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Control of an unstable reaction–diffusion PDE with long input delay

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ABSTRACT

A Smith Predictor-like design for compensation of arbitrarily long input delays is available for general, controllable, possibly unstable LTI finite-dimensional systems. Such a design has not been proposed previously for problems where the plant is a PDE. We present a design and stability analysis for a prototype problem, where the plant is a reaction–diffusion (parabolic) PDE, with boundary control. The plant has an arbitrary number of unstable eigenvalues and arbitrarily long delay, with an unbounded input operator. The predictor-based feedback design extends fairly routinely, within the framework of infinite-dimensional backstepping. However, the stability analysis contains interesting features that do not arise in predictor problems when the plant is an ODE. The unbounded character of the input operator requires that the stability be characterized in terms of the H_1 (rather than the usual L_2) norm of the actuator state. The analysis involves an interesting structure of interconnected PDEs, of parabolic and first-order hyperbolic types, where the feedback gain kernel for the undelayed problem becomes an initial condition in a PDE arising in the compensator design for the problem with input delay. Space and time variables swap their roles in an interesting manner throughout the analysis.

1. Introduction

ODE systems with input and/or output delays have been studied successfully for several decades in the framework of predictor-based control design [1–27].

Control of PDEs with input delays, and control of more general PDE–PDE cascades, is an interesting area that is just opening up for research. An example of a relevant effort is the stabilization of a beam equation with output delay by Guo and Chang [28], which is motivated by the interest to address the lack of delay-robustness identified by Datko [29].

In this note we deal with a prototypical problem in control of PDE systems with input delays (Fig. 1). We consider a reaction–diffusion equation, which may have an arbitrary number of unstable eigenvalues in the uncontrolled case, and we approach it with boundary control, with an arbitrarily long delay at the input.

We design a feedback law, which at its core has our backstepping boundary controller from [30], and which incorporates compensation of arbitrarily long delay at the input. Philosophically, this feedback law is a direct extension of our design for ODEs with input delay from [11]. However, there are several significant challenges that arise when the plant is a PDE, rather than an ODE.

Stability analysis for cascades of stable PDEs from different classes, when interconnected through a boundary, virtually explodes in complexity, despite the seemingly simple structure where one PDE is autonomous and exponentially stable and feeds into the other PDE. The difficulty arises for two reasons. One is that the connectivity through the boundary gives rise to an unbounded input operator in the interconnection. The second reason is that the two subsystems are from different PDE classes, with different numbers of derivatives in space or time (or both). This requires delicate combinations of norms in the Lyapunov functions for the overall systems.

The structure of the note is simple. In Section 2 we present a control design and state the stability result. In Section 3 we present the proof of stability. In Section 4 we give the explicit solutions for the closed-loop system, which is possible due to the explicit character of our control design and the system transformations. In Section 5 we discuss the difference between the problems where the plant is an ODE and where the plant is a PDE. In this section we also discuss a routine extension to a general class of cascade PDEs of first-order hyperbolic and parabolic types.

2. Control design

Consider the system

$$u_t(x,t) = u_{xx}(x,t) + \lambda u(x,t) \quad (1)$$

$$u(0,t) = 0 \quad (2)$$

$$u(1,t) = U(t-D), \quad (3)$$

or, in an alternative representation,

$$u_t(x,t) = u_{xx}(x,t) + \lambda u(x,t), \quad x \in (0,1) \quad (4)$$

$$u(0,t) = 0 \quad (5)$$

$$u(1,t) = v(1,t) \quad (6)$$

$$v_t(x,t) = v_x(x,t), \quad x \in [1, 1+D) \quad (7)$$

$$v(1+D,t) = U(t), \quad (8)$$

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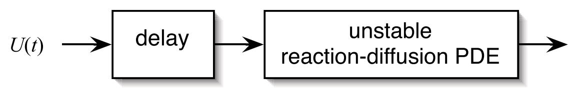

Fig. 1. Reaction–diffusion PDE system with input delay.

where $U(t)$ is the input, (u, v) is the state, and D is the delay, which is constant and known, but it can be of arbitrary length. The state of the input delay dynamics is known explicitly,

$$v(x,t) = U(t + x - 1 - D), \quad x \in [1, 1 + D]. \tag{9}$$

We consider the backstepping transformation of the form

$$w(x,t) = u(x,t) - \int_0^x k(x,y)u(y,t)\mathrm{d}y, \quad x \in [0,1] \tag{10}$$

$$z(x,t) = v(x,t) - \int_1^x p(x-y)v(y,t)\mathrm{d}y - \int_0^1 \gamma(x,y)u(y,t)\mathrm{d}y, \quad x \in [1, 1+D] \tag{11}$$

where the kernels k, p, γ need to be chosen to transform the cascade PDE system into the target system

$$w_t(x,t) = w_{xx}(x,t), \quad x \in (0,1) \tag{12}$$

$$w(0,t) = 0 \tag{13}$$

$$w(1,t) = z(1,t) \tag{14}$$

$$z_t(x,t) = z_x(x,t), \quad x \in [1, 1+D) \tag{15}$$

$$z(1+D,t) = 0, \tag{16}$$

with the control

$$U(t) = \int_1^{1+D} p(1+D-y)v(y,t)\mathrm{d}y + \int_0^1 \gamma(1+D,y)u(y,t)\mathrm{d}y. \tag{17}$$

The cascade connection

$$z \to w \tag{18}$$

is a cascade of an exponentially stable autonomous transport PDE for $z(x,t)$, feeding into the exponentially stable heat PDE for $w(x,t)$.

The change of variables

$$(u, v) \mapsto (w, z) \tag{19}$$

is defined through the three integral operator kernels, $k(x,y)$, $\gamma(x,y)$, and $p(x)$. With a lengthy calculation we show that the kernel $k(x,y)$ has to satisfy the PDE

$$k_{xx}(x,y) - k_{yy}(x,y) = \lambda k(x,y), \quad 0 \le y \le x \le 1 \tag{20}$$

$$k(x,0) = 0 \tag{21}$$

$$k(x,x) = -\frac{\lambda}{2}x, \tag{22}$$

for which the solution was found explicitly in [30] as

$$k(x,y) = -\lambda y \frac{I_1\left(\sqrt{\lambda\left(x^2 - y^2\right)}\right)}{\sqrt{\lambda\left(x^2 - y^2\right)}}, \tag{23}$$

where $I_1(\cdot)$ denotes the appropriate modified Bessel function. The kernel γ is found to be governed by the reaction–diffusion PDE

$$\gamma_x(x,y) = \gamma_{yy}(x,y) + \lambda\gamma(x,y), \quad (x,y) \in [1, 1+D] \times (0,1) \tag{24}$$

$$\gamma(x,0) = 0 \tag{25}$$

$$\gamma(x,1) = 0, \tag{26}$$

where $x \in [1, 1+D]$ should be viewed as the time variable and $y \in (0,1)$ as the space variable, and where the initial condition is given by

$$\gamma(1,y) = k(1,y). \tag{27}$$

After solving for $\gamma(x,y)$, the kernel p is obtained as

$$p(s) = -\gamma_y(1+s, 1), \quad s \in [0, D]. \tag{28}$$

Before we proceed, we make the following observation about the target system.

Proposition 1. *The spectrum of the system (12)–(16) is given by*

$$\sigma_n = -\pi^2 n^2, \quad n = 1, 2, \ldots, +\infty. \tag{29}$$

The following theorem establishes an exponential stability result in the appropriate norm for the cascade system of two PDEs which are interconnected through a boundary.

