c = 1$, and then increase toward zero as $c \to \infty$. Hence, maximal damping for each of the infinitely many eigenvalues of (109), (110), (111), (131) is achieved for $c = 1$, which results in the closed-loop eigenvalues being on a vertical line at negative infinity. ■

OBSERVERS FOR ODEs WITH SENSOR PDE DYNAMICS

We now address the dual problem of observer design for ODEs in the presence of delay or PDE sensor dynamics. In particular we focus on the heat PDE, as a model of sensor dynamics. The wave PDE is considered in [8].

Consider the LTI ODE system in cascade with diffusive sensor dynamics at the output

$$\begin{aligned} Y(t) &= u(0,t), && (132)\\ u_t(x,t) &= u_{xx}(x,t), && (133)\\ u_x(0,t) &= 0, && (134)\\ u(D,t) &= CX(t), && (135)\\ \dot{X}(t) &= AX(t) + BU(t), && (136) \end{aligned}$$

as depicted in Figure 13. The sensor dynamics are given by the transfer function

$$\check{Y}(s) = \frac{1}{\cosh(D\sqrt{s})} C\check{X}(s), \tag{137}$$

where $\check{Y}(s)$, $\check{X}(s)$ are the Laplace transforms of $Y(t)$, $X(t)$, respectively.

While the transfer functions (112) and (137) may appear similar, the difference introduced by the square root under the hyperbolic cosine in (137) is significant. The wave system (112) is characterized by finite propagation speed, which is similar to the finite propagation speed of a transport delay, except that reflections off of the boundaries occur in the wave system. On the other hand, the heat system (137) has lowpass characteristics, which have the effect of attenuating the higher frequencies, but which do not impose a hard limit on the propagation speed of signals, leaving the lower frequencies with almost no phase shift, while the higher frequencies are subject to higher phase shifts. Furthermore, the heat system (137) has a sufficiently strong smoothing property that its output is C^∞ even if its input is discontinuous. Hence, due to the differences in the input-output characteristics of (112) and (137), the manners in which the two types of dynamics need to be compensated differ significantly.

The observer for (133)–(136) is constructed as

$$\begin{aligned} \hat{u}_t(x,t) &= \hat{u}_{xx}(x,t) + CM(x)L(Y(t) - \hat{u}(0,t)), && (138)\\ \hat{u}_x(0,t) &= 0, && (139) \end{aligned}$$

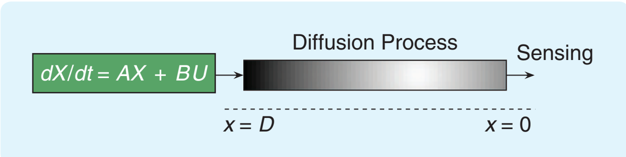

FIGURE 13 An ordinary differential equation whose output is measured through a diffusion process, namely, a heat partial differential equation. Theorem 9 guarantees that the observer (138)–(141) achieves exponential convergence of the state estimate.

The most significant open problem is extending the explicit predictor feedback design to systems with simultaneous input and state delays.

$$\hat{u}(D,t) = C\hat{X}(t), \tag{140}$$

$$\dot{\hat{X}}(t) = A\hat{X}(t) + BU(t) + M(D)L(Y(t) - \hat{u}(0,t)). \tag{141}$$

where L is chosen such that $A - LC$ is Hurwitz and

$$M(x) = [I \ \ 0]\, e^{\begin{bmatrix} 0 & A \\ I & 0 \end{bmatrix} x} \begin{bmatrix} I \\ 0 \end{bmatrix}. \tag{142}$$

The following result establishes that the error between the states of the observer (138)–(141) and of the plant (133)–(136) exponentially converges to zero.

Theorem 9

The state $(\hat{X}(t), \hat{u}(t))$ of the observer (138)–(141) converges exponentially to the state $(X(t), u(t))$ of the plant (133)–(136), that is, there exist positive constants G, g such that

$$\Gamma(t) \le Ge^{-gt}\Gamma(0) \tag{143}$$

for all $t \ge 0$, where

$$\Gamma(t) \triangleq |X(t) - \hat{X}(t)|^2 + \int_0^D (u(x,t) - \hat{u}(x,t))^2 dx. \tag{144}$$

The observer (133)–(136) estimates not only the ODE state but also the internal state of the sensor dynamics.

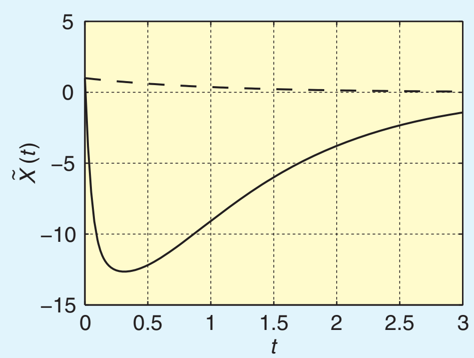

FIGURE 14 State-estimation error $\tilde{X}(t) = X(t) - \hat{X}(t)$ for the unstable and unmeasured scalar ordinary differential equation system $\dot{X}(t) = X(t) + U(t)$ in Example 6. The solid curve displays the estimation transient in the presence of sensor dynamics given by a heat equation on the domain of length $D = 3$. The dashed curve displays the nominal observer response $\tilde{X}(t) = \tilde{X}(0)\exp(-gt)$ for $D = 0$.

