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Lyapunov Stability of Linear Predictor Feedback for Time-Varying Input Delay

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***Abstract*—For linear time-invariant systems with a time-varying input delay, an explicit formula for predictor feedback was presented by Nihtila in 1991. In this note we construct a time-varying Lyapunov functional for the closed-loop system and establish exponential stability. The key challenge is the selection of a state for a transport partial differential equation, which has a non-constant propagation speed, and which is the basis of the stability analysis. We illustrate the design and its conditions with several examples. We also develop an observer equivalent of the predictor feedback design, for the case of time-varying sensor delay.**

***Index Terms*—Backstepping, delay systems, distributed parameter systems.**

I. Introduction

Systems with long input delays, even those where the plant is unstable, can be successfully controlled using predictor feedback [4]–[8], [10]–[17], [20], [21], [24]–[26] and methods based on LQ control [2], [3], [22]. All these results deal with problems where the delay is constant. In the case of systems with *time-varying* delays, various results exist for problems with state delay and no input delay. However, the case of systems with time-varying *input* delay has received very little attention.

A basic idea how to approach problems with time-varying input delay was introduced by Artstein [1], however, the design is not worked out in detail since the case of time-varying delay is considered only for plants that are time-varying, in which case explicit developments are not possible. An explicit state-feedback design for linear time-invariant (LTI) plants with time-varying input delays was presented by Nihtila [19]. (A parameter-adaptive design for a scalar system with known time-varying delay function [18] preceded the author's general result in [19].) In this note we establish exponential stability of the feedback system with the controller from [19]. We do so using an explicit construction of a (strict) Lyapunov functional. Stability is not claimed in [1], [19], where the approach employed in the design uses a transformation that relates infinite-dimensional feedback system with another finite-dimensional system. Such a construction does not yield a Lyapunov functional, though it yields a controller that compensates the time-varying delay.

The challenge in the study of stability under time-varying input delay, as compared to our result for constant input delays [12], is that one has to construct a Lyapunov functional using a backstepping transformation with time-varying kernels, and transforming the actuator state into a transport partial differential equation (PDE) with a convection speed coefficient that varies with both space and time. An additional challenge is how to define the state of the transport PDE modeling the actuator state using the past input signal.

We start in Section II with an intuitive introduction of the predictor feedback under time-varying input delay. Then in Section III we present a stability study. In Section IV we present an observer for the case of a plant with time-varying sensor delay. Finally, in Section V we present several examples, including a numerical example with a scalar unstable plant and with an oscillating time-varying input delay.

II. Predictor Feedback Design With Time-Varying Actuator Delay

We consider the system

$$\dot{X}(t) = AX(t) + BU(\phi(t)) \tag{1}$$

where $X \in \mathbb{R}^n$ is the state, U is the control input, and $\phi(t)$ is a continuously differentiable function that incorporates the actuator delay. This function will have to satisfy certain conditions that we shall impose in our development, in particular, that

$$\phi(t) \leq t, \quad \forall t \geq 0. \tag{2}$$

One can alternatively view the function $\phi(t)$ in the more standard form $\phi(t) = t - D(t)$, where $D(t) \geq 0$ is a time-varying delay. However, the formalism involving the function $\phi(t)$ turns out to be more convenient, particularly because the predictor problem requires the inverse function of $\phi(t)$, i.e., $\phi^{-1}(t)$, so we will proceed with the model (1). The invertibility of $\phi(\cdot)$ will be ensured by imposing the following assumption.

Assumption 1: $\phi : \mathbb{R}_+ \to \mathbb{R}$ is a continuously differentiable function that satisfies

$$\phi'(t) > 0, \quad \forall t \geq 0 \tag{3}$$

and such that

$$\pi_1^* = \frac{1}{\sup_{\vartheta \geq \phi^{-1}(0)} \phi'(\vartheta)} > 0. \tag{4}$$

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The meaning of the assumption is that the function $\phi(t)$ is strictly increasing, which, as we shall see, we need in several elements of our analysis.

The main premise of the predictor based design is that one generates the control input

$$U(\phi(t)) = KX(t), \quad \forall \phi(t) \geq 0 \tag{5}$$

so that the closed-loop system is $\dot{X}(t) = (A+BK)X(t)$ for all $\phi(t) \geq 0$, or, alternatively, using the inverse of $\phi(\cdot)$

$$\dot{X}(t) = (A+BK)X(t), \quad \forall t \geq \phi^{-1}(0). \tag{6}$$

The gain vector K is selected so that $A+BK$ is Hurwitz.

