Systems & Control Letters 58 (2009) 617–623

Contents lists available at ScienceDirect

Systems & Control Letters

journal homepage: www.elsevier.com/locate/sysconle

Boundary control of an anti-stable wave equation with anti-damping on the uncontrolled boundary

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ARTICLE INFO

Article history:
Received 26 September 2008
Received in revised form
6 April 2009
Accepted 26 April 2009
Available online 17 May 2009

Keywords:
Distributed parameter systems
Backstepping
Boundary control
Wave equation

ABSTRACT

Much of the boundary control of wave equations in one dimension is based on a single principle—passivity—under the assumption that control is applied through Neumann actuation on one boundary and the other boundary satisfies a homogeneous Dirichlet boundary condition. We have recently expanded the scope of tractable problems by allowing destabilizing anti-stiffness (a Robin type condition) on the uncontrolled boundary, where the uncontrolled system has a finite number of positive real eigenvalues. In this paper we go further and develop a methodology for the case where the uncontrolled boundary condition has anti-damping, which makes the real parts of all the eigenvalues of the uncontrolled system positive and arbitrarily high, i.e., the system is "anti-stable" (exponentially stable in negative time). Using a conceptually novel integral transformation, we obtain extremely simple, explicit formulae for the gain functions. For the case with only boundary sensing available (at the same end with actuation), we design backstepping observers which are dual to the backstepping controllers and have explicit output injection gains. We then combine the control and observer designs into an output-feedback compensator and prove the exponential stability of the closed-loop system.

1. Introduction

We consider the problem of stabilization of a one-dimensional wave equation which is controlled from one end and contains instability at the other (free) end. The nature of instability (negative damping) is such that all of the open-loop eigenvalues are located on the right-hand side of the complex plane, thus the system is not just unstable, it is anti-stable.

Our control design is based on the method of "backstepping" [1–3], which results in explicit formulae for the gain functions. In a recent paper [3], the backstepping method was used to design controllers and observers for an unstable wave equation with destabilizing boundary condition at the free end. However, in that paper the destabilizing term was proportional to displacement, while in this paper it is proportional to velocity, in the form of "anti-damper", which results in all eigenvalues being unstable instead of just a few. The concept of a boundary anti-damper is not of huge physical relevance, however, the design that we develop for this anti-stable system is a methodological breakthrough in boundary control of wave equations. We introduce a new integral transformation that makes the closed-loop system dynamically behave as a wave equation with well-known "passive damper" controller [4–10]. The previous results for wave equation with nonpositive damping include [11], where a distributed damping of indefinite sign is considered, and [12], where one end of the string is damped and anti-damped periodically in time.

For the case when only boundary sensing is available (at the same end with actuation), we design backstepping observers which are dual to the backstepping controllers and have explicit output injection gains. Both setups are considered: Neumann actuation/Dirichlet sensing and Dirichlet actuation/Neumann sensing. We then combine the control and observer designs into dynamic compensators and prove the exponential stability of the closed-loop system.

2. Problem formulation

We consider a one-dimensional wave equation

$$u_{tt}(x,t) = u_{xx}(x,t), \quad 0 < x < 1,\ t > 0 \tag{1}$$

$$u_x(0,t) = -qu_t(0,t), \quad t > 0 \tag{2}$$

$$u_x(1,t) = U(t), \quad t > 0, \tag{3}$$

where $U(t)$ is the control input and $q \neq \pm 1$ is a constant parameter. For $q = 0$, Eqs. (1)–(3) model a string which is free at the end $x = 0$ and is actuated at the opposite end. For $q > 0$ the free end of the string is negatively damped, with all eigenvalues located on the right-hand side of the complex plane (hence the open-loop system is "anti-stable").

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doi:10.1016/j.sysconle.2009.04.005

The objective is to exponentially stabilize the system (1)–(3) around the zero equilibrium. The case of Dirichlet actuation is considered in Section 6.

3. Control design

Consider the transformation

$$w(x,t) = u(x,t) - \int_0^x k(x,y)u(y,t)\,dy - \int_0^x s(x,y)u_t(y,t)\,dy - \int_0^x m(x,y)u_x(y,t)\,dy, \tag{4}$$

where the gains $k(x,y)$, $s(x,y)$, and $m(x,y)$ are to be determined. We want to map the system (1)–(3) into the following target system

$$w_{tt}(x,t) = w_{xx}(x,t), \quad 0 < x < 1, t > 0 \tag{5}$$

$$w_x(0,t) = cw_t(0,t), \quad t > 0 \tag{6}$$

$$w_x(1,t) = -c_0 w(1,t), \quad t > 0, \tag{7}$$

which is exponentially stable for $c > 0$ and $c_0 > 0$ (see, e.g., [3]). As will be shown later, the transformation (4) is invertible in a certain norm, so that stability of the target system ensures the stability of the closed-loop system.

Compared with the backstepping transformations for parabolic PDEs, there are two additional terms in (4)—the second and the third integrals in (4). The term with u_t is natural because hyperbolic systems are second order in time and therefore the state variable is (u, u_t) instead of just u. The need for the term with u_x is less obvious and it is in fact the main conceptual novelty of the paper.

