Input Delay Compensation for Forward Complete and Strict-Feedforward Nonlinear Systems

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***Abstract*—We present an approach for compensating input delay of arbitrary length in nonlinear control systems. This approach, which due to the infinite dimensionality of the actuator dynamics and due to the nonlinear character of the plant results in a nonlinear feedback operator, is essentially a nonlinear version of the Smith Predictor and its various predictor-based modifications for linear plants. Global stabilization in the presence of arbitrarily long delay is achieved for all nonlinear plants that are globally stabilizable in the absence of delay and that satisfy the property of forward completeness (which is satisfied by most mechanical systems, electromechanical systems, vehicles, and other physical systems). For strict-feedforward systems, one obtains the predictor-based feedback law explicitly. For the linearizable subclass of strict-feedforward systems, closed-loop solutions are also obtained explicitly. The feedback designs are illustrated through two detailed examples.**

***Index Terms*—Global stabilization.**

I. Introduction

1) Background:

SINCE the 1959 publication of Otto J. M. Smith's "Smith Predictor" paper [50], there has been continuing interest in compensation of long input delay in control systems. Smith's original result was not applicable to unstable plants, however, numerous subsequent papers have dealt with removing this limitation [3], [7], [11], [19]–[25], [37]–[40], [42], [44], [47], [62], [64]–[66], including even efforts on adaptive predictor feedback control [4], [5], [43]. In parallel, over the last ten years, innovative efforts have been ongoing on developing control design and stability analysis for nonlinear systems with state delays [9]–[12], [16], [28], [30], [45], [46], [63]. Some efforts have instead considered input delays [31], [58]. However, no attempts have been made to systematically address the problem of *compensation* of a long delay at the input of a nonlinear (possibly unstable) control system.

In [19] we launched an effort on developing a delay compensation scheme for nonlinear systems. Such ideas have already been pursued in the process control community [8], [17] for control structures that expand upon the classical Smith Predictor, which requires open-loop stability of the plant. Related work was also presented in [41], in the context of motion planning, employing linearization along the reference trajectory and time discretization. Our approach in [19] was based on ideas coming from boundary control for nonlinear PDEs [60], [61]. The scheme that we arrived at, despite employing a nonlinear infinite-dimensional feedback operator, was an exact analog of the classical extensions of the Smith Predictor for unstable LTI systems [3], [24], [25]. We dealt only with a scalar problem that highlighted the key difficulty with nonlinear plants—in the presence of input delay, the plant may have finite escape before the control 'kicks in.' In [19] we obtained a regional result, which achieves a region of attraction equal to the set of initial conditions from which plant does *not* exhibit finite escape during the control dead time.

2) Contributions: With the finite escape obstacle recognized, in this paper we focus on two classes of problems for which global stability is achievable in the presence of arbitrarily long input delay. The first class is the general class of *forward complete* systems (Section VI), which do not exhibit finite escape as long as the input remains (locally, i.e., not necessarily uniformly) bounded. This seems like a restrictive class in a mathematical sense, but includes most, if not all, mechanical and electromechanical systems.

The second class is the class of *strict-feedforward* systems (Section VII), which is a subclass of forward complete systems. For this class, which addresses a relatively limited set of applications but is important from the structural point of view, we not only obtain global stability, but also obtain an explicit formula for the *predictor state*, which is used in the nominal control law to compensate for the input delay. This is significant, because the predictor state is normally not obtainable explicitly—for example, it is generally not available in explicit form for feedback linearizable systems.

We dedicate special attention (Section VIII) to a small but nice subclass of strict-feedforward systems which area *linearizable* by coordinate transformation [18], [51]–[53]. For these systems we obtain both the feedback operator and the closed-loop solutions (which are both infinite dimensional) in closed form.

3) Organization: We start in Section II where we introduce a predictor-based delay compensation design for general stabilizable nonlinear systems. In Section V we present some important stability properties of the transport PDE in various (mostly non-standard) norms. These technical results help in the stability proof for the broad class of forward-complete systems. Then in subsequent sections we introduce a predictor feedback design for general nonlinear systems and present global stability analyses for the forward-complete and strict-feedforward systems.

4) Notation: Several norms are used in the paper, for vectors and functions. For an n-vector, the norm $|\cdot|$ denotes the

Manuscript received February 01, 2009; revised April 13, 2009 and April 22, 2009. First published December 15, 2009; current version published February 10, 2010. This work was supported by NSF and Bosch. Recommended by Associate Editor S. Celikovsky.