We now re-write (5) as

$$U(t) = KX\left(\phi^{-1}(t)\right), \quad \forall t \geq 0. \tag{7}$$

With the help of the model (1) and the variation of constants formula, the quantity $X\left(\phi^{-1}(t)\right)$ is written as

$$X\left(\phi^{-1}(t)\right) = \mathrm{e}^{A\left(\phi^{-1}(t)-t\right)}X(t) + \int_t^{\phi^{-1}(t)} \mathrm{e}^{A\left(\phi^{-1}(t)-\tau\right)}BU(\phi(\tau))d\tau. \tag{8}$$

To express the integral in terms of the signal $U(\cdot)$ rather than the signal $U(\phi(\cdot))$, we introduce the change of the integration variable, $\theta = \phi(\tau)$, i.e., $\tau = \phi^{-1}(\theta)$. Recalling the basic differentiation rule for the inverse of a function, $(d/d\theta)\phi^{-1}(\theta) = 1/\phi'\left(\phi^{-1}(\theta)\right)$, where $\phi'(\cdot)$ denotes the derivative of the function $\phi(\cdot)$, we get

$$X\left(\phi^{-1}(t)\right) = \mathrm{e}^{A\left(\phi^{-1}(t)-t\right)}X(t) + \int_{\phi(t)}^{t} \mathrm{e}^{A\left(\phi^{-1}(t)-\phi^{-1}(\theta)\right)}B\frac{U(\theta)}{\phi'\left(\phi^{-1}(\theta)\right)}d\theta. \tag{9}$$

Substituting this expression into the control law (7), we obtain the predictor feedback

$$U(t) = K\left[\mathrm{e}^{A\left(\phi^{-1}(t)-t\right)}X(t) + \int_{\phi(t)}^{t} \mathrm{e}^{A\left(\phi^{-1}(t)-\phi^{-1}(\theta)\right)}B\frac{U(\theta)}{\phi'\left(\phi^{-1}(\theta)\right)}d\theta\right], \quad \forall t \geq 0. \tag{10}$$

The division by $\phi'\left(\phi^{-1}(\theta)\right)$ safe thanks to assumption (3).

We refer to the quantity $t-\phi(t)$ as the *delay time* and to the quantity $\phi^{-1}(t)-t$ as the *prediction time*.

Remark 2.1: To make sure the above discussion is completely clear, we point out that, when the system has a constant delay, $\phi(t) = t-D$, we have $\phi^{-1}(t) = t+D$ and $\phi'\left(\phi^{-1}(\theta)\right) = 1$. Hence, controller (10) reduces to the standard predictor feedback for constant delay [1], [12]–[14].

III. Stability Analysis

In our stability analysis we will us the transport equation representation of the delay and a Lyapunov construction. First we introduce the following fairly non-obvious choice for the state of the transport equation:

$$u(x,t) = U\left(\phi\left(t + x\left(\phi^{-1}(t)-t\right)\right)\right). \tag{11}$$

This choice yields boundary values $u(0,t) = U(\phi(t))$ and $u(1,t) = U(t)$. System (1) can now be represented as

$$\dot{X}(t) = AX(t) + Bu(0,t) \tag{12}$$

$$u_t(x,t) = \pi(x,t)u_x(x,t) \tag{13}$$

$$u(1,t) = U(t) \tag{14}$$

where the speed of propagation of the transport equation is given by

$$\pi(x,t) = \frac{1 + x\left(\frac{d\left(\phi^{-1}(t)\right)}{dt} - 1\right)}{\phi^{-1}(t)-t}. \tag{15}$$

To obtain a meaningful stability result, we need the propagation speed function $\pi(x,t)$ to be strictly positive and uniformly bounded from below and from above by finite constants. Guided by the concern for boundedness from above, we examine the denominator $\phi^{-1}(t)-t$. Since we assumed that $\phi(t)$ is strictly increasing (and continuous), so is $\phi^{-1}(t)$. We also recall the assumption (2). We need to make this inequality strict, since if $\phi(t) = t$, i.e., $\phi^{-1}(t) = t$, for any t, the propagation speed is infinite at that time instant and the transport PDE representation does not make sense for the study of the stability problem. Hence, we assume the following.

Assumption 2: $\phi(t) < t$ for all $t \geq 0$ and

$$\pi_0^* = \frac{1}{\sup_{\vartheta \geq \phi^{-1}(0)}(\vartheta - \phi(\vartheta))} > 0. \tag{16}$$

Assumption 2 can be alternatively stated as $\phi^{-1}(t)-t > 0$. The implication on the delay time and the prediction time functions is that they are both positive and uniformly bounded.

Now we return to the system (12)–(14), the definition of the transport PDE state (11), and the control law (10). The control law (10) is written in terms of $u(x,t)$ as

$$u(1,t) = K\left[\mathrm{e}^{A\left(\phi^{-1}(t)-t\right)}X(t) + \int_0^1 \mathrm{e}^{A(1-y)\left(\phi^{-1}(t)-t\right)}Bu(y,t)\left(\phi^{-1}(t)-t\right)dy\right]. \tag{17}$$

In order to study exponential stability of the system $(X(t), u(x,t), x \in [0,1])$, we introduce the initial condition $u_0(x) = u(x,0) = U\left(\phi\left(\phi^{-1}(0)x\right)\right)$, $x \in [0,1]$, and $X_0 = X(0)$. Now we establish the following stability result.