Theorem 2. *Consider the closed-loop system consisting of the plant (4)–(8) and the control law*

$$U(t) = \int_0^1 \gamma(D,y)u(y,t)\mathrm{d}y - \int_1^{1+D} \gamma_y(1+D-y,1)v(y,t)\mathrm{d}y. \tag{30}$$

If the initial conditions are such that $(u_0, v_0) \in L_2[0,1] \times H_1[1, 1+D]$, then the system has a unique solution $(u(\cdot,t), v(\cdot,t)) \in C([0,\infty), L_2[0,1] \times H_1[1, 1+D])$ and there exists a positive continuous function $M : \mathbb{R}^2 \to \mathbb{R}_+$ such that

$$\Upsilon(t) \le M(\lambda, D)\mathrm{e}^{cD}\Upsilon(0)\mathrm{e}^{-\min\{2,c\}t}, \quad \forall t \ge 0 \tag{31}$$

for any $c > 0$, where

$$\Upsilon(t) = \int_0^1 u^2(x,t)\mathrm{d}x + \int_1^{1+D} \left(v^2(x,t) + v_x^2(x,t)\right)\mathrm{d}x. \tag{32}$$

3. Proof of stability

First, we seek the inverse transformation $(u, v) \mapsto (w, z)$. We postulate it in the form

$$u(x,t) = w(x,t) + \int_0^x l(x,y)w(y,t)\mathrm{d}y, \quad x \in [0,1] \tag{33}$$

$$v(x,t) = z(x,t) + \int_1^x q(x-y)z(y,t)\mathrm{d}y + \int_0^1 \delta(x,y)w(y,t)\mathrm{d}y, \quad x \in [1, 1+D]. \tag{34}$$

With a lengthy calculation we show that the kernel $l(x,y)$ has to satisfy the PDE

$$l_{xx}(x,y) - l_{yy}(x,y) = -\lambda l(x,y), \quad 0 \le y \le x \le 1 \tag{35}$$

$$l(x,0) = 0 \tag{36}$$

$$l(x,x) = -\frac{\lambda}{2}x, \tag{37}$$

for which the solution was found explicitly in [30] as

$$l(x,y) = -\lambda y \frac{J_1\left(\sqrt{\lambda\left(x^2 - y^2\right)}\right)}{\sqrt{\lambda\left(x^2 - y^2\right)}}, \tag{38}$$

Proof. We start with

$$\|w(t)\|^2 = \|\omega(t)\|^2 + 2z(1,t)\int_0^1 x\omega(x,t)\mathrm{d}x + z^2(1,t)\int_0^1 x^2\mathrm{d}x$$
$$\leq \|\omega(t)\|^2 + \frac{2}{\sqrt{3}}|z(1,t)|\|\omega(t)\| + \frac{1}{3}z^2(1,t), \tag{66}$$

where we used the Cauchy–Schwartz inequality. With Young's and Agmon's inequalities we get

$$\|w(t)\|^2 \leq 2\|\omega(t)\|^2 + \frac{2}{3}z^2(1,t)$$
$$\leq 2\|\omega(t)\|^2 + \frac{8}{3}\int_1^{1+D} z_x^2(x,t)\mathrm{d}x$$
$$\leq 3\left(\|\omega(t)\|^2 + \int_1^{1+D} z_x^2(x,t)\mathrm{d}x\right). \tag{67}$$

Since

$$\|\omega(t)\|^2 + \int_1^{1+D} z_x^2(x,t)\mathrm{d}x \leq 12\Pi(t), \tag{68}$$

we arrive at the result of the lemma. □

Now we obtain a stability result in terms of the state of the (w, z) system.

Lemma 6.

$$\Xi(t) \leq 72\mathrm{e}^{cD}\Xi(0)\mathrm{e}^{-\min\{2,c\}t}, \quad \forall t \geq 0 \tag{69}$$

where

$$\Xi(t) = \int_0^1 w^2(x,t)\mathrm{d}x + \int_1^{1+D}\left(z^2(x,t) + z_x^2(x,t)\right)\mathrm{d}x. \tag{70}$$

Proof. The result of this lemma follows immediately from the last three lemmas and from the fact that

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_1^{1+D} \mathrm{e}^{c(x-1)}z^2(x,t)\mathrm{d}x$$
$$= -z(1,t)^2 - c\int_1^{1+D}\mathrm{e}^{c(x-1)}z^2(x,t)\mathrm{d}x. \quad \square \tag{71}$$

Next, with several applications of the Cauchy–Schwartz inequality we get the following lemma.

Lemma 7.

$$\Xi(t) \leq \alpha_1\Upsilon(t) \tag{72}$$
$$\Upsilon(t) \leq \alpha_2\Xi(t), \tag{73}$$

where

$$\alpha_1 = 2\left(1 + \int_0^1\int_0^x k^2(x,y)\mathrm{d}y\mathrm{d}x\right) + 3\int_1^{1+D}\int_0^1 \gamma^2(x,y)\mathrm{d}y\mathrm{d}x$$
$$+4\int_1^{1+D}\int_0^1 \gamma_x^2(x,y)\mathrm{d}y\mathrm{d}x + 3\left(1 + D\int_1^{1+D}\gamma_y^2(x,1)\mathrm{d}x\right)$$
$$+4\left(\gamma_y^2(1,1) + D\int_1^{1+D}\gamma_{xy}^2(x,1)\mathrm{d}x\right) + 4 \tag{74}$$

$$\alpha_2 = 2\left(1 + \int_0^1\int_0^x l^2(x,y)\mathrm{d}y\mathrm{d}x\right) + 3\int_1^{1+D}\int_0^1 \delta^2(x,y)\mathrm{d}y\mathrm{d}x$$
$$+4\int_1^{1+D}\int_0^1 \delta_x^2(x,y)\mathrm{d}y\mathrm{d}x + 3\left(1 + D\int_1^{1+D}\delta_y^2(x,1)\mathrm{d}x\right)$$
$$+4\left(\delta_y^2(1,1) + D\int_1^{1+D}\delta_{xy}^2(x,1)\mathrm{d}x\right) + 4. \tag{75}$$

Proof. We just highlight several steps in the proof of the first half of the lemma. The proof of the second half is identical. First, from $w(x,t) = u(x,t) - \int_0^x k(x,y)u(y,t)\mathrm{d}y$ we can obtain

$$\|w(t)\|^2 \leq 2\left(1 + \int_0^1\int_0^x k^2(x,y)\mathrm{d}y\mathrm{d}x\right)\|u(t)\|^2. \tag{76}$$

Then, from $z(x,t) = v(x,t) + \int_1^x \gamma_y(1+x-y,1)v(y,t)\mathrm{d}y - \int_0^1 \gamma(x,y)u(y,t)\mathrm{d}y$ we get

$$\int_1^{1+D} z^2(x,t)\mathrm{d}x \leq 3\left(1 + \int_1^{1+D}\int_0^{x-1}\gamma_y^2(1+s,1)\mathrm{d}s\mathrm{d}x\right)$$
$$\times\int_1^{1+D} v^2(x,t)\mathrm{d}x + 3\left(\int_1^{1+D}\int_0^1\gamma^2(x,y)\mathrm{d}y\mathrm{d}x\right)\|u(t)\|^2. \tag{77}$$