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Digital Object Identifier 10.1109/TAC.2009.2034923

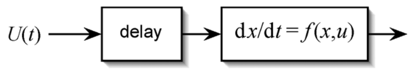

Fig. 1. Nonlinear system with input delay.

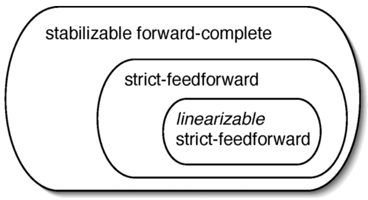

Fig. 2. Relation between forward-complete systems, strict-feedforward systems, and strict-feedforward systems that are feedback linearizable. For all these systems predictor feedback achieves global stabilization.

usual Euclidean norm. For functions $u : [0, D] \times \mathbb{R} \to \mathbb{R}$, the norm $\|u(t)\|_{L_p[0,D]} = \left(\int_0^D |u(x,t)|^p dx\right)^{1/p}$ for $p \in [1, \infty)$ denotes the spatial L_p norm. For $p = 2$ we simplify the notation to $\|u(t)\| = \|u(t)\|_{L_2[0,D]}$. The norm $\|u(t)\|_{L_\infty[0,D]} = \sup_{0 \le x \le D} |u(x,t)|$ is the spatial L_∞ norm, which is, in some cases where the context is clear, written more compactly as $\|u(t)\|_\infty$. For functions $U : \mathbb{R} \to \mathbb{R}$, we use the norm $\|U\|_{L_\infty[t-D,t]} = \sup_{t-D \le \theta \le t} |U(\theta)|$. Finally, for vector valued functions $p : [0, D] \times \mathbb{R} \to \mathbb{R}^n$, we use a spatial L_∞-norm $\|p(t)\|_{L_\infty[0,D]} = \sup_{0 \le x \le D} \left(p_1^2(x,t) + \cdots + p_n^2(x,t)\right)^{1/2}$. The types of solutions that the closed-loop system has depend on the initial state and input. If they satisfy a 'compatibility' condition (the initial input is equal to the value of the feedback law applied to the initial ODE state and actuator state), the solutions are *classical* (continuously differentiable in time). Otherwise, the system has *mild* solutions (measurable and locally essentially bounded).

II. Predictor Feedback for General Nonlinear Systems

Consider the system

$$\dot{Z}(t) = f(Z(t), U(t-D)) \tag{1}$$

where $Z \in \mathbb{R}^n$ is the state vector and U is a scalar control input, as given in Fig. 1. Denote the D-second ahead prediction $P(t) = Z(t+D)$, or, alternatively, as $P(t) = \Phi(t, t+D, Z(t))$, where $\Phi(t_0, t, X(0))$ is the flow of the system $\dot{X}(t) = f(X(t), U(t-D))$ (see Fig. 2). Assuming that a continuous function $\kappa(Z)$ is known such that $\dot{Z} = f(Z, \kappa(Z))$ is globally asymptotically stable at $Z = 0$, we define our delay-compensating nonlinear predictor-based controller as

$$U(t) = \kappa(P(t)) \tag{2}$$

$$P(t) = \int_{t-D}^{t} f(P(\theta), U(\theta))\, d\theta + Z(t) \tag{3}$$

where the initial condition for the integral equation for $P(t)$ is defined as

$$P(\theta) = \int_{-D}^{\theta} f(P(\sigma), U(\sigma))\, d\sigma + Z(0), \quad \theta \in [-D, 0]. \tag{4}$$

The predictor state $P(t)$ is given by the implicit relation (3), which can be solved using various approximation strategies for the integral on the right-hand side, the simplest one being the explicit expression $P_k = \sum_{i=k-N}^{k-1} f(P_i, U_i)h + Z_k$, where $h = D/N$ is the discretization step for the integral, N is an integer the user chooses, and $P_k = P(hk)$, $Z_k = Z(hk)$, $U_k = U(hk)$. Convergence of this—or any other approximation algorithm—for (3) is an important question but beyond the scope of the present paper which concentrates on basic continuous-time designs.