Theorem 1: Consider the closed-loop system consisting of the plant (12)–(14) and the controller (17) and let Assumptions 1 and 2 hold. There exists a positive constants G, and a positive constant g independent of the function $\phi(\cdot)$, such that

$$|X(t)|^2 + \|u(t)\|^2 \leq G\mathrm{e}^{-gt}\left(|X_0|^2 + \|u_0\|^2\right), \quad \forall t \geq 0. \tag{18}$$

Proof: Consider the transformation of the transport PDE state given by

$$w(x,t) = u(x,t) - K\mathrm{e}^{Ax\left(\phi^{-1}(t)-t\right)}X(t) - K\int_0^x \mathrm{e}^{A(x-y)\left(\phi^{-1}(t)-t\right)}Bu(y,t)\left(\phi^{-1}(t)-t\right)dy. \tag{19}$$

Taking the derivatives of $w(x,t)$ with respect to t and x we get

$$\begin{aligned} w_t(x,t) = {} & u_t(x,t) - KAx\left(\frac{d\left(\phi^{-1}(t)\right)}{dt} - 1\right)\mathrm{e}^{Ax\left(\phi^{-1}(t)-t\right)}X(t) \\ & - K\mathrm{e}^{Ax\left(\phi^{-1}(t)-t\right)}\left(X(t) + Bu(0,t)\right) \\ & - K\int_0^x \mathrm{e}^{A(x-y)\left(\phi^{-1}(t)-t\right)}\left(A(x-y)\left(\phi^{-1}(t)-t\right) + I\right) \\ & \times \left(\frac{d\left(\phi^{-1}(t)\right)}{dt} - 1\right)Bu(y,t)dy \end{aligned}$$

$$
\begin{aligned}
&- K\int_0^x \mathrm{e}^{A(x-y)\left(\phi^{-1}(t)-t\right)}Bu_t(y,t)\left(\phi^{-1}(t)-t\right)dy\\
&= u_t(x,t) - \left(1 + x\left(\frac{d\left(\phi^{-1}(t)\right)}{dt}-1\right)\right)\\
&\times K\Big[A\mathrm{e}^{Ax\left(\phi^{-1}(t)-t\right)}X(t)\\
&+A\int_0^x \mathrm{e}^{A(x-y)\left(\phi^{-1}(t)-t\right)}Bu(y,t)\left(\phi^{-1}(t)-t\right)dy\\
&+Bu(x,t)\Big]
\end{aligned}
\tag{20}
$$

where we have used integration by parts, and

$$
\begin{aligned}
w_x(x,t) = u_x(x,t) - \left(\phi^{-1}(t)-t\right)K\Bigg[&A\mathrm{e}^{Ax\left(\phi^{-1}(t)-t\right)}X(t)\\
&+A\int_0^x \mathrm{e}^{A(x-y)\left(\phi^{-1}(t)-t\right)}Bu(y,t)\left(\phi^{-1}(t)-t\right)dy\\
&+Bu(x,t)\Bigg].
\end{aligned}
\tag{21}
$$

With the help of (17) we also obtain $w(1,t)=0$ and hence we arrive at the "target system"

$$\dot{X}(t) = (A+BK)X(t) + Bw(0,t) \tag{22}$$
$$w_t(x,t) = \pi(x,t)w_x(x,t) \tag{23}$$
$$w(1,t) = 0. \tag{24}$$

This is a cascade configuration $w \to X$. We focus first on the Lyapunov analysis of the w-subsystem. We take a Lyapunov function

$$L(t) = \frac{1}{2}\int_0^1 \mathrm{e}^{bx}w^2(x,t)dx \tag{25}$$

where b is any positive constant. The time derivative of $L(t)$ is

$$
\begin{aligned}
\dot{L}(t) &= \int_0^1 \mathrm{e}^{bx}w(x,t)w_t(x,t)dx\\
&= \int_0^1 \mathrm{e}^{bx}w(x,t)\pi(x,t)w_x(x,t)dx\\
&= \frac{1}{2}\int_0^1 \mathrm{e}^{bx}\pi(x,t)dw^2(x,t)\\
&= \frac{\mathrm{e}^{bx}}{2}\pi(x,t)w^2(x,t)\Big|_0^1\\
&\quad - \frac{1}{2}\int_0^1 \left(b\pi(x,t)+\pi_x(x,t)\right)\mathrm{e}^{bx}w^2(x,t)dx\\
&= -\frac{\pi(0,t)}{2}w^2(0,t)\\
&\quad - \frac{1}{2}\int_0^1 \left(b\pi(x,t)+\pi_x(x,t)\right)\mathrm{e}^{bx}w^2(x,t)dx.
\end{aligned}
\tag{26}
$$

Noting that $\pi(0,t) = 1/\phi^{-1}(t)-t \geq \pi_0^*$, we get

$$
\begin{aligned}
\dot{L}(t) \leq &-\frac{\pi_0^*}{2}w^2(0,t)\\
&-\frac{1}{2}\int_0^1 \left(b\pi(x,t)+\pi_x(x,t)\right)\mathrm{e}^{bx}w^2(x,t)dx.
\end{aligned}
\tag{27}
$$