Next, from $z(x,t) = v(x,t) + \int_1^x \gamma_y(1+x-y,1)v(y,t)\mathrm{d}y - \int_0^1 \gamma(x,y)u(y,t)\mathrm{d}y$ we derive

$$z_x(x,t) = v_x(x,t) + \gamma_y(1,1)v(x,t) + \int_1^x \gamma_{xy}(1+x-y,1)$$
$$\times v(y,t)\mathrm{d}y - \int_0^1 \gamma_x(x,y)u(y,t)\mathrm{d}y, \tag{78}$$

which yields

$$\int_1^{1+D} z_x^2(x,t)\mathrm{d}x \leq 4\int_1^{1+D} v_x^2(x,t)\mathrm{d}x$$
$$+4\left(\gamma_y^2(1,1) + \int_1^{1+D}\int_0^{x-1}\gamma_{xy}^2(1+s,1)\mathrm{d}s\mathrm{d}x\right)$$
$$\times\int_1^{1+D} v^2(x,t)\mathrm{d}x + 4\left(\int_1^{1+D}\int_0^1\gamma_x^2(x,y)\mathrm{d}y\mathrm{d}x\right)\|u(t)\|^2. \tag{79}$$

Combining the above steps, along with the fact that

$$\int_1^{1+D}\int_0^{x-1}\gamma_y^2(1+s,1)\mathrm{d}s\mathrm{d}x = \int_1^{1+D}(1+D-x)\gamma_y^2(x,1)\mathrm{d}x$$
$$\leq D\int_1^{1+D}\gamma_y^2(x,1)\mathrm{d}x \tag{80}$$

$$\int_1^{1+D}\int_0^{x-1}\gamma_{xy}^2(1+s,1)\mathrm{d}s\mathrm{d}x = \int_1^{1+D}(1+D-x)\gamma_{xy}^2(x,1)\mathrm{d}x$$
$$\leq D\int_1^{1+D}\gamma_{xy}^2(x,1)\mathrm{d}x, \tag{81}$$

we obtain the first half of the lemma. In a similar manner we also prove the second half. □

The constants α_1 and α_2 are finite since the γ-system and the δ-system are parabolic PDEs which generate analytic semigroups, whereas their respective initial conditions $\gamma(1,y) = k(1,y)$ and $\delta(1,y) = l(1,y)$ are C^∞ in y. This establishes Theorem 2 with

$$M(\lambda, D) = 72\alpha_1\alpha_2. \tag{82}$$

Some parts of α_1 and α_2 can even be calculated analytically, as given by the next lemma.

Lemma 8.

$$\gamma_y(1,1) = \frac{\lambda^2}{8} - \frac{\lambda}{2} \tag{83}$$

$$\delta_y(1,1) = -\frac{\lambda^2}{8} - \frac{\lambda}{2}. \tag{84}$$

Proof. By calculating

$$k_y(1,y) = -\lambda \frac{I_1\left(\sqrt{\lambda\left(1-y^2\right)}\right)}{\sqrt{\lambda\left(1-y^2\right)}} + \lambda^2 y^2 \frac{I_2\left(\sqrt{\lambda\left(1-y^2\right)}\right)}{\lambda\left(1-y^2\right)} \tag{85}$$

$$l_y(1,y) = -\lambda \frac{J_1\left(\sqrt{\lambda\left(1-y^2\right)}\right)}{\sqrt{\lambda\left(1-y^2\right)}} - \lambda^2 y^2 \frac{J_2\left(\sqrt{\lambda\left(1-y^2\right)}\right)}{\lambda\left(1-y^2\right)} \tag{86}$$

and using the facts that

$$\lim_{\xi\to 0} \frac{I_n(\xi)}{\xi^n} = \frac{1}{2^n n!} \tag{87}$$

$$\lim_{\xi\to 0} \frac{J_n(\xi)}{\xi^n} = \frac{1}{2^n n!} \tag{88}$$

for all $n \in \mathbb{N}$. □

For other parts of α_1 and α_2 a bound can be easily calculated, as given in the next lemma.

Lemma 9. *The following holds:*

$$\int_1^{1+D}\int_0^1 \gamma^2(x,y)\mathrm{d}y\mathrm{d}x \le \frac{1}{2|\lambda|-\pi^2/2}\left(\mathrm{e}^{(2|\lambda|-\pi^2/2)D}-1\right) \times \int_0^1 k^2(1,y)\mathrm{d}y \tag{89}$$

$$\int_1^{1+D}\int_0^1 \gamma_x^2(x,y)\mathrm{d}y\mathrm{d}x \le \frac{1}{|\lambda|-\pi^2/4}\left(\mathrm{e}^{(2|\lambda|-\pi^2/2)D}-1\right) \times \left(\int_0^1 k_{yy}^2(1,y)\mathrm{d}y + |\lambda|\int_0^1 k^2(1,y)\mathrm{d}y\right) \tag{90}$$

$$\int_1^{1+D}\int_0^1 \delta^2(x,y)\mathrm{d}y\mathrm{d}x \le \frac{2}{\pi^2}\left(1-\mathrm{e}^{D\pi^2/2}\right)\int_0^1 l^2(1,y)\mathrm{d}y \tag{91}$$

$$\int_1^{1+D}\int_0^1 \delta_x^2(x,y)\mathrm{d}y\mathrm{d}x \le \frac{2}{\pi^2}\left(1-\mathrm{e}^{D\pi^2/2}\right)\int_0^1 l_{yy}^2(1,y)\mathrm{d}y, \tag{92}$$

where

$$k(1,y) = -\lambda y \frac{I_1\left(\sqrt{\lambda\left(1-y^2\right)}\right)}{\sqrt{\lambda\left(1-y^2\right)}} \tag{93}$$

$$k_{yy}(1,y) = 3\lambda^2 y \frac{I_2\left(\sqrt{\lambda\left(1-y^2\right)}\right)}{\lambda\left(1-y^2\right)} - \lambda^3 y^3 \frac{I_3\left(\sqrt{\lambda\left(1-y^2\right)}\right)}{\left(\sqrt{\lambda\left(1-y^2\right)}\right)^3} \tag{94}$$

$$l(1,y) = -\lambda y \frac{J_1\left(\sqrt{\lambda\left(1-y^2\right)}\right)}{\sqrt{\lambda\left(1-y^2\right)}} \tag{95}$$

$$l_{yy}(1,y) = -3\lambda^2 y \frac{J_2\left(\sqrt{\lambda\left(1-y^2\right)}\right)}{\lambda\left(1-y^2\right)} - \lambda^3 y^3 \frac{J_3\left(\sqrt{\lambda\left(1-y^2\right)}\right)}{\left(\sqrt{\lambda\left(1-y^2\right)}\right)^3} \tag{96}$$

are continuous functions with

$$k(1,1) = -\lambda \tag{97}$$

$$k_{yy}(1,1) = -\frac{\lambda^3}{48} + \frac{3\lambda^2}{8} \tag{98}$$

$$l(1,1) = -\lambda \tag{99}$$

$$l_{yy}(1,1) = -\frac{\lambda^3}{48} - \frac{3\lambda^2}{8}. \tag{100}$$

Proof. We prove the results only for $\gamma(x,y)$. The results for $\delta(x,y)$ are similar. We start from the fact that

$$\frac{\mathrm{d}}{\mathrm{d}x}\frac{1}{2}\|\gamma(x)\|^2 = -\|\gamma_y(x)\|^2 + \lambda\|\gamma(x)\|^2 \tag{101}$$

$$\le \left(|\lambda| - \frac{\pi^2}{4}\right)\|\gamma(x)\|^2, \tag{102}$$

where we have used the Wirtinger inequality and where the norm $\|\cdot\|$ is taken with respect to y. Then we get

$$\|\gamma(x)\|^2 \le \mathrm{e}^{(2|\lambda|-\pi^2/2)(x-1)}\|\gamma(1)\|^2. \tag{103}$$