Substituting (4) into (5)–(7) we obtain

$$\begin{aligned} 0 &= w_{tt}(x,t) - w_{xx}(x,t) \\ &= u(x,t)\left(\frac{d}{dx}k(x,x) + k_x(x,x)\right) \\ &\quad + u_t(x,t)\left(\frac{d}{dx}s(x,x) + s_x(x,x)\right) \\ &\quad + u_x(x,t)\left(\frac{d}{dx}m(x,x) + m_x(x,x)\right) + \int_0^x k_{xx}(x,y)u(y,t)\,dy \\ &\quad - \int_0^x k(x,y)u_{yy}(y,t)\,dy + \int_0^x s_{xx}(x,y)u_t(y,t)\,dy \\ &\quad - \int_0^x s(x,y)u_{tyy}(y,t)\,dy \\ &\quad + \int_0^x m_{xx}(x,y)u_y(y,t)\,dy - \int_0^x m(x,y)u_{yyy}(y,t)\,dy \\ &\quad - k(x,x)u(x,t) - s(x,x)u_t(x,t) - m(x,x)u_x(x,t). \end{aligned} \tag{8}$$

Using integration by parts (twice) in the appropriate integrals and the identity $k_x(x,x) + k_y(x,x) = (d/dx)k(x,x)$, we obtain

$$\begin{aligned} 0 &= 2u(x,t)\frac{d}{dx}k(x,x) + 2u_t(x,t)\frac{d}{dx}s(x,x) \\ &\quad + 2u_x(x,t)\frac{d}{dx}m(x,x) + \int_0^x (k_{xx}(x,y) - k_{yy}(x,y))u(y,t)\,dy \\ &\quad + \int_0^x (s_{xx}(x,y) - s_{yy}(x,y))u_t(y,t)\,dy \\ &\quad + \int_0^x (m_{xx}(x,y) - m_{yy}(x,y))u_y(y,t)\,dy \\ &\quad - k_y(x,0)u(0,t) + [qm_y(x,0) - s_y(x,0) - qk(x,0)]u_t(0,t) \\ &\quad + [m(x,0) - qs(x,0)]u_{xx}(0,t). \end{aligned} \tag{9}$$

Matching all the terms we get the following PDE for $k(x,y)$:

$$k_{xx}(x,y) = k_{yy}(x,y), \quad 0 < y < x < 1, \tag{10}$$

$$k_y(x,0) = 0, \quad x \in [0,1), \tag{11}$$

$$\frac{d}{dx}k(x,x) = 0, \quad x \in [0,1), \tag{12}$$

and two coupled PDEs for $s(x,y)$ and $m(x,y)$:

$$s_{xx}(x,y) = s_{yy}(x,y), \quad 0 < y < x < 1, \tag{13}$$

$$s_y(x,0) = qm_y(x,0) - qk(x,0), \quad x \in [0,1), \tag{14}$$

$$\frac{d}{dx}s(x,x) = 0, \quad x \in [0,1) \tag{15}$$

and

$$m_{xx}(x,y) = m_{yy}(x,y), \quad 0 < y < x < 1, \tag{16}$$

$$m(x,0) = qs(x,0), \quad x \in [0,1), \tag{17}$$

$$\frac{d}{dx}m(x,x) = 0, \quad x \in [0,1). \tag{18}$$

Substituting (4) into the boundary condition (6) we get

$$\begin{aligned} 0 &= w_x(0,t) - cw_t(0,t) \\ &= [qm(0,0) - s(0,0) - q - c]u_t(0,t) - k(0,0)u(0,t), \end{aligned} \tag{19}$$

which gives two more conditions:

$$k(0,0) = 0, \tag{20}$$

$$qm(0,0) = s(0,0) + q + c. \tag{21}$$

The solution to (10)–(12), (20) is simply $k(x,y) \equiv 0$. This is a consequence of not changing a stiffness of the system (i.e., if we were to add the term proportional to $w(0,t)$ to the boundary condition at $x = 0$, the gain $k(x,y)$ would not be zero). To solve the PDEs for s and m, we note that a general solution to (13) and (15) is $s(x,y) = \phi(x-y)$ and similarly for (16), (18) we have $m(x,y) = \psi(x-y)$ for arbitrary functions ϕ and ψ. Using (14) and (17) we obtain

$$\phi'(x) = q\psi'(x), \tag{22}$$

$$\psi(x) = q\phi(x). \tag{23}$$

Integrating (22) from 0 to x and using (23), we obtain

$$(q^2 - 1)(\phi(x) - \phi(0)) = 0. \tag{24}$$

Since $q^2 \neq 1$, we get that both $\phi(x)$ and $\psi(x)$ are constant functions of x. Finally, using the relationship (21) together with (23), we get

$$s(x,y) \equiv \frac{q+c}{q^2-1}, \tag{25}$$

$$m(x,y) \equiv \frac{q(q+c)}{q^2-1}. \tag{26}$$

The transformation (4) can therefore be written in one of the two forms, either as

$$w(x,t) = u(x,t) - \frac{q(q+c)}{q^2-1}\int_0^x u_x(y,t)\,dy - \frac{q+c}{q^2-1}\int_0^x u_t(y,t)\,dy, \tag{27}$$

or as

$$w(x,t) = -\frac{1+qc}{q^2-1}u(x,t) + \frac{q(q+c)}{q^2-1}u(0,t) - \frac{q+c}{q^2-1}\int_0^x u_t(y,t)\,dy. \tag{28}$$

Differentiating (28) with respect to x, setting $x = 1$, and using the boundary condition (7), we obtain the following controller

$$U(t) = \frac{c_0 q(q+c)}{1+qc}u(0,t) - c_0 u(1,t) - \frac{q+c}{1+qc}u_t(1,t) - \frac{c_0(q+c)}{1+qc}\int_0^1 u_t(y,t)\,dy. \tag{29}$$

Note that this controller is non-local since it uses spatially-averaged velocity.

We will use the following two spaces in the paper:

$$H = H^1(0,1) \times L^2(0,1), \qquad X = H^2(0,1) \times H^1(0,1). \tag{30}$$

Our main result on stabilization is given by the following theorem.