A crucial ingredient in the stability analysis for the control law (2), (3) is a *backstepping transformation*, and its inverse

$$W(t) = U(t) - \kappa(P(t)) \tag{5}$$

$$U(t) = W(t) + \kappa(\Pi(t)) \tag{6}$$

where $\Pi(t)$ is defined via the integral equation

$$\Pi(t) = \int_{t-D}^{t} f(\Pi(\theta), \kappa(\Pi(\theta)) + W(\theta))\, d\theta + Z(t) \tag{7}$$

with initial condition

$$\Pi(\theta) = \int_{-D}^{\theta} f(\Pi(\sigma), \kappa(\Pi(\sigma)) + W(\sigma))\, d\sigma + Z(0), \quad \theta \in [-D, 0]. \tag{8}$$

The backstepping transformation results in a closed-loop system (*target system*) of the form

$$\dot{Z}(t) = f(Z(t), \kappa(Z(t)) + W(t-D)) \tag{9}$$

$$W(t) \equiv 0, \quad \text{for } t \ge 0. \tag{10}$$

The target system is obtained by shifting the time back by D in (5) which yields (9) from (1). Equation (10) follows trivially from (2) and (5) for $t \ge 0$. For $t \in [-D, 0]$, $W(t)$ is nonzero and defined by (5) with P given by (4), hence, it depends only on the initial condition $Z(0)$ and the initial actuator state, $U(\sigma)$, $\sigma \in [-D, 0]$.

Note that $P = \Pi$, however, depending on which of the two definitions is considered, namely, depending on whether it is governed by input $U(t)$ or $W(t)$, this variable plays two different roles. The mapping (5) represents the direct backstepping transformation $U \mapsto W$, whereas (6) represents the inverse backstepping transformation $W \mapsto U$. Both transformations are nonlinear and infinite dimensional.

Throughout the paper we make an assumption that the plant $\dot{x} = f(x, \omega)$ is *forward complete*, namely, that, for every initial condition and every measurable locally essentially bounded input signal ω the corresponding solution is defined for all $t \ge 0$, i.e., the maximal interval of existence is $T_{\max} = +\infty$.

Forward completeness may seem as a restrictive assumption because some basic globally stabilizable systems are not forward complete—it is only under stabilizing feedback that they become forward complete. For instance, the scalar system $\dot{Z}(t) = Z^2(t) + U(t-D)$ fails to be globally stabilizable for $D > 0$ because it can exhibit finite escape for $t \in [0, D)$. In [19] we estimated its region of attraction under predictor feedback. Unfortunately, many systems within popular classes such as feedback linearizable systems, or strict-feedback systems $\dot{Z}_i = Z_{i+1} + \varphi_i(Z_1, \ldots, Z_i)$, $Z_{N+1}(t) = U(t-D)$, $i = 1, \ldots, N$, are not globally stabilizable for $D > 0$ because they are not forward complete. Hence, we look among forward-complete systems to find globally stabilizable nonlinear systems with long input delay. However, we note that Hypothesis (A2) in [16] allows to study some strict-feedback systems in the framework developed in the present paper.

The *plant-predictor* system (3) and the *target-predictor* system (7) play crucial roles in determining whether a closed-loop system under predictor feedback is globally stable or not. If the plant is forward-complete, the plant-predictor system is globally well defined, and so is the direct backstepping transformation $W = U - \kappa(P[U, Z])$. If the plant is globally stabilizable, then the target-predictor system is globally well defined, and so is the inverse backstepping transformation $U = W + \kappa(\Pi[W, Z])$.

For global stabilization via predictor feedback we require all of the following three ingredients:

1) target system is globally asymptotically stable;
2) direct backstepping transformation is globally well defined;
3) inverse backstepping transformation is globally well defined.

The ingredients 1 and 3 are automatically satisfied by the existence of a globally stabilizing feedback in the absence of input delay ($D = 0$). As for ingredient 2, this ingredient is missing from the scalar example in [19] which is not forward-complete and thus not globally stabilizable.

To summarize our conclusions, which at this point are not supposed to be obvious but should be helpful in guiding the reader through the coming sections:

- For general systems that are globally stabilizable in the absence of input delay, including feedback linearizable systems and systems in the strict-feedback form, the target-predictor system and the inverse backstepping transformation will be globally well defined, but this is not necessarily the case for the plant-predictor system and the direct backstepping transformation. Consequently, predictor feedback will not be globally (but only regionally) stabilizing within this broad class of systems.
- For forward-complete systems that are globally stabilizable in the absence of input delay, both the plant-predictor and the target-predictor systems, and both the direct and inverse backstepping transformations, will be globally well defined. Consequently, predictor feedback will be globally stabilizing within this class, including its subclass of strict-feedforward systems.