Next, we observe that

$$\pi_x(x,t) = \frac{\frac{d\left(\phi^{-1}(t)\right)}{dt}-1}{\phi^{-1}(t)-t} \tag{28}$$

is a function of t only. Hence

$$
\begin{aligned}
&b\pi(x,t)+\pi_x(x,t)\\
&\quad= \frac{b\left[1+x\left(\frac{d\left(\phi^{-1}(t)\right)}{dt}-1\right)\right]+\frac{d\left(\phi^{-1}(t)\right)}{dt}-1}{\phi^{-1}(t)-t}\\
&\quad= \frac{b-1+\frac{d\left(\phi^{-1}(t)\right)}{dt}+bx\left(\frac{d\left(\phi^{-1}(t)\right)}{dt}-1\right)}{\phi^{-1}(t)-t}.
\end{aligned}
\tag{29}
$$

Since this is a linear function of x, it follows that it has a minimum either at $x=0$ or $x=1$, so we get

$$
\begin{aligned}
&b\pi(x,t)+\pi_x(x,t)\\
&\quad\geq \frac{\min\left\{b-1+\frac{d\left(\phi^{-1}(t)\right)}{dt},\ (b+1)\frac{d\left(\phi^{-1}(t)\right)}{dt}-1\right\}}{\phi^{-1}(t)-t}.
\end{aligned}
\tag{30}
$$

Next we note that $d\left(\phi^{-1}(t)\right)/dt = 1/\phi'\left(\phi^{-1}(t)\right) \geq 1/\sup_{\vartheta\geq\phi^{-1}(0)}\phi'(\vartheta) = \pi_1^*$, which yields

$$b\pi(x,t)+\pi_x(x,t) \geq \frac{\min\left\{b-1+\pi_1^*,\ (b+1)\pi_1^*-1\right\}}{\phi^{-1}(t)-t}. \tag{31}$$

Choosing $b \geq (1-\pi_1^*)\max\{1,\ 1/\pi_1^*\}$, we get $b\pi(x,t)+\pi_x(x,t) \geq \pi_0^*\beta^*$, where $\beta^* = \min\{b-1+\pi_1^*,\ (b+1)\pi_1^*-1\} > 0$. So, returning to $\dot{L}(t)$, we have that

$$\dot{L}(t) \leq -\frac{\pi_0^*}{2}w^2(0,t) - \pi_0^*\beta^*L(t). \tag{32}$$

Let us now turn our attention to the X-subsystem. We have

$$\frac{d}{dt}\left(X(t)^TPX(t)\right) = -X^T(t)QX(t)+2X^T(t)PBw(0,t) \tag{33}$$

where P satisfies a Lyapunov equation $P(A+BK)+(A+BK)^TP = -Q$. With a usual completion of squares, we get

$$\frac{d}{dt}(X(t)^TPX(t)) \leq -\lambda_{\min}(Q)|X(t)|^2 + \frac{2|PB|^2}{\lambda_{\min}(Q)}w^2(0,t). \tag{34}$$

Now we take the Lyapunov functional

$$V(t) = X(t)^TPX(t) + \frac{4|PB|^2}{\pi_0^*\lambda_{\min}(Q)}L(t). \tag{35}$$

Its derivative is

$$\dot{V}(t) \leq -\frac{\lambda_{\min}(Q)}{2}|X(t)|^2 - \pi_0^*\beta^*\frac{4|PB|^2}{\pi_0^*\lambda_{\min}(Q)}L(t). \tag{36}$$

Finally, with the definition of $V(t)$ we get $\dot{V}(t) \leq -\mu V(t)$, where

$$\mu = \min\left\{\pi_0^*\beta^*,\ \frac{\lambda_{\min}(Q)}{2\lambda_{\max}(P)}\right\}. \tag{37}$$

Thus we obtain $V(t) \leq \mathrm{e}^{-\mu t}V(0)$ for all $t \geq 0$. Let us now denote $\Omega(t) = |X(t)|^2 + \int_0^1 w^2(x,t)dx$. We show that $\psi_1\Omega(t) \leq V(t) \leq \psi_2\Omega(t)$, where

$$\psi_1 = \min\left\{\lambda_{\min}(P), \frac{2|PB|^2}{\pi_0^*\lambda_{\min}(Q)}\right\} \tag{38}$$
$$\psi_2 = \max\left\{\lambda_{\max}(P), \frac{2|PB|^2}{\pi_0^*\lambda_{\min}(Q)}\mathrm{e}^b\right\}. \tag{39}$$

It then follows that $\Omega(t) \leq (\psi_2/\psi_1)\mathrm{e}^{-\mu t}\Omega(0)$ for all $t \geq 0$. Now we consider the norm $\Xi(t) = |X(t)|^2 + \int_0^1 u^2(x,t)dx$. We recall the backstepping transformation (19) and introduce its inverse

$$\begin{aligned}u(x,t) &= w(x,t) + K\mathrm{e}^{(A+BK)x(\phi^{-1}(t)-t)}X(t)\\ &+ K\int_0^x \mathrm{e}^{(A+BK)(x-y)(\phi^{-1}(t)-t)}Bu(y,t)\left(\phi^{-1}(t)-t\right)dy.\end{aligned} \tag{40}$$