Theorem 1. *Consider the system (1)–(3) with the controller (29) under the assumption $c \neq 1$, $c > 0$, $qc \neq -1$. For any initial data $(u(\cdot,0), u_t(\cdot,0)) \in H$ the closed-loop system has a unique solution $(u, u_t) \in C([0,\infty), H)$, which is exponentially stable in the sense of the norm*

$$\left(\int_0^1 u_x(x,t)^2 dx + \int_0^1 u_t(x,t)^2 dx + u(1,t)^2\right)^{1/2}. \tag{31}$$

If, in addition, the initial conditions are compatible with the boundary conditions and belong to X, then $(u, u_t) \in C^1([0,\infty), H) \cap C([0,\infty), X)$ is the classical solution of the closed-loop system.

Proof. First, let us establish the stability of the target system. With the Lyapunov function

$$V_1(t) = \frac{1}{2}\int_0^1 w_x(x,t)^2 dx + \frac{1}{2}\int_0^1 w_t(x,t)^2 dx + \frac{c_0}{2}w(1,t)^2 + \delta\int_0^1 (x-2)w_x(x,t)w_t(x,t)\,dx, \tag{32}$$

where δ is sufficiently small (so that V_1 is positive definite), we obtain

$$\dot{V}_1 = -\frac{\delta}{2}\int_0^1 (w_x(x,t)^2 + w_t(x,t)^2)\,dx - \left[c - \delta(1+c^2)\right] w_t(0,t)^2 - \frac{\delta}{2}\left[w_t(1,t)^2 + c_0^2 w(1,t)^2\right] \le -\omega V_1, \tag{33}$$

where $\omega > 0$. Since

$$a_1 V_2 \le V_1 \le a_2 V_2, \tag{34}$$

where $a_1 = \min\{1/2 - \delta, c_0/2\}$, $a_2 = \max\{1/2 + \delta, c_0/2\}$, and

$$V_2(t) = \int_0^1 w_x(x,t)^2 dx + \int_0^1 w_t(x,t)^2 dx + w(1,t)^2, \tag{35}$$

we obtain

$$V_2(t) \le \frac{a_2}{a_1} e^{-\omega t} V_2(0). \tag{36}$$

Let us differentiate the transformation (28) with respect to t and x. We get

$$w_t(x,t) = -\frac{1+qc}{q^2-1}u_t(x,t) - \frac{q+c}{q^2-1}u_x(x,t), \tag{37}$$

and

$$w_x(x,t) = -\frac{1+qc}{q^2-1}u_x(x,t) - \frac{q+c}{q^2-1}u_t(x,t). \tag{38}$$

From (37) and (38) it easily follows that

$$\int_0^1 (w_x(x,t)^2 + w_t(x,t)^2)\,dx \le \frac{(c+1)^2}{(q-1)^2}\int_0^1 (u_x(x,t)^2 + u_t(x,t)^2)\,dx. \tag{39}$$

Using the transformation (27) with $x = 1$, we get

$$w(1,t)^2 \le 2u(1,t)^2 + \frac{4(q+c)^2}{(|q|-1)^2}\int_0^1 (u_x(x,t)^2 + u_t(x,t)^2)\,dx. \tag{40}$$

From (39) and (40) we obtain

$$V_2 \le 4\frac{(q+c)^2 + (c+1)^2}{(|q|-1)^2}V_3, \tag{41}$$

where

$$V_3(t) = \int_0^1 u_x(x,t)^2 dx + \int_0^1 u_t(x,t)^2 dx + u(1,t)^2. \tag{42}$$

One can easily show that the inverse transformation to (37)–(38) is

$$u_t(x,t) = -\frac{1+qc}{c^2-1}w_t(x,t) + \frac{q+c}{c^2-1}w_x(x,t), \tag{43}$$

and

$$u_x(x,t) = -\frac{1+qc}{c^2-1}w_x(x,t) + \frac{q+c}{c^2-1}w_t(x,t). \tag{44}$$

Setting $x = 1$ in (27) and using (43), (44) to express $u(1,t)$ in terms of $w(1,t)$, w_t, and w_x, we obtain

$$V_3 \le 4\frac{(q+c)^2 + (q+1)^2}{(c-1)^2}V_2. \tag{45}$$

From (36), (41) and (45) one gets

$$V_3(t) \le \frac{16a_2}{a_1}\frac{((q+c)^2 + (c+1)^2)((q+c)^2 + (q+1)^2)}{(c-1)^2(|q|-1)^2} \times e^{-\omega t}V_3(0). \tag{46}$$

The existence and uniqueness of the solution follow by standard arguments as in [3]. First, the abstract operator describing the system is introduced. It is dissipative due to the estimates above and it is easy to show that it has a bounded inverse. The result then follows from the Lumer–Phillips theorem. □

4. Observer design

In this section, we design an observer for the system (1)–(3) when only boundary measurements are available. We assume that displacement and velocity at the end $x = 1$ are measured (i.e., $u(1,t)$ and $u_t(1,t)$).

Since we expect this observer to be dual to the controller designed in the previous section, it is natural to assume that the observer gains are also constant. We propose the following observer[1]

$$\hat{u}_{tt}(x,t) = \hat{u}_{xx}(x,t) + p_1[u(1,t) - \hat{u}(1,t)] + p_2[u_t(1,t) - \hat{u}_t(1,t)] \tag{47}$$

$$\hat{u}_x(0,t) = -q\hat{u}_t(0,t) + p_3[u(1,t) - \hat{u}(1,t)] + p_4[u_t(1,t) - \hat{u}_t(1,t)] \tag{48}$$

$$\hat{u}_x(1,t) = U(t) + p_5[u(1,t) - \hat{u}(1,t)] + p_6[u_t(1,t) - \hat{u}_t(1,t)]. \tag{49}$$

[1] For the rest of the paper we do not explicitly specify that every PDE evolves on $(x,t): 0 < x < 1, t > 0$ and that boundary conditions are valid for $t > 0$.