Since $\gamma(1,y) = k(1,y)$, we get

$$\|\gamma(x)\|^2 \le \mathrm{e}^{(2|\lambda|-\pi^2/2)(x-1)}\int_0^1 k^2(1,y)\mathrm{d}y. \tag{104}$$

Finally,

$$\int_1^{1+D}\int_0^1 \gamma^2(x,y)\mathrm{d}y\mathrm{d}x = \int_1^{1+D}\|\gamma(x)\|^2\mathrm{d}x \le \frac{1}{2|\lambda|-\pi^2/2}\left(\mathrm{e}^{(2|\lambda|-\pi^2/2)D}-1\right)\int_0^1 k^2(1,y)\mathrm{d}y. \tag{105}$$

This proves the first inequality in the lemma. To prove the second inequality, we use the fact that $\gamma_x = \gamma_{yy} + \lambda\gamma$. With the boundary conditions $\gamma(x,0) = \gamma(x,1) \equiv 0$, we get the system

$$\gamma_{yyx} = \gamma_{yyyy} + \lambda\gamma_{yy} \tag{106}$$

$$\gamma_{yy}(x,0) = 0 \tag{107}$$

$$\gamma_{yy}(x,1) = 0. \tag{108}$$

Then, using a similar calculation as for obtaining the first inequality, we get

$$\|\gamma_{yy}(x)\|^2 \le \mathrm{e}^{(2|\lambda|-\pi^2/2)(x-1)}\int_0^1 k_{yy}^2(1,y)\mathrm{d}y, \tag{109}$$

and finally

$$\int_1^{1+D}\int_0^1 \gamma_{yy}^2(x,y)\mathrm{d}y\mathrm{d}x \le \frac{1}{2|\lambda|-\pi^2/2}\left(\mathrm{e}^{(2|\lambda|-\pi^2/2)D}-1\right) \times \int_0^1 k_{yy}^2(1,y)\mathrm{d}y. \tag{110}$$

By combining the above results for γ_{yy} and γ we get the second inequality in the lemma, for γ_x. The proof of the inequalities for δ mimic those for γ. □

Remark 1. Alternative bounds can be derived which do not involve L_2 bounds on $k_{yy}(1,y)$ and $l_{yy}(1,y)$ but only on $k_y(1,y)$ and $l_y(1,y)$. First, one would integrate (101) in x and obtain

$$\int_1^{1+D}\int_0^1 \gamma_y^2(x,y)\mathrm{d}y\mathrm{d}x \le \lambda\int_1^{1+D}\int_0^1 \gamma^2(x,y)\mathrm{d}y\mathrm{d}x + \frac{1}{2}\int_0^1 \gamma^2(1,y)\mathrm{d}y. \tag{111}$$

Then, one would consider the system

$$\gamma_{yx} = \gamma_{yyy} + \lambda\gamma_y \tag{112}$$

$$\gamma_{yy}(x,0) = 0 \tag{113}$$

$$\gamma_{yy}(x,1) = 0 \tag{114}$$

and obtain

$$\frac{d}{dx}\frac{1}{2}\|\gamma_y(x)\|^2 = -\|\gamma_{yy}(x)\|^2 + \lambda\|\gamma_y(x)\|^2, \tag{115}$$

which, upon integration in x, yields

$$\int_1^{1+D}\int_0^1 \gamma_{yy}^2(x,y)\mathrm{d}y\mathrm{d}x \le \lambda\int_1^{1+D}\int_0^1 \gamma_y^2(x,y)\mathrm{d}y\mathrm{d}x + \frac{1}{2}\int_0^1 \gamma_y^2(1,y)\mathrm{d}y. \tag{116}$$

Substituting (111) into (116), we get

$$\begin{aligned}\int_1^{1+D}\int_0^1 \gamma_{yy}^2(x,y)\mathrm{d}y\mathrm{d}x &\le \lambda^2\int_1^{1+D}\int_0^1 \gamma^2(x,y)\mathrm{d}y\mathrm{d}x \\ &\quad +\frac{\lambda}{2}\int_0^1 \gamma^2(1,y)\mathrm{d}y + \frac{1}{2}\int_0^1 \gamma_y^2(1,y)\mathrm{d}y \\ &\le \left[\frac{\lambda^2}{2|\lambda|-\pi^2/2}\left(\mathrm{e}^{(2|\lambda|-\pi^2/2)D}-1\right)+\frac{\lambda}{2}\right] \\ &\quad \times\int_0^1 k^2(1,y)\mathrm{d}y + \frac{1}{2}\int_0^1 k_y^2(1,y)\mathrm{d}y.\end{aligned} \tag{117}$$

Finally, using $\gamma_x = \gamma_{yy} + \lambda\gamma$, we get

$$\begin{aligned}\int_1^{1+D}\int_0^1 \gamma_x^2(x,y)\mathrm{d}y\mathrm{d}x &\le \left[\frac{2\lambda^2}{|\lambda|-\pi^4/2}\left(\mathrm{e}^{(2|\lambda|-\pi^2/2)D}-1\right)+\lambda\right] \\ &\quad\times\int_0^1 k^2(1,y)\mathrm{d}y + \int_0^1 k_y^2(1,y)\mathrm{d}y.\end{aligned} \tag{118}$$

For δ_x we get

$$\int_1^{1+D}\int_0^1 \delta_x^2(x,y)\mathrm{d}y\mathrm{d}x \le \frac{1}{2}\int_0^1 l_y^2(1,y)\mathrm{d}y. \tag{119}$$

To complete the proof of the main theorem, we need to provide estimates for the norms $\int_1^{1+D}\gamma_y^2(x,1)\mathrm{d}x$, $\int_1^{1+D}\gamma_{xy}^2(x,1)\mathrm{d}x$, $\int_1^{1+D}\delta_y^2(x,1)\mathrm{d}x$, $\int_1^{1+D}\delta_{xy}^2(x,1)\mathrm{d}x$. First we do it for the latter two quantities as they are easier to obtain.

Lemma 10. *The following is true*

$$\int_1^{1+D}\delta_y^2(x,1)\mathrm{d}x \le \int_0^1 l^2(1,y)\mathrm{d}y + \frac{1}{2}\int_0^1 l_y^2(1,y)\mathrm{d}y \tag{120}$$

$$\int_1^{1+D}\delta_{yx}^2(x,1)\mathrm{d}x \le \int_0^1 l_{yy}^2(1,y)\mathrm{d}y + \frac{1}{2}\int_0^1 l_{yyy}^2(1,y)\mathrm{d}y, \tag{121}$$

where

$$\begin{aligned}l_{yyy}(1,y) &= -3\lambda^2\frac{J_2\left(\sqrt{\lambda\left(1-y^2\right)}\right)}{\lambda\left(1-y^2\right)} - 6\lambda^3y^2\frac{J_3\left(\sqrt{\lambda\left(1-y^2\right)}\right)}{\left(\lambda\left(1-y^2\right)\right)^{3/2}} \\ &\quad -\lambda^4y^4\frac{J_4\left(\sqrt{\lambda\left(1-y^2\right)}\right)}{\left(\lambda\left(1-y^2\right)\right)^2}\end{aligned} \tag{122}$$

is a continuous function with

$$l_{yy}(1,1) = -\frac{\lambda^4}{384} - \frac{\lambda^3}{8} - \frac{3\lambda^2}{8}. \tag{123}$$