It can be show that $\|w(t)\|^2 \leq \alpha_1(t)\|u(t)\|^2 + \alpha_2|X(t)|^2$ and $\|u(t)\|^2 \leq \beta_1(t)\|w(t)\|^2 + \beta_2|X(t)|^2$, where

$$\alpha_1(t) = 3\left(1+\int_0^1 \left(KM\left(x\left(\phi^{-1}(t)-t\right)\right)B\left(\phi^{-1}(t)-t\right)\right)^2 dx\right) \tag{41}$$

$$\alpha_2(t) = 3\int_0^1 \left|KM\left(x\left(\phi^{-1}(t)-t\right)\right)\right|^2 dx \tag{42}$$

$$\beta_1(t) = 3\left(1+\int_0^1 \left(KN\left(x\left(\phi^{-1}(t)-t\right)\right)B\left(\phi^{-1}(t)-t\right)\right)^2 dx\right) \tag{43}$$

$$\beta_2(t) = 3\int_0^1 \left|KN\left(x\left(\phi^{-1}(t)-t\right)\right)\right|^2 dx \tag{44}$$

and where $M(s) = \mathrm{e}^{As}$ and $N(s) = \mathrm{e}^{(A+BK)s}$. Furthermore, we can show that

$$\alpha_1(t) \leq \bar{\alpha}_1 = 3\left(1+|K|^2|B|^2\frac{\mathrm{e}^{2|A|/\pi_0^*}-1}{2\pi_0^*|A|}\right) \tag{45}$$

$$\alpha_2(t) \leq \bar{\alpha}_2 = 3|K|^2\pi_0^*\frac{\mathrm{e}^{2|A|/\pi_0^*}-1}{2|A|} \tag{46}$$

$$\beta_1(t) \leq \bar{\beta}_1 = 3\left(1+|K|^2|B|^2\frac{\mathrm{e}^{2|A+BK|/\pi_0^*}-1}{2\pi_0^*|A+BK|}\right) \tag{47}$$

$$\beta_2(t) \leq \bar{\beta}_2 = 3|K|^2\pi_0^*\frac{\mathrm{e}^{2|A+BK|/\pi_0^*}-1}{2|A+BK|}. \tag{48}$$

With a few substitutions we obtain that $\phi_1\Xi(t) \leq \Omega(t) \leq \phi_2\Xi(t)$, where $\phi_1 = 1/\max\left\{\bar{\beta}_1, \bar{\beta}_2+1\right\}$ and $\phi_2 = \max\left\{\bar{\alpha}_1, \bar{\alpha}_2+1\right\}$. Finally, we get $\Xi(t) \leq \phi_2\psi_2/\phi_1\psi_1\mathrm{e}^{-\mu t}\Xi(0), \quad \forall t \geq 0$, which completes the proof of the theorem with $G = \phi_2\psi_2/\phi_1\psi_1$ and with $g = \mu$. By choosing $\beta^* \geq \lambda_{\min}(Q)/2\pi_0^*\lambda_{\max}(P)$, i.e., by picking b positive and such that

$$b \geq \left(1-\pi_1^*+\frac{\lambda_{\min}(Q)}{2\pi_0^*\lambda_{\max}(P)}\right)\max\left\{1,\ \frac{1}{\pi_1^*}\right\} \tag{49}$$

we get $g = \lambda_{\min}(Q)/2\lambda_{\max}(P)$, so g is independent of $\phi(\cdot)$. ∎

While Theorem 1 provides a stability result in terms of the system norm $|X(t)|^2 + \int_0^1 u^2(x,t)dx$, we would like to also get a stability result in terms of the norm $|X(t)|^2 + \int_{\phi(t)}^t U^2(\theta)d\theta$. Towards that end, we first observe that

$$\int_{\phi(t)}^t U^2(\theta)d\theta = \left(\phi^{-1}(t)-t\right) \cdot \int_0^1 \phi'\left(t+x\left(\phi^{-1}(t)-t\right)\right)u^2(x,t)dx \tag{50}$$

$$\int_0^2 u_0^2(x)dx = \frac{1}{\phi^{-1}(0)}\int_{\phi(0)}^0 \frac{1}{\phi'\left(\phi^{-1}(\theta)\right)}U^2(\theta)d\theta. \tag{51}$$

With these identities and Theorem 1 we obtain the following.

Theorem 2: Consider the closed-loop system consisting of the plant (12)–(14) and the controller (17) and let Assumptions 1 and 2 hold. There exist positive constants G and g (the latter one being independent of ϕ) such that

$$|X(t)|^2+\int_{\phi(t)}^t U^2(\theta)d\theta \leq hG\mathrm{e}^{-gt}\left(|X_0|^2+\int_{\phi(0)}^0 U^2(\theta)d\theta\right) \tag{52}$$

for all $t \geq 0$, where G is as in the proof of Theorem 1 and

$$h = \frac{\sup_{\tau\geq 0}\phi'(\tau)}{\pi_0^*\phi^{-1}(0)\inf_{\tau\in\left[0,\phi^{-1}(0)\right]}\phi'(\tau)}. \tag{53}$$

IV. Observer Design With Time-Varying Sensor Delay

We give a brief presentation of an observer design for an LTI systems with a time-varying sensor delay:

$$\dot{X}(t) = AX(t) + BU(t) \tag{54}$$

$$Y(t) = CX(\phi(t)). \tag{55}$$

We approach the observer design in a two-step manner:

- Design an observer for the delay state $X(\phi(t))$ since the output $Y(t) = CX(\phi(t))$ is delayed.
- Use a model-based predictor to advance the estimate of $X(\phi(t))$ by the delay time $t-\phi(t)$.