The observer error $\tilde{u} = u - \hat{u}$ satisfies

$$\tilde{u}_{tt}(x,t) = \tilde{u}_{xx}(x,t) - p_1\tilde{u}(1,t) - p_2\tilde{u}_t(1,t) \tag{50}$$

$$\tilde{u}_x(0,t) = -q\tilde{u}_t(0,t) - p_3\tilde{u}(1,t) - p_4\tilde{u}_t(1,t) \tag{51}$$

$$\tilde{u}_x(1,t) = -p_5\tilde{u}(1,t) - p_6\tilde{u}_t(1,t). \tag{52}$$

Consider the transformation

$$\tilde{u}(x,t) = \tilde{w}(x,t) + \alpha\int_x^1 \tilde{w}(y,t)\,dy + \beta\int_x^1 \tilde{w}_t(y,t)\,dy + \gamma\int_x^1 \tilde{w}_x(y,t)\,dy. \tag{53}$$

Note that unlike in the control transformation (27), here the integrals run from x to 1. This is because the input and the output are collocated.

We map the observer error system into the system

$$\tilde{w}_{tt}(x,t) = \tilde{w}_{xx}(x,t) \tag{54}$$

$$\tilde{w}_x(0,t) = \tilde{c}\tilde{w}_t(0,t) \tag{55}$$

$$\tilde{w}_x(1,t) = -c_0\tilde{w}(1,t), \tag{56}$$

which is exponentially stable for $\tilde{c} > 0$ and $c_0 > 0$.

First, we differentiate the transformation (53) twice w.r.t. time (note that $\tilde{u}(1,t) = \tilde{w}(1,t)$),

$$\begin{aligned}\tilde{u}_{tt}(x,t) &= \tilde{u}_{xx}(x,t) + \alpha\tilde{w}_x(1,t) + \beta\tilde{w}_{xt}(1,t) + \gamma\tilde{w}_{tt}(1,t)\\ &= \tilde{u}_{xx}(x,t) - \alpha c_0\tilde{u}(1,t) - \beta c_0\tilde{u}_t(1,t) + \gamma\tilde{u}_{tt}(1,t).\end{aligned} \tag{57}$$

Comparing the above with (50) we get

$$\gamma = 0, \qquad p_1 = \alpha c_0, \qquad p_2 = \beta c_0. \tag{58}$$

From the transformation (53) we also get

$$\begin{aligned}\tilde{u}_x(1,t) &= \tilde{w}_x(1,t) - \alpha\tilde{w}(1,t) - \beta\tilde{w}_t(1,t)\\ &= -(c_0+\alpha)\tilde{u}(1,t) - \beta\tilde{u}_t(1,t)\end{aligned} \tag{59}$$

and

$$\tilde{u}_x(0,t) + q\tilde{u}(0,t) = -\alpha w(0,t) + q\alpha\int_0^1 w_t(y,t)\,dy + \tilde{w}_t(0,t)[\tilde{c} - \beta + q - q\beta\tilde{c}] - q\beta c_0\tilde{u}(1,t). \tag{60}$$

Comparing (59) and (60) with (51) and (52), respectively, we get

$$p_5 = c_0 + \alpha, \qquad p_6 = \beta, \tag{61}$$

$$p_3 = q\beta c_0, \qquad p_4 = 0, \qquad \alpha = 0, \qquad \tilde{c} + q = \beta(1 + q\tilde{c}). \tag{62}$$

Using (58), (61) and (62), we get the observer

$$\hat{u}_{tt}(x,t) = \hat{u}_{xx}(x,t) + \frac{c_0(q+\tilde{c})}{1+q\tilde{c}}[u_t(1,t) - \hat{u}_t(1,t)] \tag{63}$$

$$\hat{u}_x(0,t) = -q\hat{u}_t(0,t) + \frac{c_0 q(q+\tilde{c})}{1+q\tilde{c}}[u(1,t) - \hat{u}(1,t)] \tag{64}$$

$$\hat{u}_x(1,t) = U(t) + c_0[u(1,t) - \hat{u}(1,t)] + \frac{q+\tilde{c}}{1+q\tilde{c}}[u_t(1,t) - \hat{u}_t(1,t)] \tag{65}$$

and the transformation is

$$\tilde{u}(x,t) = \tilde{w}(x,t) + \frac{q+\tilde{c}}{1+q\tilde{c}}\int_x^1 \tilde{w}_t(y,t)\,dy. \tag{66}$$

Note the duality of 4 observer gains in (63)–(65) to 4 control gains in (29) (for $\tilde{c} = c$), even though the control and observer transformations are different.

Theorem 2. *For any initial data $(\tilde{u}(\cdot,0), \tilde{u}_t(\cdot,0)) \in H$ the system (50)–(52) with $\tilde{c} > 0$, $\tilde{c} \neq 1$, $\tilde{c}q \neq -1$, and $p_1 - p_6$ given by (58), (61)–(62), has a unique solution $(\tilde{u}, \tilde{u}_t) \in C([0,\infty), H)$, which is exponentially stable in the sense of the norm*

$$\left(\int_0^1 \tilde{u}_x(x,t)^2 dx + \int_0^1 \tilde{u}_t(x,t)^2 dx + \tilde{u}(1,t)^2\right)^{1/2}. \tag{67}$$

If, in addition, the initial conditions are compatible with the boundary conditions and belong to X, then $(\tilde{u}, \tilde{u}_t) \in C^1([0,\infty), H) \cap C([0,\infty), X)$ is the classical solution of the system (50)–(52).