III. Stability Proof Without a Lyapunov Function for Forward Complete Systems

The property of predictor feedback that it exactly compensates the input delay, and that after D seconds the closed-loop system evolves as if no delay were present, allows to prove stability without using Lyapunov functions. This approach would not be possible in the presence of even the most innocuous modeling uncertainties such as disturbances. For this reason, in Section VI we revisit the problem of stability proof using Lyapunov functions.

Theorem 1: Consider the closed-loop system (1), (2), (3), (4) with $f(0,0) = 0$, $\kappa(0) = 0$ and with an initial condition $Z_0 = Z(0)$ and $U_0(\theta) = U(\theta), \theta \in [-D, 0]$. Let $\dot{Z} = f(Z, \omega)$ be forward complete and $\dot{Z} = f(Z, \kappa(Z))$ be globally asymptotically stable at $Z = 0$. Then there exists a function $\hat{\beta} \in \mathcal{KL}$ such that

$$|Z(t)| + \|U\|_{L_\infty[t-D,t]} \leq \hat{\beta}\left(|Z(0)| + \|U_0\|_{L_\infty[-D,0]}, t\right) \quad (11)$$

for all $(Z_0, U_0) \in \mathbb{R}^n \times L_\infty[-D, 0]$ and for all $t \geq 0$.

Proof: From the forward completeness of $\dot{Z} = f(Z, \omega)$, from [15, Lemma 3.5], using the fact that $f(0,0) = 0$ which allows to set $R = 0$, we get that $|Z(t)| \leq \nu(t)\psi(|Z(0)| + \sup_{\theta \in [-D, t-D]} |U(\theta)|)$, with a continuous positive-valued monotonically increasing function $\nu(\cdot)$ and a function $\psi(\cdot)$ in class $\mathcal{K}$. It follows that $|Z(t)| \leq \nu(D)\psi(|Z(0)| + \sup_{\theta \in [-D,0]} |U(\theta)|)$ for all $t \in [0, D]$. Using the fact that $U(t) = \kappa(P(t)) = \kappa(Z(t+D))$ for all $t \geq 0$, and using the fact that $\dot{Z} = f(Z, \kappa(Z))$ is globally asymptotically stable at the origin, there exists a class $\mathcal{KL}$ function $\hat{\sigma}$ such that $|Z(t)| \leq \hat{\sigma}(|Z(D)|, t - D)$ for all $t \geq D$. It follows that:

$$|Z(t)| \leq \hat{\sigma}\left(\nu(D)\psi\left(|Z(0)| + \sup_{\theta \in [-D,0]} |U(\theta)|\right), \max\{0, t-D\}\right) \quad (12)$$

for all $t \geq 0$, where we have used the fact that $\hat{\sigma}(s, 0) \geq s$ for all $s \geq 0$. Due to continuity of $\kappa(\cdot)$, there exists $\hat{\rho} \in \mathcal{K}_\infty$ such that $|\kappa(p)| \leq \hat{\rho}(|p|)$. With the above expressions we get $|U(t)| \leq \hat{\rho}(\hat{\sigma}(|Z(D)|, t)) \leq \hat{\rho}(\hat{\sigma}(\nu(D)\psi(|Z(0)| + \sup_{\theta \in [-D,0]} |U(\theta)|), t))$ for all $t \geq 0$, which also implies that $\sup_{\theta \in [t-D,t]} |U(t)| \leq \hat{\rho}(\hat{\sigma}(\nu(D)\psi(|Z(0)| + \sup_{\theta \in [-D,0]} |U(\theta)|), t - D))$ for all $t \geq D$. Now we turn our attention to estimating $\sup_{\theta \in [t-D,t]} |U(\theta)|$ over $t \in [0, D]$. We split the interval $[t-D, t]$ in the following manner:

$$\begin{aligned}
&\sup_{\theta \in [t-D,t]} |U(\theta)| \\
&\quad \leq \sup_{\theta \in [t-D,0]} |U(\theta)| + \sup_{\theta \in [0,t]} |U(\theta)| \\
&\quad \leq \sup_{\theta \in [-D,0]} |U(\theta)| + \sup_{\theta \in [0,t]} |U(\theta)| \\
&\quad \leq \sup_{\theta \in [-D,0]} |U(\theta)| \\
&\qquad + \hat{\rho}\left(\hat{\sigma}\left(\nu(D)\psi\left(|Z(0)| + \sup_{\theta \in [-D,0]} |U(\theta)|\right), 0\right)\right) \\
&\quad \leq |Z(0)| + \sup_{\theta \in [-D,0]} |U(\theta)| \\
&\qquad + \hat{\rho}\left(\hat{\sigma}\left(\nu(D)\psi\left(|Z(0)| + \sup_{\theta \in [-D,0]} |U(\theta)|\right), 0\right)\right) \\
&\quad = \hat{\zeta}\left(|Z(0)| + \sup_{\theta \in [-D,0]} |U(\theta)|\right), \quad \forall t \in [0, D]
\end{aligned} \quad (13)$$

where $\hat{\zeta}(s) = s + \hat{\rho}(\hat{\sigma}(\nu(D)\psi(s), 0))$. Let us now consider the function $\eta : \mathbb{R}_+^2 \to \mathbb{R}_+$

$$\hat{\eta}(r,t) = \begin{cases} \hat{\zeta}(r), & t \in [0,D] \\ \hat{\rho}\left(\hat{\sigma}\left(\nu(D)\psi(r), t-D\right)\right), & t \geq D. \end{cases} \tag{14}$$

Since $\hat{\sigma}$ is a class $\mathcal{KL}$ function and $\hat{\zeta}, \hat{\rho}, \psi$ are class $\mathcal{K}$, there exists a class $\mathcal{KL}$ function $\hat{\xi}$ such that $\hat{\xi}(r,t) \geq \hat{\eta}(r,t)$ for all $(r,t) \in \mathbb{R}_+^2$. For example, the function $\hat{\xi}(r,t)$ can be chosen as

$$\hat{\xi}(r,t) = \begin{cases} \hat{\zeta}(r), & t \in [0,D] \\ \hat{\zeta}(r)\frac{\hat{\rho}(\hat{\sigma}(\nu(D)\psi(r),t-D))}{\hat{\rho}(\hat{\sigma}(\nu(D)\psi(r),0))}, & t > D, r > 0 \\ 0, & t > D, r = 0. \end{cases} \tag{15}$$

Hence, $\sup_{\theta\in[t-D,t]} |U(\theta)| \leq \hat{\xi}(|Z(0)| + \sup_{\theta\in[-D,0]} |U(\theta)|, t)$ for all $t \geq 0$. Adding now the bound (12), we get $|Z(t)| + \sup_{\theta\in[t-D,t]} |U(\theta)| \leq \hat{\sigma}(\nu(D)\psi(|Z(0)|+\sup_{\theta\in[-D,0]} |U(\theta)|), \max\{0, t-D\}) + \hat{\xi}(|Z(0)|+\sup_{\theta\in[-D,0]} |U(\theta)|, t)$ for all $t \geq 0$. Denoting $\hat{\beta}(r,t) = \hat{\sigma}(\nu(D)\psi(r), \max\{0, t-D\}) + \hat{\xi}(r,t)$, we complete the proof of the theorem. ■

IV. A Transport PDE Representation of the Infinite-Dimensional Backstepping Transformation

To develop Lyapunov-based tools for studying stability of nonlinear predictor feedback, we introduce a transport PDE formalism for representing the actuator state. We represent the plant as

$$\dot{Z}(t) = f\left(Z(t), u(t,0)\right) \tag{16}$$
$$u_t(x,t) = u_x(x,t) \tag{17}$$
$$u(D,t) = U(t) \tag{18}$$

and the *target system* as

$$\dot{Z}(t) = f\left(Z(t), \kappa\left(Z(t)\right) + w(t,0)\right) \tag{19}$$
$$w_t(x,t) = w_x(x,t) \tag{20}$$
$$w(D,t) = 0. \tag{21}$$