We start by writing (54) as $dX(\phi(t))/d\phi(t) = AX(\phi(t)) + BU(\phi(t))$. Then we introduce a state estimator $\Sigma(\phi(t))$ for $X(\phi(t))$ as $d\Sigma(\phi(t))/d\phi(t) = A\Sigma(\phi(t)) + BU(\phi(t)) + L\left(Y(t) - C\Sigma(\phi(t))\right)$, where L is selected so that the matrix $A-LC$ is Hurwitz, i.e., so that the system $d\zeta(\phi(t))/d\phi(t) = (A-LC)\zeta(\phi(t))$, where $\zeta(\phi(t)) = X(\phi(t)) - \Sigma(\phi(t))$, is exponentially stable in the time variable $\phi(t)$. Let us now denote $\xi(t) = \Sigma(\phi(t))$. This variable is governed by the differential equation

$$\begin{aligned}\dot{\xi}(t) &= \frac{d\Sigma(\phi(t))}{d\phi(t)}\phi'(t)\\ &= \phi'(t)\left[A\Sigma(\phi(t)) + BU(\phi(t)) + L(Y(t) - C\Sigma(\phi(t)))\right]\\ &= \phi'(t)\left[A\xi(t) + BU(\phi(t)) + L(Y(t) - C\xi(t))\right].\end{aligned} \tag{56}$$

Now we take $\xi(t) = \Sigma(\phi(t))$, which is an estimate of the past state $X(\phi(t))$, and advance it by the delay time $t-\phi(t)$, obtaining the estimate of the current state $X(t)$ as $\hat{X}(t) = \mathrm{e}^{A(t-\phi(t))}\xi(t) + \int_{\phi(t)}^t \mathrm{e}^{A(t-\tau)}BU(\tau)d\tau$.

To summarize, the observer is given by the equations

$$\dot{\xi}(t) = \phi'(t)\left[A\xi(t) + BU(\phi(t)) + L(Y(t) - C\xi(t))\right] \tag{57}$$

$$\hat{X}(t) = \mathrm{e}^{A(t-\phi(t))}\xi(t) + \int_{\phi(t)}^t \mathrm{e}^{A(t-\tau)}BU(\tau)d\tau. \tag{58}$$

This observer has a structure that displays duality with respect to the predictor-based controller (10) in two interesting ways:

- While controller (10) employs prediction over the future period $\left[t,\phi^{-1}(t)\right]$, the observer (57)–(58) employs prediction over the past period $[\phi(t), t]$.
- While controller (10) involves a time derivative of $\phi^{-1}(t)$, the observer (57)–(58) involves a time derivative of $\phi(t)$.

In the case of a constant sensor delay, $\phi(t) = t - D$, the observer (57)–(58) reduces to [9], [12], [25].

V. Examples

The first two examples violate some of the assumptions of the theory but are valuable in illustrating the design principle. The other two examples fit the assumptions.

Example 5.1: (Linearly growing delay.) We consider $\phi_1(t) = t/2$, which means that the delay time is linearly growing and is unbounded.

(In addition, the assumption that the delay is strictly positive for all time is violated, but this assumption is less essential.) So, the predictor feedback (10) assumes the form

$$U(t) = K\left[e^{At}X(t) + \int_{t/2}^{t} 2e^{2A(t-\theta)}BU(\theta)d\theta\right]. \quad (59)$$

It is interesting to observe that this system has zero dead time, since the initial delay is zero and the controller continues to compensate the delay for $t \geq 0$. The control signal is $U(t) = KX(2t)$ and, since $X(t) = e^{(A+BK)t}X_0$, the control signal remains a bounded function $U(t) = Ke^{(A+BK)2t}X_0$ in spite of the delay growing unbounded. This is potentially confusing as some difficulty should arise in a system where the delay is growing unbounded. The difficulty manifests itself when the system is subject to a disturbance or modeling error. In that case the control signal will not be given by $U(t) = Ke^{(A+BK)2t}X_0$ but will be governed by the feedback law (59). In this feedback law the gains grow exponentially with time. In the presence of a persistent disturbance, which prevents $X(t)$ from settling, the control signal will grow unbounded as its gains grow unbounded.

Example 5.2: (Prediction time grows exponentially.) We consider $\phi_2(t) = \ln(1+t)$. In this case

$$U(t)=K\left[e^{A\left(e^{t}-1-t\right)}X(t)+\int_{\ln(1+t)}^{t} e^{-\theta}e^{A\left(e^{t}-e^{\theta}\right)}BU(\theta)d\theta\right]. \quad (60)$$

The gain growth in t is more pronounced than in Example 5.2—the gains grow as an exponential of an exponential.