Proof. With the Lyapunov function

$$V_4 = \frac{1}{2}\int_0^1 \tilde{w}_x(x,t)^2 dx + \frac{1}{2}\int_0^1 \tilde{w}_t(x,t)^2 dx + \frac{c_0}{2}\tilde{w}(1,t)^2 + \delta_2\int_0^1 (x-2)\tilde{w}_x(x,t)\tilde{w}_t(x,t)\,dx, \tag{68}$$

where δ_2 is sufficiently small, exactly the same calculation as in (33) shows that

$$\dot{V}_4 \le -\tilde{\omega} V_4, \tag{69}$$

where $\tilde{\omega} > 0$.

Differentiating the transformation (66) we get

$$\tilde{u}_x(x,t) = \tilde{w}_x(x,t) - \frac{q+\tilde{c}}{1+q\tilde{c}}\tilde{w}_t(x,t), \tag{70}$$

and

$$\tilde{u}_t(x,t) = \tilde{w}_t(x,t) - \frac{q+\tilde{c}}{1+q\tilde{c}}\tilde{w}_x(x,t) - \frac{c_0(q+\tilde{c})}{1+q\tilde{c}}\tilde{w}(1,t). \tag{71}$$

Note also that $\tilde{u}(1,t) = \tilde{w}(1,t)$. Therefore,

$$\|\tilde{u}_x\|^2 + \|\tilde{u}_t\|^2 + \tilde{u}(1)^2 \le M_6(\|\tilde{w}_x\|^2 + \|\tilde{w}_t\|^2 + \tilde{w}(1)^2), \tag{72}$$

where $M_6 = 3 + 3\max\{c_0^2, 1\}(q+\tilde{c})^2/(1+q\tilde{c})^2$.

The inverse to (70) and (71) is

$$\tilde{w}_x(x,t) = \frac{(1+q\tilde{c})^2\tilde{u}_x(x,t) + (q+\tilde{c})(1+q\tilde{c})\tilde{u}_t(x,t)}{(q^2-1)(c^2-1)} + \frac{c_0(q+\tilde{c})^2}{(q^2-1)(c^2-1)}\tilde{u}(1,t) \tag{73}$$

$$\tilde{w}_t(x,t) = \frac{(1+q\tilde{c})^2\tilde{u}_t(x,t) + (q+\tilde{c})(1+q\tilde{c})\tilde{u}_x(x,t)}{(q^2-1)(c^2-1)} + \frac{c_0(q+\tilde{c})(1+q\tilde{c})}{(q^2-1)(c^2-1)}\tilde{u}(1,t). \tag{74}$$

Therefore,

$$\|\tilde{w}_x\|^2 + \|\tilde{w}_t\|^2 + \tilde{w}(1)^2 \le M_7(\|\tilde{u}_x\|^2 + \|\tilde{u}_t\|^2 + \tilde{u}(1)^2), \tag{75}$$

where $M_7 = 3\max\{c_0^2, 1\}((q+\tilde{c})^2 + (1+q\tilde{c})^2)^2(q^2-1)^{-2}(\tilde{c}^2-1)^{-2}$. From (69), (72), and (75) we get that the norm (67) decays exponentially. The existence and uniqueness of the solution of the observer error system are obtained as in the proof of Theorem 1. □

5. Output feedback

In this section we combine the controller and the observer designed in the previous two sections to solve the output-feedback problem.

Theorem 3. *Consider the system (1)–(3) with the observer (63)–(65) and the controller*

$$U(t) = \frac{c_0 q(q+c)}{1+qc}\hat{u}(0,t) - c_0 u(1,t) - \frac{q+c}{1+qc}u_t(1,t) - \frac{c_0(q+c)}{1+qc}\int_0^1 \hat{u}_t(y,t)\,dy \tag{76}$$

under the assumptions $c > 0$, $\tilde{c} > 0$, $c \neq 1$, $\tilde{c} \neq 1$, $cq \neq -1$, $\tilde{c}q \neq -1$. For any initial data $(u(\cdot,0), u_t(\cdot,0), \hat{u}(\cdot,0), \hat{u}_t(\cdot,0)) \in H \times H$, the closed-loop system is exponentially stable in the sense of the norm

$$\left(\int_0^1 u_x(x,t)^2 dx + \int_0^1 u_t(x,t)^2 dx + u(1,t)^2 + \int_0^1 \hat{u}_x(x,t)^2 dx + \int_0^1 \hat{u}_t(x,t)^2 dx + \hat{u}(1,t)^2\right)^{1/2}. \quad (77)$$

If, in addition, the initial conditions are compatible with the boundary conditions and belong to $X \times X$, then $(u, u_t, \hat{u}, \hat{u}_t) \in C^1([0,\infty), H \times H) \cap C([0,\infty), X \times X)$ is the classical solution of the closed-loop system.

Proof. Consider two transformations: (66) and

$$\hat{w}(x,t) = -\frac{1+qc}{q^2-1}\hat{u}(x,t) + \frac{q(q+c)}{q^2-1}\hat{u}(0,t) - \frac{q+c}{q^2-1}\int_0^x \hat{u}_t(y,t)\,dy. \quad (78)$$

It is straightforward to show that these transformations along with the control law (76) map the system $(\hat{u}, \tilde{u})$ into the $(\hat{w}, \tilde{w})$-system, where the $\tilde{w}$ part is given by (54)–(56) and $\hat{w}$ satisfies the following PDE:

$$\hat{w}_{tt}(x,t) = \hat{w}_{xx}(x,t) + A(q\tilde{w}_t(1,t) - x\tilde{w}_{tt}(1,t)) \quad (79)$$

$$\hat{w}_x(0,t) = c\hat{w}_t(0,t) - A\frac{q(1+qc)}{q+c}\tilde{w}(1,t) \quad (80)$$

$$\hat{w}_x(1,t) = -c_0\hat{w}(1,t) + \frac{\tilde{c}-c}{1+q\tilde{c}}\tilde{w}_t(1,t), \quad (81)$$

where

$$A = \frac{c_0(q+\tilde{c})(q+c)}{(q^2-1)(1+q\tilde{c})}. \quad (82)$$

Note that the PDE (79)–(81) contains the terms proportional to $\tilde{w}_t(1,t)$ and $\tilde{w}_{tt}(1,t)$, which are in H^2 and H^3 respectively, while our Lyapunov functions used in control and observer designs are H^1 norms. To overcome this difficulty, we introduce a new variable $\breve{w}(x,t) = \hat{w}(x,t) + Ax\tilde{w}(1,t)$, which eliminates the term $\tilde{w}_{tt}(1,t)$. The remaining $\tilde{w}_t(1,t)$-terms are handled using the term $-\tilde{w}_t(1,t)^2$ in $\dot{V}_4$, which was simply discarded in the estimate (69) (see (33)).