Proof. We start with the PDEs $\delta_x = \delta_{yy}$ and $\delta_{xx} = \delta_{xyy}$ and multiply them, respectively, by $2y\delta_y(x,y)$ and $2y\delta_{xy}(x,y)$, obtaining

$$2y\delta_x(x,y)\delta_y(x,y) = 2y\delta_y(x,y)\delta_{yy}(x,y) \tag{124}$$

$$2y\delta_{xx}(x,y)\delta_{xy}(x,y) = 2y\delta_{xy}(x,y)\delta_{xyy}(x,y). \tag{125}$$

Integrating both sides in y and applying integration by parts on the right side, we get

$$2\int_0^1 y\delta_x(x,y)\delta_y(x,y)\mathrm{d}y = \delta_y^2(x,1) - \int_0^1 \delta_y^2(x,y)\mathrm{d}y \tag{126}$$

$$2\int_0^1 y\delta_{xx}(x,y)\delta_{xy}(x,y)\mathrm{d}y = \delta_{xy}^2(x,1) - \int_0^1 \delta_{xy}^2(x,y)\mathrm{d}y. \tag{127}$$

Applying Young's inequality, we obtain

$$\delta_y^2(x,1) \le 2\int_0^1 \delta_y^2(x,y)\mathrm{d}y + \int_0^1 \delta_x^2(x,y)\mathrm{d}y \tag{128}$$

$$\delta_{xy}^2(x,1) \le 2\int_0^1 \delta_{xy}^2(x,y)\mathrm{d}y + \int_0^1 \delta_{xyy}^2(x,y)\mathrm{d}y. \tag{129}$$

The four quantities on the right hand sides of the two inequalities are bounded by

$$\int_1^{1+D}\int_0^1 \delta_y^2(x,y)\mathrm{d}y\mathrm{d}x \le \frac{1}{2}\int_0^1 l^2(1,y)\mathrm{d}y \tag{130}$$

$$\int_1^{1+D}\int_0^1 \delta_x^2(x,y)\mathrm{d}y\mathrm{d}x \le \frac{1}{2}\int_0^1 l_y^2(1,y)\mathrm{d}y \tag{131}$$

$$\int_1^{1+D}\int_0^1 \delta_{xy}^2(x,y)\mathrm{d}y\mathrm{d}x \le \frac{1}{2}\int_0^1 l_{yy}^2(1,y)\mathrm{d}y \tag{132}$$

$$\int_1^{1+D}\int_0^1 \delta_{xyy}^2(x,y)\mathrm{d}y\mathrm{d}x \le \frac{1}{2}\int_0^1 l_{yyy}^2(1,y)\mathrm{d}y. \tag{133}$$

We do not prove all of them but only the last one. From the PDE $\delta_{yxx} = \delta_{yyyx}$ with boundary conditions $\delta_{yyx}(x,0) = \delta_{yyx}(x,1) \equiv 0$, we get

$$\frac{d}{dx}\frac{1}{2}\int_0^1 \delta_{yx}^2(x,y)\mathrm{d}y = -\int_0^1 \delta_{yyx}^2(x,y)\mathrm{d}y. \tag{134}$$

Integrating this equation in x, we get

$$\int_1^{1+D}\int_0^1 \delta_{xyy}^2(x,y)\mathrm{d}y\mathrm{d}x \le \frac{1}{2}\int_0^1 \delta_{yyy}^2(1,y)\mathrm{d}y, \tag{135}$$

yielding (133) with the initial condition $\delta_{yyy}(1,y) = l_{yyy}(1,y)$. Integrating the inequalities (128) and (129) and substituting (130)–(133), we complete the proof of the lemma. □

Finally, we provide estimates for the norms $\int_1^{1+D}\gamma_y^2(x,1)\mathrm{d}x$ and $\int_1^{1+D}\gamma_{xy}^2(x,1)\mathrm{d}x$.

Lemma 11. *The following is true*

$$\begin{aligned}\int_1^{1+D}\gamma_y^2(x,1)\mathrm{d}x &\le \left((4\lambda^2+\lambda)\frac{\mathrm{e}^{(2|\lambda|-\pi^2/2)D}-1}{2|\lambda|-\pi^2/2}+\lambda+1\right) \\ &\quad\times\int_0^1 k^2(1,y)\mathrm{d}y + \int_0^1 k_y^2(1,y)\mathrm{d}y\end{aligned} \tag{136}$$

$$\begin{aligned}\int_1^{1+D}\gamma_{yx}^2(x,1)\mathrm{d}x &\le 4\lambda^2(\lambda+1) \\ &\quad\times\left[\frac{2\lambda}{|\lambda|-\pi^4/2}\left(\mathrm{e}^{(2|\lambda|-\pi^2/2)D}-1\right)+\frac{1}{2}\right]\int_0^1 k^2(1,y)\mathrm{d}y\end{aligned}$$

$$+(2\lambda^2+1)\int_0^1 k_y^2(1,y)\mathrm{d}y$$
$$+2\left((\lambda+1)\int_0^1 k_{yy}^2(1,y)\mathrm{d}y+\int_0^1 k_{yyy}^2(1,y)\mathrm{d}y\right), \tag{137}$$

where

$$k_{yyy}(1,y)=3\lambda^2\frac{I_2\left(\sqrt{\lambda\left(1-y^2\right)}\right)}{\lambda\left(1-y^2\right)}-6\lambda^3y^2\frac{I_3\left(\sqrt{\lambda\left(1-y^2\right)}\right)}{\left(\lambda\left(1-y^2\right)\right)^{3/2}}$$
$$-\lambda^4y^4\frac{I_4\left(\sqrt{\lambda\left(1-y^2\right)}\right)}{\left(\lambda\left(1-y^2\right)\right)^2} \tag{138}$$

is a continuous function with

$$k_{yy}(1,1)=\frac{\lambda^4}{384}-\frac{\lambda^3}{8}-\frac{3\lambda^2}{8}. \tag{139}$$

Proof. We start with the same steps as at the beginning of the proof of Lemma 10 and obtain

$$\gamma_y^2(x,1)=\int_0^1\gamma_y^2(x,y)\mathrm{d}y+\lambda\int_0^1\gamma^2(x,y)\mathrm{d}y$$
$$-\lambda\gamma^2(x,1)+2\int_0^1 y\gamma_x(x,y)\gamma_y(x,y)\mathrm{d}y \tag{140}$$

$$\gamma_{xy}^2(x,1)=\int_0^1\gamma_{xy}^2(x,y)\mathrm{d}y+\lambda\int_0^1\gamma_x^2(x,y)\mathrm{d}y$$
$$-\lambda\gamma_x^2(x,1)+2\int_0^1 y\gamma_{xx}(x,y)\gamma_{xy}(x,y)\mathrm{d}y. \tag{141}$$

From (140) we obtain

$$\gamma_y^2(x,1)\le\lambda\int_0^1\gamma^2(x,y)\mathrm{d}y+2\int_0^1\gamma_y^2(x,y)\mathrm{d}y$$
$$+\int_0^1\gamma_x^2(x,y)\mathrm{d}y. \tag{142}$$

By integrating both sides in x and substituting a long sequence of various inequalities that we have derived so far, we obtain (136). From (141) we get

$$\gamma_{xy}^2(x,1)\le\lambda\int_0^1\gamma_x^2(x,y)\mathrm{d}y+2\int_0^1\gamma_{xy}^2(x,y)\mathrm{d}y$$
$$+\int_0^1\gamma_{xx}^2(x,y)\mathrm{d}y. \tag{143}$$