Example 5.3: (Bounded delay with a constant limit.) We consider $\phi_3(t) = t - (1+t)/(1+2t)$, where the prediction time is

$$\phi_3^{-1}(t) - t = \frac{-t + \sqrt{(t+1)^2+1}}{2} = \frac{t+1}{\sqrt{(t+1)^2+1}+t}.$$

The initial value of the prediction time is $\sqrt{2}/2$, the final value is $1/2$, and the uniform bound on the prediction time is $\sqrt{2}/2$. Furthermore, the uniform bound on the quantity $d\left(\phi_3^{-1}(t)\right)/dt = (1/2)\left(1 + 1/\sqrt{1+1/(1+t)^2}\right)$ is 1. Hence the feedback law

$$U(t) = K\left[e^{A\left(\phi_3^{-1}(t)-t\right)}X(t) + \int_{\phi_3(t)}^{t} \frac{d\left(\phi_3^{-1}(\theta)\right)}{d\theta}e^{A\left(\phi_3^{-1}(t)-\phi_3^{-1}(\theta)\right)}BU(\theta)d\theta\right]$$

employs bounded gains and achieves exponential stability (it also achieves a finite disturbance-to-state gain).

Example 5.4: (Bounded delay function without a limit.) In the past three examples the function $\phi'(t)$ was monotonic and it had a limit (in two of the three examples the function $\phi(t)$ itself also had a limit). Now we consider an example where $\phi(t)$ is oscillatory. Let $\rho(t) = t + 1 + (1/2)\cos t$ and denote $\phi_4(t) = \rho^{-1}(t)$. So, the gains of the predictor feedback

$$U(t) = K\left[e^{A(1+(1/2)\cos t)}X(t) + \int_{\rho^{-1}(t)}^{t}\left(1 - \frac{1}{2}\sin\theta\right)e^{A(t+(1/2)\cos t-\theta-(1/2)\cos\theta)}BU(\theta)d\theta\right] \quad (61)$$

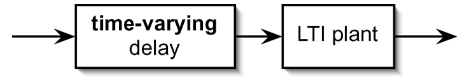

Fig. 1. Linear system $\dot{X}(t) = AX(t) + BU(\phi(t))$ with time-varying actuator delay $\delta(t) = t - \phi(t)$.

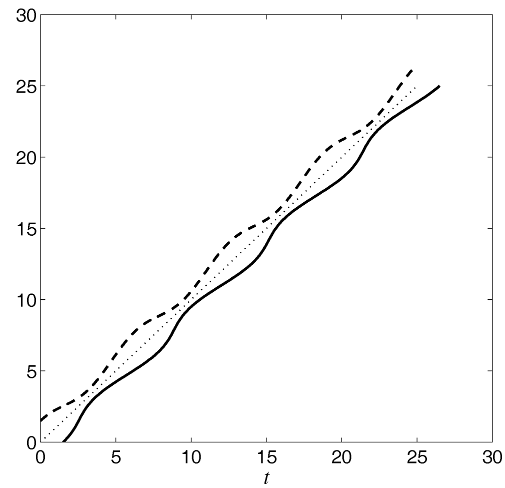

Fig. 2. Oscillating delay function in Example 5.4. Solid: $\phi_4(t)$, dashed: $\rho(t)$.

are uniformly bounded. Now we consider a specific first-order example

$$\dot{X}(t) = X(t) + U\left(\rho^{-1}(t)\right) \quad (62)$$

namely, $A = B = 1$. In closed loop with the control law

$$U(t) = -(1+c)\left[e^{1+(1/2)\cos t}X(t) + \int_{\rho^{-1}(t)}^{t}\left(1 - \frac{1}{2}\sin\theta\right)e^{t+(1/2)\cos t-\theta-(1/2)\cos\theta}U(\theta)d\theta\right]$$

where $c > 0$, the plant (62) has an explicit solution

$$X(t) = X_0\begin{cases} e^{t}, & t \in [0, \rho(0)) \\ e^{-c(t-\rho(0))+\rho(0)}, & t \geq \rho(0)|. \end{cases} \quad (63)$$

The explicit form of the control signal is $U(t) = -(1+c)e^{\rho(0)-c\rho(t)}X_0, \quad \forall t \geq 0$ (see Fig. 1). The explicit formulae for both $X(t)$ and $U(t)$ require $\rho(0)$, which is given by $\rho(0) = 3/2$. Figs. 2 and 3 show the graphs of the delay, state, and control functions. The gain is chosen as $c = 0.13$ to achieve visual clarity about the LTV character of the overall system, particularly about the response of $U(t)$, which has a "wavy" character to achieve compensation of the oscillating delay function.