The variable $\breve{w}(x,t)$ satisfies the following PDE:

$$\breve{w}_{tt}(x,t) = \breve{w}_{xx}(x,t) + Aq\tilde{w}_t(1,t) \quad (83)$$

$$\breve{w}_x(0,t) = c\breve{w}_t(0,t) - \frac{c_0c(q+\tilde{c})}{(1+q\tilde{c})}\tilde{w}(1,t) \quad (84)$$

$$\breve{w}_x(1,t) = -c_0\breve{w}(1,t) + \frac{\tilde{c}-c}{1+q\tilde{c}}\tilde{w}_t(1,t) + (c_0+1)A\tilde{w}(1,t). \quad (85)$$

With the Lyapunov function

$$V_5 = \frac{1}{2}\int_0^1 \breve{w}_x(x,t)^2 dx + \frac{1}{2}\int_0^1 \breve{w}_t(x,t)^2 dx + \frac{c_0}{2}\breve{w}(1,t)^2 + \delta_1\int_0^1 (x-2)\breve{w}_x(x,t)\breve{w}_t(x,t)\,dx + KV_4 \quad (86)$$

we get

$$\begin{aligned}\dot{V}_5 = &-\frac{\delta_1}{2}(\|\breve{w}_x\|^2 + \|\breve{w}_t\|^2) - \frac{\delta_1}{2}\breve{w}_t(1,t)^2 - \frac{\delta_1}{2}\breve{w}_x(1,t)^2 + \frac{\tilde{c}-c}{1+q\tilde{c}}\breve{w}_t(1,t)\tilde{w}_t(1,t) \\ &+ (c_0+1)A\breve{w}_t(1,t)\tilde{w}(1,t) - c\breve{w}_t(0,t)^2 + \frac{c_0c(q+\tilde{c})}{1+q\tilde{c}}\breve{w}_t(0,t)\tilde{w}(1,t) \\ &+ \delta_1\breve{w}_t(0,t)^2 + \delta_1\breve{w}_x(0,t)^2 + Aq\tilde{w}_t(1,t)\int_0^1 \breve{w}_t(x,t)\,dx \\ &+ Aq\delta_1\tilde{w}_t(1,t)\int_0^1 (1+x)\breve{w}_x(x,t)\,dx \\ &- K[\tilde{c} - \delta_2(1+\tilde{c}^2)]\tilde{w}(0,t)^2 - K\frac{\delta_2}{2}\tilde{w}_t(1,t)^2 \\ &- K\frac{\delta_2}{2}c_0^2\tilde{w}(1,t)^2 - K\frac{\delta_2}{2}(\|\tilde{w}_x\|^2 + \|\tilde{w}_t\|^2). \end{aligned} \quad (87)$$

Using Young's inequality, we estimate

$$\begin{aligned}\dot{V}_5 \le &-\frac{\delta_1}{4}(\|\breve{w}_x\|^2 + \|\breve{w}_t\|^2) - \frac{\delta_1}{2}c_0^2\breve{w}(1,t)^2 \\ &- \left[\frac{c}{2} - \delta_1(1+2c^2)\right]\breve{w}_t(0,t)^2 \\ &- K\frac{\delta_2}{2}(\|\tilde{w}_x\|^2 + \|\tilde{w}_t\|^2) - K[\tilde{c} - \delta_2(1+\tilde{c}^2)]\tilde{w}(0,t)^2 \\ &- \left[K\frac{\delta_2}{2} - \frac{(\delta_1+2)(\tilde{c}-c)^2}{\delta_1(1+q\tilde{c})^2} - \frac{2A^2q^2}{\delta_1}(1+4\delta_1^2)\right]\tilde{w}_t(1,t)^2 \\ &- \left[K\frac{\delta_2}{2}c_0^2 - \frac{4A^2(1+\delta_1)}{\delta_1}(c_0+1)^2 - \frac{c_0^2c(1+4c)(q+\tilde{c})^2}{(1+q\tilde{c})^2}\right]\tilde{w}(1,t)^2. \end{aligned} \quad (88)$$

For large enough K and small enough δ_1 and δ_2 (independent of initial conditions) we get

$$\dot{V}_5 \le -\omega_1 V_5 - \alpha\tilde{w}_t(1,t)^2, \quad \omega_1 > 0, \alpha > 0. \quad (89)$$

Going back to the old variable $\hat{w}$, and using (89) we obtain the exponential stability in $(\tilde{w}, \hat{w})$ variables. From the transformations (66) and (78) and their inverses we obtain the exponential stability in $(\tilde{u}, \hat{u})$ variables, and therefore in $(u, \hat{u})$ variables. The existence and uniqueness of the solutions is proved as in Theorem 1. □

6. Dirichlet actuation and Neumann sensing

In this section we use the control and observer transformations derived in Sections 3 and 4 to design the controller and the observer for the case of Dirichlet actuation and Neumann sensing.