Of the three terms on the right, for the first one, $\int_0^1\gamma_x^2(x,y)\mathrm{d}y$, we have computed an integral bound in (118). For the second terms we have

$$\int_1^{1+D}\int_0^1\gamma_{xy}^2(x,y)\mathrm{d}y\mathrm{d}x$$
$$\le\lambda\int_1^{1+D}\int_0^1\gamma_x^2(x,y)\mathrm{d}y\mathrm{d}x+\frac{1}{2}\int_0^1\gamma_x^2(1,y)\mathrm{d}y$$
$$\le\lambda\int_1^{1+D}\int_0^1\gamma_x^2(x,y)\mathrm{d}y\mathrm{d}x$$
$$+\lambda^2\int_0^1 k^2(1,y)\mathrm{d}y+\int_0^1 k_{yy}^2(1,y)\mathrm{d}y. \tag{144}$$

Again, $\int_0^1\gamma_x^2(x,y)\mathrm{d}y$ is bounded by (118). Finally, we estimate the integral in x of the term $\int_0^1\gamma_{xx}^2(x,y)\mathrm{d}y$. First, from the PDE $\gamma_{xxy}=\gamma_{xyyy}+\lambda xy$ with boundary conditions $\gamma_{xyy}(x,0)=\gamma_{xyy}(x,1)\equiv 0$, we get

$$\int_1^{1+D}\int_0^1\gamma_{xyy}^2(x,y)\mathrm{d}y\mathrm{d}x\le\lambda\int_1^{1+D}\int_0^1\gamma_{xy}^2(x,y)\mathrm{d}y\mathrm{d}x$$
$$+\frac{1}{2}\int_0^1\gamma_{xy}^2(1,y)\mathrm{d}y. \tag{145}$$

Then, with a chain of inequalities, whose details we omit, we obtain

$$\int_1^{1+D}\int_0^1\gamma_{xx}^2(x,y)\mathrm{d}y\mathrm{d}x\le 4\lambda^2\int_1^{1+D}\int_0^1\gamma_x^2(x,y)\mathrm{d}y\mathrm{d}x$$
$$+2\left(\lambda^3\int_0^1 k^2(1,y)\mathrm{d}y+\lambda^2\int_0^1 k_y^2(1,y)\mathrm{d}y\right.$$
$$\left.+\lambda\int_0^1 k_{yy}^2(1,y)\mathrm{d}y+\int_0^1 k_{yyy}^2(1,y)\mathrm{d}y\right). \tag{146}$$

Collecting the results from all of the above inequalities, we obtain

$$\int_1^{1+D}\gamma_{yx}^2(x,1)\mathrm{d}x\le(4\lambda^2+3\lambda)\int_1^{1+D}\int_0^1\gamma_x^2(x,y)\mathrm{d}y\mathrm{d}x$$
$$+2\left((\lambda^3+\lambda^2)\int_0^1 k^2(1,y)\mathrm{d}y+\lambda^2\int_0^1 k_y^2(1,y)\mathrm{d}y\right.$$
$$\left.+(\lambda+1)\int_0^1 k_{yy}^2(1,y)\mathrm{d}y+\int_0^1 k_{yyy}^2(1,y)\mathrm{d}y\right). \tag{147}$$

With a substitution of (118), we arrive at the result (137) of the lemma. □

With all the lemmas in this section, we prove Theorem 2 with explicit expressions for α_1 and α_2 in $M(\lambda,D)=72\alpha_1\alpha_2$.

4. Explicit solutions of the closed-loop system

In this section we determine the explicit closed-loop solutions.

Lemma 12. *The solution of the equation for $\gamma(x,y)$ is given explicitly as*

$$\gamma(x,y)=2\sum_{n=1}^{\infty}\mathrm{e}^{(\lambda-\pi^2n^2)(x-1)}\sin(\pi ny)\int_0^1\sin(\pi n\xi)k(1,\xi)\mathrm{d}\xi$$
$$=-2\sum_{n=1}^{\infty}\mathrm{e}^{(\lambda-\pi^2n^2)(x-1)}\sin(\pi ny)$$
$$\times\int_0^1\sin(\pi n\xi)\lambda\xi\frac{I_1\left(\sqrt{\lambda\left(1-\xi^2\right)}\right)}{\sqrt{\lambda\left(1-\xi^2\right)}}\mathrm{d}\xi \tag{148}$$

and yields the following expression for $\gamma_y(x,1)$:

$$\gamma_y(x,1)=2\sum_{n=1}^{\infty}\mathrm{e}^{(\lambda-\pi^2n^2)(x-1)}\pi n(-1)^n\int_0^1\sin(\pi n\xi)k(1,\xi)\mathrm{d}\xi$$
$$=-2\sum_{n=1}^{\infty}\mathrm{e}^{(\lambda-\pi^2n^2)(x-1)}\pi n(-1)^n$$
$$\times\int_0^1\sin(\pi n\xi)\lambda\xi\frac{I_1\left(\sqrt{\lambda\left(1-\xi^2\right)}\right)}{\sqrt{\lambda\left(1-\xi^2\right)}}\mathrm{d}\xi. \tag{149}$$

Substituting the gain functions $\gamma(1+D,y)$ and $\gamma_y(1+D-\eta,1)$ into the feedback law

$$U(t)=\int_0^1\gamma(D,y)u(y,t)\mathrm{d}y-\int_{t-D}^{t}\gamma_y(t-\theta,1)U(\theta)\mathrm{d}\theta, \tag{150}$$

we obtain

$$U(t) = 2\int_0^1 \sin(\pi n\xi)\lambda\xi \frac{I_1\left(\sqrt{\lambda\left(1-\xi^2\right)}\right)}{\sqrt{\lambda\left(1-\xi^2\right)}}d\xi \times \sum_{n=1}^{\infty}\left(-e^{\left(\lambda-\pi^2n^2\right)D}\int_0^1 \sin(\pi ny)u(y,t)dy + \pi n(-1)^n \int_{t-D}^{t} e^{\left(\lambda-\pi^2n^2\right)(t-\theta)}U(\theta)d\theta\right). \quad (151)$$

So, by explicitly determining the kernel functions $k(x,y)$, $\gamma(x,y)$, and $p(x)$, we have not only found the control law explicitly, but we have also found the transformation $(u,v) \mapsto (w,z)$ explicitly. Now we seek $l(x,y)$, $\delta(x,y)$, and $q(x)$ explicitly, so we can find the transformation $(w,z) \mapsto (u,v)$ explicitly.