Example 6

Consider the unstable scalar plant $A = B = C = 1$. Since the nominal $(D = 0)$ observer error system is governed by the system matrix $A - CL = 1 - L$, we choose the observer gain $L = 1 + g$, where $g > 0$. The observer is thus given by

$$\hat{u}_t(x,t) = \hat{u}_{xx}(x,t) + (1+g)\sinh(x)(Y(t) - \hat{u}(0,t)), \tag{145}$$

$$\hat{u}_x(0,t) = 0, \tag{146}$$

$$\hat{u}(D,t) = \hat{X}(t), \tag{147}$$

$$\dot{\hat{X}}(t) = \hat{X}(t) + U(t) + (1+g)\sinh(D)(Y(t) - \hat{u}(0,t)). \tag{148}$$

We note that the observer gain grows with D and that the gain on the sensor state is the highest on the part of the sensor state that is the farthest away from the sensor location $(x = 0)$. Figure 14 shows simulation results for the state estimation error $\tilde{X}(t) = X(t) - \hat{X}(t)$ for the scalar ODE state for parameter values $D = 3$ and $g = 1$. The initial conditions are $X(0) = 1$, $\hat{X}(0) = 0$, $u(x,0) \equiv 0$, and $\hat{u}(x,0) \equiv 0$. ■

When the sensor dynamics are governed by pure delay, then the observer simplifies to

$$\hat{u}_t(x,t) = \hat{u}_x(x,t) + Ce^{Ax}L(Y(t) - \hat{u}(0,t)), \tag{149}$$

$$\hat{u}(D,t) = C\hat{X}(t), \tag{150}$$

$$\dot{\hat{X}}(t) = A\hat{X}(t) + BU(t) + e^{AD}L(Y(t) - \hat{u}(0,t)), \tag{151}$$

which can be equivalently written as

$$\dot{\Xi}(t) = A\Xi(t) + BU(t-D) + L(Y(t) - C\Xi(t)), \tag{152}$$

$$\hat{X}(t) = e^{AD}\Xi(t) + \int_{t-D}^{t} e^{A(t-\theta)}BU(\theta)d\theta, \tag{153}$$

$$\hat{u}(x,t) = C\left[e^{ADx}\Xi(t) + \int_{t-D}^{t+D(x-1)} e^{A(t+D(x-1)-\theta)}BU(\theta)d\theta\right]. \tag{154}$$

The observer representation (152), (153) can be arrived at without resorting to the backstepping design. The variable $\Xi(t)$ is the estimate of $X(t-D)$, whereas the variable $\hat{X}(t)$ is obtained by advancing $\Xi(t)$ in time by the mount D with the help of the variation of constants formula. On the other hand, observer design when the sensor dynamics are governed by the heat PDE cannot be completed using similar intuition. The observer (138)–(141) is derived using the PDE backstepping approach.

CONCLUSIONS: TOOLS AND OPEN PROBLEMS

The PDE backstepping approach is a potentially powerful tool for advancing design techniques for systems with input and output delays. Three key ideas are presented in this article. The first idea is the construction of backstepping transformations that facilitate treatment of delays and PDE dynamics at the input, as well as in more complex plant structures such as systems in the lower triangular form, with delays or PDE dynamics affecting the integrators. The second idea is the construction of Lyapunov functionals and explicit stability estimates, with the help of direct and inverse backstepping transformations. The third idea is the connection between delay systems and an array of other classes of PDE systems, which can be approached in a similar manner, each introducing a different type of challenge.

All the results presented here are constructive. The constants and bounds that are not stated explicitly, such as the robustness margin δ in Theorem 3 and the convergence speeds in all the theorems, are estimated explicitly in the proofs of these theorems. However, these estimates are conservative and have limited value as guidance on the achievable convergence speeds or robustness margins.

A wealth of future opportunities exists for research in this area. The most significant open problem is extending the explicit predictor feedback design to systems with simultaneous input and state delays. Another open problem is developing predictor feedback for systems with state-dependent input delays, which are related to, but not a subclass, of the case of time-varying delays. Developing design tools for nonlinear systems with input dynamics governed by heat or wave PDEs is of lesser practical significance, but it is of considerable theoretical and methodological significance. Finally, it is of interest to develop feedback laws for more general PDE-PDE cascades, such as wave-heat (or structure-fluid) and other interconnections.

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