Consider the system

$$u_{tt}(x,t) = u_{xx}(x,t) \quad (90)$$

$$u_x(0,t) = -qu_t(0,t) \quad (91)$$

$$u(1,t) = U(t). \quad (92)$$

Using the transformation (28), we map this system into the target system

$$w_{tt}(x,t) = w_{xx}(x,t) \quad (93)$$

$$w_x(0,t) = cw_t(0,t) \quad (94)$$

$$w(1,t) = 0, \quad (95)$$

which is exponentially stable for $c > 0$. The controller is obtained by setting $x = 1$ in (28):

$$U(t) = \frac{q(q+c)}{1+qc}u(0,t) - \frac{q+c}{1+qc}\int_0^1 u_t(y,t)\,dy. \quad (96)$$

Theorem 4. *Consider the system (90)–(92) with the controller (96) under the assumption $c > 0$, $c \neq 1$, $cq \neq -1$. For any initial* *data* $(u(\cdot,0),u_t(\cdot,0))\in X$ *compatible with the boundary conditions, the closed-loop system has a unique classical solution* $(u,u_t)\in C^1([0,\infty),H)\cap C([0,\infty),X)$, *which is exponentially stable in the sense of the norm*

$$\left(\int_0^1 u_x(x,t)^2\mathrm{d}x+\int_0^1 u_t(x,t)^2\mathrm{d}x\right)^{1/2}. \qquad (97)$$

Proof. Starting with the Lyapunov function

$$V_6(t)=\frac{1}{2}\int_0^1 w_x(x,t)^2\mathrm{d}x+\frac{1}{2}\int_0^1 w_t(x,t)^2\mathrm{d}x +\delta\int_0^1(x-2)w_x(x,t)w_t(x,t)\,\mathrm{d}x, \qquad (98)$$

where δ is sufficiently small, we obtain

$$\dot V_6=-\frac{\delta}{2}\int_0^1(w_x(x,t)^2+w_t(x,t)^2)\,\mathrm{d}x -\left[c-\delta(1+c^2)\right]w_t(0,t)^2-\frac{\delta}{2}w_x(1,t)^2 \le -\omega V_6,\quad \omega>0. \qquad (99)$$

The rest of the proof is very similar to the proof of Theorem 1. □

When only measurements of $u_x(1,t)$ and $u_{xt}(1,t)$ are available, we design the observer

$$\hat u_{tt}(x,t)=\hat u_{xx}(x,t)+p_1[u_x(1,t)-\hat u_x(1,t)] +p_2[u_{xt}(1,t)-\hat u_{xt}(1,t)] \qquad (100)$$

$$\hat u_x(0,t)=-q\hat u_t(0,t)+p_3[u_x(1,t)-\hat u_x(1,t)] +p_4[u_{xt}(1,t)-\hat u_{xt}(1,t)] \qquad (101)$$

$$\hat u(1,t)=U(t), \qquad (102)$$

where p_1,p_2,p_3,p_4 are constants to be chosen.

The observer error system is

$$\tilde u_{tt}(x,t)=\tilde u_{xx}(x,t)-p_1\tilde u_x(1,t)-p_2\tilde u_{xt}(1,t) \qquad (103)$$

$$\tilde u_x(0,t)=-q\tilde u_t(0,t)-p_3\tilde u_x(1,t)-p_4\tilde u_{xt}(1,t) \qquad (104)$$

$$\tilde u(1,t)=0. \qquad (105)$$

We use the observer transformation (66), derived for the case of the Dirichlet sensing to map the observer error system into the following system:

$$\tilde w_{tt}(x,t)=\tilde w_{xx}(x,t) \qquad (106)$$

$$\tilde w_x(0,t)=\tilde c\tilde w_t(0,t) \qquad (107)$$

$$\tilde w(1,t)=0. \qquad (108)$$

Substituting (66) into (103)–(105), we get the following conditions:

$$p_1=p_2=0,\qquad p_3=-q\frac{q+\tilde c}{1+q\tilde c},\qquad p_4=-\frac{q+\tilde c}{1+q\tilde c}. \qquad (109)$$

Therefore, the observer is

$$\hat u_{tt}(x,t)=\hat u_{xx}(x,t)-\frac{q+\tilde c}{1+q\tilde c}[u_{xt}(1,t)-\hat u_{xt}(1,t)] \qquad (110)$$

$$\hat u_x(0,t)=-q\hat u_t(0,t)-\frac{q(q+\tilde c)}{1+q\tilde c}[u_x(1,t)-\hat u_x(1,t)] \qquad (111)$$

$$\hat u(1,t)=U(t). \qquad (112)$$

Note the duality of two observer gains in (110), (111) to the two control gains in (96) (for $\tilde c=c$).

Theorem 5. *Consider the system* (90)–(92) *with the observer* (110)–(112) *and the controller*

$$U(t)=\frac{q(q+c)}{1+qc}\hat u(0,t)-\frac{q+c}{1+qc}\int_0^1\hat u_t(y,t)\,\mathrm{d}y \qquad (113)$$

under the assumptions $c>0,\tilde c>0,c\neq1,\tilde c\neq1,cq\neq-1,\tilde cq\neq-1$. *For any initial data* $(u(\cdot,0),u_t(\cdot,0),\hat u(\cdot,0),\hat u_t(\cdot,0))\in X\times X$ *compatible with the boundary conditions, the closed-loop system has a unique classical solution* $(u,u_t,\hat u,\hat u_t)\in C^1([0,\infty),H\times H)\cap C([0,\infty),X\times X)$, *which is exponentially stable in the sense of the norm*

$$\left(\int_0^1 u_x(x,t)^2\mathrm{d}x+\int_0^1 u_t(x,t)^2\mathrm{d}x+\int_0^1 u_{xx}(x,t)^2\mathrm{d}x +\int_0^1 u_{xt}(x,t)^2\mathrm{d}x+\int_0^1\hat u_x(x,t)^2\mathrm{d}x+\int_0^1\hat u_t(x,t)^2\mathrm{d}x\right)^{1/2}. \qquad (114)$$