Lemma 13. *The solution of the equation for $\delta(x,y)$ is given explicitly as*

$$\delta(x,y) = 2\sum_{n=1}^{\infty} e^{-\pi^2n^2(x-1)}\sin(\pi ny)\int_0^1 \sin(\pi n\xi)l(1,\xi)d\xi = -2\sum_{n=1}^{\infty} e^{-\pi^2n^2(x-1)}\sin(\pi ny) \times \int_0^1 \sin(\pi n\xi)\lambda\xi \frac{J_1\left(\sqrt{\lambda\left(1-\xi^2\right)}\right)}{\sqrt{\lambda\left(1-\xi^2\right)}}d\xi \quad (152)$$

and yields the following expression for $\delta_y(x,1)$:

$$\delta_y(x,1) = 2\sum_{n=1}^{\infty} e^{-\pi^2n^2(x-1)}\pi n(-1)^n\int_0^1 \sin(\pi n\xi)l(1,\xi)d\xi = -2\sum_{n=1}^{\infty} e^{-\pi^2n^2(x-1)}\pi n(-1)^n \times \int_0^1 \sin(\pi n\xi)\lambda\xi \frac{J_1\left(\sqrt{\lambda\left(1-\xi^2\right)}\right)}{\sqrt{\lambda\left(1-\xi^2\right)}}d\xi. \quad (153)$$

Lemma 14. *The explicit solutions of the system*

$$w_t(x,t) = w_{xx}(x,t) \quad (154)$$

$$w(0,t) = 0 \quad (155)$$

$$w(1,t) = z(1,t) \quad (156)$$

$$z_t(x,t) = z_x(x,t) \quad (157)$$

$$z(1+D,t) = 0 \quad (158)$$

from the initial conditions (w_0, z_0) are given by

$$z(x,t) = \begin{cases} z_0(t+x), & t\in[0,D] \\ 0, & t>D \end{cases} \quad (159)$$

and

$$w(x,t) = 2\sum_{n=1}^{\infty} e^{-\pi^2n^2t}\sin(\pi nx)\left(\int_0^1 \sin(\pi ny)w_0(y)dy + \int_0^1 \sin(\pi ny)\pi^2n^2ydy\left(\int_0^t e^{\pi^2n^2\tau}z_0(1+\tau)d\tau\right)\right) \quad (160)$$

for $t\in[0,D]$ and

$$w(x,t) = 2\sum_{n=1}^{\infty} e^{-\pi^2n^2t}\sin(\pi nx)\int_0^1 \sin(\pi ny)w(y,D)dy \quad (161)$$

for $t > D$.

Proof. First we observe that

$$z_t(1,t) = \begin{cases} z_0'(1+t), & t\in[0,D] \\ 0, & t>D \end{cases} \quad (162)$$

Then, from (45)–(47) we get

$$\begin{aligned}\omega(x,t) &= 2\sum_{n=1}^{\infty} e^{-\pi^2n^2t}\sin(\pi nx)\int_0^1 \sin(\pi ny)\omega_0(y)dy - 2\sum_{n=1}^{\infty}\int_0^t e^{-\pi^2n^2(t-\tau)}\sin(\pi nx)\int_0^1 \sin(\pi ny)ydyz_t(1,\tau)d\tau \\ &= 2\sum_{n=1}^{\infty} e^{-\pi^2n^2t}\sin(\pi nx)\int_0^1 \sin(\pi ny)\omega_0(y)dy - 2\sum_{n=1}^{\infty} e^{-\pi^2n^2t}\sin(\pi nx)\int_0^1 \sin(\pi ny)ydy \times \left(e^{\pi^2n^2t}z(1,t) - z(1,0) - \pi^2n^2\int_0^t e^{\pi^2n^2\tau}z(1,\tau)d\tau\right) \\ &= 2\sum_{n=1}^{\infty} e^{-\pi^2n^2t}\sin(\pi nx)\int_0^1 \sin(\pi ny)\omega_0(y)dy + 2\sum_{n=1}^{\infty} e^{-\pi^2n^2t}\sin(\pi nx)\int_0^1 \sin(\pi ny)ydy \times \left(z(1,0) + \pi^2n^2\int_0^t e^{\pi^2n^2\tau}z(1,\tau)d\tau\right) - \left(2\sum_{n=1}^{\infty}\sin(\pi nx)\int_0^1 \sin(\pi ny)ydy\right)z(1,t).\end{aligned} \quad (163)$$

Using the Fourier series representation of x on $[0, 1]$ we get

$$\omega(x,t) = 2\sum_{n=1}^{\infty} e^{-\pi^2n^2t}\sin(\pi nx)\int_0^1 \sin(\pi ny)\omega_0(y)dy + 2\sum_{n=1}^{\infty} e^{-\pi^2n^2t}\sin(\pi nx)\int_0^1 \sin(\pi ny)ydy \times \left(z(1,0) + \pi^2n^2\int_0^t e^{\pi^2n^2\tau}z(1,\tau)d\tau\right) - xz(1,t). \quad (164)$$

Using (44), i.e., the fact that $w(x,t) = \omega(x,t) + xz(1,t)$, we obtain

$$w(x,t) = 2\sum_{n=1}^{\infty} e^{-\pi^2n^2t}\sin(\pi nx)\int_0^1 \sin(\pi ny)\omega_0(y)dy + 2\sum_{n=1}^{\infty} e^{-\pi^2n^2t}\sin(\pi nx)\int_0^1 \sin(\pi ny)ydy \times \left(z(1,0) + \pi^2n^2\int_0^t e^{\pi^2n^2\tau}z(1,\tau)d\tau\right). \quad (165)$$

Further, with $\omega_0(y) + yz(1,0) = w_0(y)$ we get

$$w(x,t) = 2\sum_{n=1}^{\infty} e^{-\pi^2n^2t}\sin(\pi nx)\int_0^1 \sin(\pi ny)w_0(y)dy + 2\sum_{n=1}^{\infty} e^{-\pi^2n^2t}\sin(\pi nx)\int_0^1 \sin(\pi ny)ydy\pi^2n^2 \times \int_0^t e^{\pi^2n^2\tau}z(1,\tau)d\tau. \quad (166)$$

Recalling that $z(1,t) = z_0(1+t)$ for $t\in[0,D]$ we complete the proof of (160). Finally, to obtain (161) we observe that for $t > D$ the

w-system is just the heat equation with homogeneous boundary conditions, which completes the proof of the lemma. □

We have thus established the following.

Proposition 15. *The closed-loop system consisting of the plant*

$$u_t(x,t) = u_{xx}(x,t) + \lambda u(x,t) \tag{167}$$

$$u(0,t) = 0 \tag{168}$$

$$u(1,t) = v(1,t) \tag{169}$$

$$v_t(x,t) = v_x(x,t) \tag{170}$$

$$v(1+D,t) = U(t), \tag{171}$$

and the control law

$$U(t) = 2\int_0^1 \sin(\pi n\xi)\lambda\xi \frac{I_1\left(\sqrt{\lambda\left(1-\xi^2\right)}\right)}{\sqrt{\lambda\left(1-\xi^2\right)}}\mathrm{d}\xi \times \sum_{n=1}^{\infty}\left(-\mathrm{e}^{\left(\lambda-\pi^2n^2\right)D}\int_0^1 \sin(\pi ny)u(y,t)\mathrm{d}y + \pi n(-1)^n\int_1^{1+D}\mathrm{e}^{\left(\lambda-\pi^2n^2\right)(1+D-y)}v(y,t)\mathrm{d}y\right), \tag{172}$$

and starting from the initial condition $(u_0(x), v_0(x))$, has solutions given by

$$u(x,t) = w(x,t) - \int_0^x \lambda y \frac{J_1\left(\sqrt{\lambda\left(x^2-y^2\right)}\right)}{\sqrt{\lambda\left(x^2-y^2\right)}}w(y,t)\mathrm{d}y \tag{173}$$

$$v(x,t) = z(x,t) + 2\left(\int_0^1 \sin(\pi n\xi)\lambda\xi \frac{J_1\left(\sqrt{\lambda\left(1-\xi^2\right)}\right)}{\sqrt{\lambda\left(1-\xi^2\right)}}\mathrm{d}\xi\right) \times \sum_{n=1}^{\infty}\left(\pi n(-1)^n\int_1^x \mathrm{e}^{-\pi^2n^2(x-y)}z(y,t)\mathrm{d}y - \mathrm{e}^{-\pi^2n^2(x-1)}\int_0^1 \sin(\pi ny)w(y,t)\mathrm{d}y\right), \tag{174}$$