Example 5.5: (Observer.) We illustrate the observer design (57), (58) for a second order system with $A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$, $B = 0, C = [1 \ \ 0]$, and $L = [1 \ \ 0]^T$. The resulting observer is

$$\dot{\xi}_1(t) = \phi'(t)\left(-\xi_1(t) + \xi_2(t) + Y(t)\right) \quad (64)$$

$$\dot{\xi}_2(t) = -\phi'(t)\xi_1(t) \quad (65)$$

$$\hat{X}_1(t) = \xi_1(t)\cos(t-\phi(t)) + \xi_2(t)\sin(t-\phi(t)) \quad (66)$$

$$\hat{X}_2(t) = -\xi_1(t)\sin(t-\phi(t)) + \xi_2(t)\cos(t-\phi(t)). \quad (67)$$

VI. Conclusion

For predictor feedback for LTI systems with a time-varying delay, we have proved exponential stability under several conditions on the delay function $\delta(t) = t - \phi(t)$:

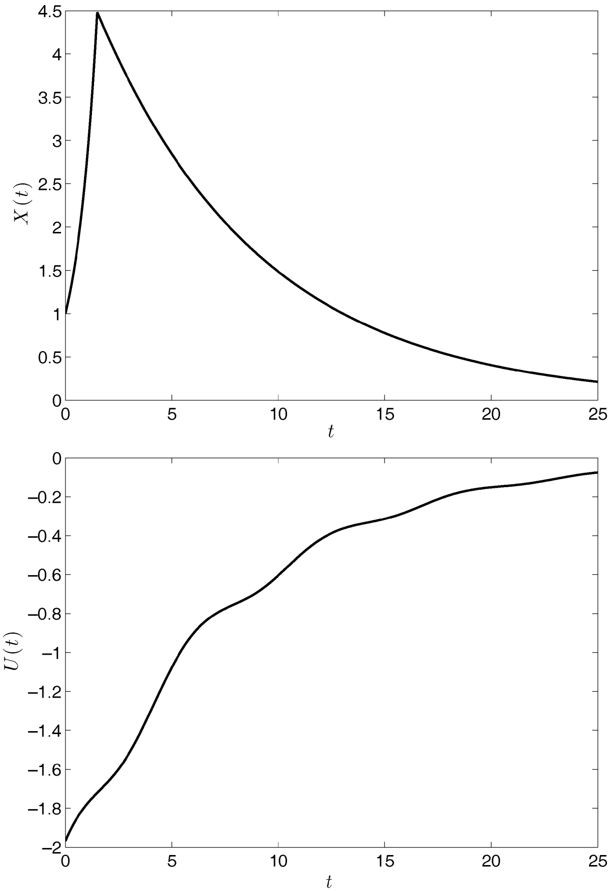

Fig. 3. State and control signal evolution for Example 5.4. The control "kicks in" at $\rho(0) = 3/2$ sec. The "waviness" in the control $U(t)$ is for the purpose of compensation of the time-varying (oscillating) delay.

- the delay function $\delta(t)$ is strictly positive (technical condition which ensures that the state space of the input dynamics can be defined);
- the delay function $\delta(t)$ is uniformly bounded from above;
- the delay rate function, $\delta'(t)$, is strictly smaller than 1, i.e., the delay may *increase* at a rate smaller than 1;
- the delay rate function $\delta'(t)$ is uniformly bounded from below (by a possibly negative finite constant), i.e., the delay may *decrease* at a uniformly bounded rate.

These four conditions need to be satisfied simultaneously but they are not restrictive and they have two natural implications on the growth and decrease of the delay. First, the delay can *grow* at a rate strictly smaller than 1 but not indefinitely, because the delay must remain uniformly bounded. Second, the delay may *decrease* at any uniformly bounded rate but not indefinitely, because the delay must remain positive.

It may be somewhat disappointing, though inherent in the problem, that the delay function $\delta(t) = t - \phi(t)$ needs to be known sufficiently far in advance in order to be able to compute $\phi^{-1}(t)$, which is needed in the controller (10). If the delay is bounded by $\bar{D}$, it is sufficient that the function $\phi^{-1}(t)$ be know $\bar{D}$ seconds in advance. For instance, in Example 5.4 and in Fig. 2, $\bar{D} = 1.5$. An approximate real-time computation of $\phi^{-1}(t)$ can be conducted using the differential equation $\epsilon\dot{\gamma}(t) = -\phi(\gamma(t)) + t$, where $0 < \epsilon \ll 1$ and $\gamma(0) > 0$. This idea is based on the singular perturbation approach and will result in a stable mainainance of $\gamma(t) \approx \phi^{-1}(t)$ thanks to the boundary layer being exponentially stable since $\phi'(\cdot) > 0$.

A small error in the knowledge of $\phi^{-1}(t)$ can be tolerated (the error needs to be sufficiently small and sufficiently slow). This robustness result is provable due to the fact that the nominal closed-loop system under predictor feedback is exponentially stable. The topology of the system in the robustness proof involves an H_1 norm of $U(t)$, rather than the L_2 norm.

Another approach to studying stability in the presence of delays is the invariance principle [23, Theorem IV.4.2]. However, with the approach we pursue, which involves a strict Lyapunov functional and explicit norm estimates, we avoid a separate study of orbital precompactness [23, Theorem IV.5.2].

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