Proof. The transformations (66) and

$$\hat w(x,t)=-\frac{1+qc}{q^2-1}\hat u(x,t)+\frac{q(q+c)}{q^2-1}\hat u(0,t) -\frac{q+c}{q^2-1}\int_0^x\hat u_t(y,t)\,\mathrm{d}y \qquad (115)$$

map (103)–(105), (110)–(112) into (106)–(108) and the following system

$$\hat w_{tt}(x,t)=\hat w_{xx}(x,t)-\frac{q+\tilde c}{1+q\tilde c}\tilde w_{xt}(1,t) +\frac{(q+c)(q+\tilde c)}{(q^2-1)(1+q\tilde c)}x\tilde w_{xtt}(1,t) \qquad (116)$$

$$\hat w_x(0,t)=c\hat w_t(0,t)+\frac{q(1+qc)(q+\tilde c)}{(q^2-1)(1+q\tilde c)}\tilde w_x(1,t) \qquad (117)$$

$$\hat w(1,t)=0. \qquad (118)$$

First, we establish the exponential stability of the system (106)–(108) with the Lyapunov function

$$V_7(t)=\frac{1}{2}\int_0^1\tilde w_x(x,t)^2\,\mathrm{d}x+\frac{1}{2}\int_0^1\tilde w_t(x,t)^2\,\mathrm{d}x +\delta_1\int_0^1(x-2)\tilde w_x(x,t)\tilde w_t(x,t)\,\mathrm{d}x +\frac{1}{2}\int_0^1\tilde w_{xx}(x,t)^2\,\mathrm{d}x+\frac{1}{2}\int_0^1\tilde w_{xt}(x,t)^2\,\mathrm{d}x +\delta_2\int_0^1(x-2)\tilde w_{xx}(x,t)\tilde w_{xt}(x,t)\,\mathrm{d}x. \qquad (119)$$

It is straightforward to show that

$$\dot V_7\le-\omega V_7-\alpha\tilde w_{xt}(1,t)^2,\quad \omega>0,\alpha>0. \qquad (120)$$

Note that, unlike in the case of Dirichlet sensing, here the Lyapunov function has to contain H^2 norms for us to be able to show the exponential stability of the observer error system. This is due to the H^2 nature of the terms $\tilde u_x(1,t)$ appearing in (103)–(105).

To eliminate the term proportional to $\tilde w_{xtt}(1,t)$ in (116) we introduce a new variable

$$\check w(x,t)=\hat w(x,t)-\frac{(q+c)(q+\tilde c)}{(q^2-1)(1+q\tilde c)}x\tilde w_x(1,t). \qquad (121)$$

With the Lyapunov function

$$V_8(t)=\frac{1}{2}\int_0^1\check w_x(x,t)^2\,\mathrm{d}x+\frac{1}{2}\int_0^1\check w_t(x,t)^2\,\mathrm{d}x +\delta\int_0^1(x-2)\check w_x(x,t)\check w_t(x,t)\,\mathrm{d}x+KV_7(t),$$

we obtain

$$\dot{V}_8 \le -\omega_2 V_8, \quad \omega_2 > 0 \tag{122}$$

for sufficiently large K and sufficiently small δ, δ_1, and δ_2 (independent of initial conditions). The rest of the proof is similar to the proof of Theorem 3. □

7. Conclusions

In this paper we introduced a new integral transformation for wave equations and used it to obtain explicit controllers and observers for a wave equation with negative damping at the boundary. The application of the presented approach to other hyperbolic systems is very promising and will be the subject of future work.

References

[1] A. Smyshlyaev, M. Krstic, Closed form boundary state feedbacks for a class of 1-D partial integro-differential equations, IEEE Trans. Automa. Control 49 (2004) 2185–2202.
[2] A. Smyshlyaev, M. Krstic, Backstepping observers for a class of parabolic PDEs, Systems Control Lett. 54 (2005) 613–625.
[3] M. Krstic, B.Z. Guo, A. Balogh, A. Smyshlyaev, Output-feedback stabilization of an unstable wave equation, Automatica 44 (2008) 63–74.
[4] C. Bardos, L. Halpern, G. Lebeau, J. Rauch, E. Zuazua, Stabilization of the wave equation by Dirichlet type boundary feedback, Asymptot. Anal. 4 (4) (1991) 285–291.
[5] G. Chen, Energy decay estimates and exact boundary value controllability for the wave equation in a bounded domain, J. Math. Pure Appl. 58 (1979) 249–273.
[6] M.S. de Queiroz, C.D. Rahn, Boundary control of vibration and noise in distributed parameter systems: And overview, Mech. Systems Signal Processing 16 (2002) 19–38.
[7] V. Komornik, E. Zuazua, A direct method for the boundary stabilization of the wave equation, J. Math. Pure Appl. 69 (1990) 33–54.
[8] J. Lagnese, Boundary Stabilization of Thin Plates, SIAM, 1989.
[9] I. Lasiecka, R. Triggiani, Uniform stabilization of the wave equation with Dirichlet or Neumann feedback control without geometrical conditions, Appl. Math. Optim. 25 (1992) 189–224.
[10] Z.-H. Luo, B.-Z. Guo, O. Morgul, Stability and Stabilization of Infinite Dimensional Systems with Applications, Springer, 1998.
[11] P. Freitas, E. Zuazua, Stability results for the wave equation with indefinite damping, J. Differential Equations 132 (2) (1996) 338–352.
[12] P. Martinez, J. Vancostenoble, Stabilization of the wave equation by on-off and positive-negative feedbacks, ESAIM Control Optim. Calc. Var. 7 (2002) 335–377.