where $(w(x,t), z(x,t))$ are given by (159)–(161) with initial conditions

$$w_0(x) = u_0(x) + \int_0^x \lambda y \frac{I_1\left(\sqrt{\lambda\left(x^2-y^2\right)}\right)}{\sqrt{\lambda\left(x^2-y^2\right)}}u_0(y)\mathrm{d}y \tag{175}$$

$$z_0(x) = v_0(x) - 2\left(\int_0^1 \sin(\pi n\xi)\lambda\xi \frac{I_1\left(\sqrt{\lambda\left(1-\xi^2\right)}\right)}{\sqrt{\lambda\left(1-\xi^2\right)}}\mathrm{d}\xi\right) \times \sum_{n=1}^{\infty}\left(\pi n(-1)^n\int_1^x \mathrm{e}^{\left(\lambda-\pi^2n^2\right)(x-y)}v_0(y)\mathrm{d}y - \mathrm{e}^{\left(\lambda-\pi^2n^2\right)(x-1)}\int_0^1 \sin(\pi ny)u_0(y)\mathrm{d}y\right). \tag{176}$$

It can be noted that $v(x,t)$ can be written in a form that is even more direct, using the orthogonality of the basis functions $\sin(\pi ny)$. We get that for $t \in [0, D]$

$$v(x,t) = z_0(x+t) - 2\left(\int_0^1 \sin(\pi n\xi)\lambda\xi \frac{J_1\left(\sqrt{\lambda\left(1-\xi^2\right)}\right)}{\sqrt{\lambda\left(1-\xi^2\right)}}\mathrm{d}\xi\right) \times \sum_{n=1}^{\infty}\left(\pi n(-1)^{n+1}\int_1^x \mathrm{e}^{-\pi^2n^2(x-y)}z_0(y+t)\mathrm{d}y + \int_0^1 \sin(\pi ny)\pi^2n^2y\mathrm{d}y\int_0^t \mathrm{e}^{-\pi^2n^2(t-\tau+x-1)}z_0(1+\tau)\mathrm{d}\tau + \mathrm{e}^{-\pi^2n^2(t+x-1)}\int_0^1 \sin(\pi ny)w_0(y)\mathrm{d}y\right), \tag{177}$$

whereas for $t > D$,

$$v(x,t) = -2\left(\int_0^1 \sin(\pi n\xi)\lambda\xi \frac{J_1\left(\sqrt{\lambda\left(1-\xi^2\right)}\right)}{\sqrt{\lambda\left(1-\xi^2\right)}}\mathrm{d}\xi\right) \times \sum_{n=1}^{\infty}\mathrm{e}^{-\pi^2n^2(t+x-1)}\int_0^1 \sin(\pi ny)w(y,D)\mathrm{d}y. \tag{178}$$

5. Closing comments

After our lengthy stability analysis it is fair to ask what is the crucial difference between the result for a delay-PDE cascade in this paper and the general delay-ODE cascade result in [11]. The stability result for delay-ODE systems of the form

$$\dot{X}(t) = AX(t) + BU(t-D) \tag{179}$$

is

$$|X(t)|^2 + \int_{t-D}^t U^2(\theta)\mathrm{d}\theta \le \frac{\phi_2\psi_2}{\phi_1\psi_1}\mathrm{e}^{-\mu t} \times \left(|X(0)|^2 + \int_{-D}^0 U^2(\theta)\mathrm{d}\theta\right), \tag{180}$$

where μ is a positive constant,

$$\frac{\phi_2}{\phi_1} = \max\left\{3\left(1+D\|m\|^2\right), 1+3\|KM\|^2\right\} \times \max\left\{3\left(1+D\|n\|^2\right), 1+3\|KN\|^2\right\} \tag{181}$$

$$\frac{\psi_2}{\psi_1} = \frac{\max\left\{\lambda_{\max}(P), a\right\}}{\min\left\{\lambda_{\min}(P), \frac{a(1+D)}{2}\right\}} \tag{182}$$

$$m(x) = K\mathrm{e}^{Ax}B \tag{183}$$

$$n(x) = K\mathrm{e}^{(A+BK)x}B \tag{184}$$

$$M(x) = \mathrm{e}^{Ax} \tag{185}$$

$$N(x) = \mathrm{e}^{(A+BK)x}, \tag{186}$$

a is a positive constant, and P is a solution of a Lyapunov equation for the Hurwitz matrix $A + BK$. The critical portion of these estimates is the dependence on the L_2 norms $\|m\|$ and $\|n\|$. The functions $m(x)$ and $n(x)$ depend on the input vector B. In the boundary-controlled PDE problems, such as the one in this paper, the input operator B is unbounded, so a much more delicate analysis, with involvement of the H_1 norm of the delay state in the system norm, namely $\int_{t-D}^t \dot{U}^2(\theta)\mathrm{d}\theta$, was needed here.

To a reader who has navigated the details of Section 3, the analysis may seem as a daunting maze of PDEs. To a reader who has successfully digested this analysis, this interconnection of various PDEs may seem fascinating. We show a diagram of the interconnections in Fig. 2. For example, the k-PDE is autonomous, it is a second-order hyperbolic PDE in the Goursat form, and it has an explicit solution via Bessel functions. The solution to the k-PDE acts as an initial condition to the γ-PDE, which is of parabolic type. A similar relation exists between the l-PDE and δ-PDE. The k, l, γ, and δ-PDEs appear as kernels (i.e., multiplicatively) in the transformations between the (u, v) and (w, z) PDE systems.

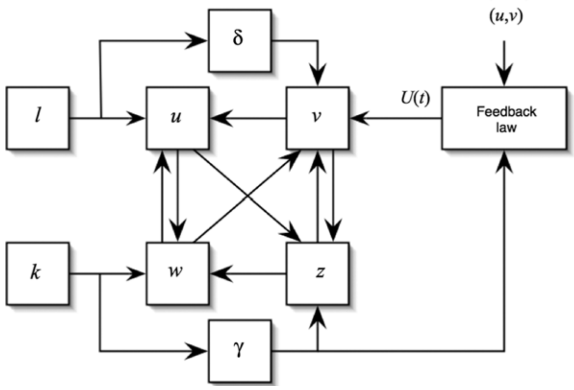

Fig. 2. Interconnection of various PDEs in the analysis of the feedback system for a reaction–diffusion PDE with input delay.

It should be clear that we focused on the 'simple' plant (4)–(8) only for notational simplicity. It is straightforward to extend the result of this chapter to the system

$$u_t(x,t) = u_{xx}(x,t) + b(x)u_x(x,t) + \lambda_1(x)u(x,t) + g_1(x)u(0,t) + \int_0^x f_1(x,y)u(y,t)\mathrm{d}y, \quad x \in (0,1) \tag{187}$$

$$u(0,t) = 0 \tag{188}$$

$$u(1,t) = v(1,t) \tag{189}$$

$$v_t(x,t) = v_x(x,t) + \lambda_2(x)v(x,t) + g_2(x)v(0,t) + \int_0^x f_2(x,y)v(y,t)\mathrm{d}y, \quad x \in [1, 1+D) \tag{190}$$

$$v(1+D,t) = U(t), \tag{191}$$

where $b(x)$, $\lambda_1(x)$, $\lambda_2(x)$, $g_1(x)$, $g_2(x)$, $f_1(x,y)$, $f_2(x,y)$ are arbitrary continuous functions. The tools used in this extension are those in [30,11,33].

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