The predictor variables are represented by the following integral equations:

$$p(x,t) = \int_0^x f\left(p(\xi,t)u(\xi,t)\right) d\xi + Z(t) \tag{22}$$

$$\pi(x,t) = \int_0^x f\left(\pi(\xi,t)\kappa\left(\pi(x,t)\right) + w(\xi,t)\right) d\xi + Z(t) \tag{23}$$

where $p : [0,D] \times \mathbb{R}_+ \to \mathbb{R}^n$. It should be noted that, in the (22) and (23), t acts as a parameter. It is helpful *not* to view in its role as a time variable when thinking about solutions of these two nonlinear integral equations. The alternative form of these integral equations is as differential equations, with appropriate initial conditions, given as

$$p_x(x,t) = f\left(p(x,t), u(x,t)\right), \quad p(0,t) = Z(t) \tag{24}$$

$$\pi_x(x,t) = f\left(\pi(x,t), \kappa\left(\pi(x,t)\right) + w(x,t)\right)$$
$$\pi(0,t) = Z(t) \tag{25}$$

where we reiterate that these are equations in only one independent variable, x, so they are not PDEs but ODEs, despite our use of partial derivative notation. The variables $p(x,t)$ and $\pi(x,t)$ are used to define the backstepping transformations (direct and inverse) as

$$w(x,t) = u(x,t) - \kappa\left(p(x,t)\right) \tag{26}$$
$$u(x,t) = w(x,t) + \kappa\left(\pi(x,t)\right) \tag{27}$$

with $x \in [0,D]$.

Lemma 2: The functions $(Z(t), u(x,t))$ satisfy the (16), (17) if and only if the functions $(Z(t), w(x,t))$ satisfy the (19), (20), where the three functions $Z(t), u(x,t), w(x,t)$ are related through (22)–(27).

Proof: This result is immediate by noting that $u(x,t)$ and $w(x,t)$ are functions of only one variable, $x+t$, and therefore so are $p(x,t)$ and $\pi(x,t)$ based on the ODEs (24), (25). This implies that $p_t(x,t) = p_x(x,t)$ and $\pi_t(x,t) = \pi_x(x,t)$, from which it follows that $w_t(x,t) - w_x(x,t) = u_t(x,t) - (\partial\kappa(p(x,t))/\partial p)p_t(x,t) - (u_x(x,t) - (\partial\kappa(p(x,t))/\partial p)p_x(x,t)) = 0$ and $u_t(x,t) - u_x(x,t) = w_t(x,t) + (\partial\kappa(\pi(x,t))/\partial\pi)\pi_t(x,t) - (w_x(x,t) + (\partial\kappa(\pi(x,t))/\partial\pi)\pi_x(x,t)) = 0$, which completes the proof. ■

The variables $p(x,t)$ and $\pi(x,t)$ are used to generate the plant-predictor state $P(t) = p(D,t)$ and the target predictor state $\Pi(t) = \pi(D,t)$. From (26) and (21) the backstepping control law is given by $U(t) = u(D,t) = \kappa(p(D,t))$.

V. Lyapunov Functions for the Transport PDE

In order to be able to construct Lypunov functions for the closed-loop nonlinear system under predictor feedback, we need various Lyapunov functions for the target system's transport PDE subsystem

$$w_t(x,t) = w_x(x,t) \tag{28}$$
$$w(D,t) = 0 \tag{29}$$

where $w_0(x) = w(x,0)$ denotes its initial condition.

The following results on stability and Lyapunov functions for this system will be useful in this paper.

Theorem 3: Consider the functional $V(t) = \int_0^D \mathrm{e}^{gx}\delta(|w(x,t)|)dx$, where $w(x,t)$ is the classical solution of the system (28), (29), g is any positive constant, and δ is any function in class $\mathcal{K}$. Then, for all $t \geq 0$

$$\dot{V}(t) = -\delta\left(|w(0,t)|\right) - gV(t) \tag{30}$$

$$\int_0^D \delta\left(|w(x,t)|\right) dx \leq \mathrm{e}^{g(D-t)} \int_0^D \delta\left(|w_0(x)|\right) dx. \tag{31}$$

Proof: The derivative of $V(t)$ is

$$\dot{V}(t) = \int_0^D \mathrm{e}^{gx}\delta'\left(|w(x,t)|\right) \operatorname{sgn}\left\{w(x,t)\right\} w_t(x,t)dx$$
$$= \int_0^D \mathrm{e}^{gx}\delta'\left(|w(x,t)|\right) \operatorname{sgn}\left\{w(x,t)\right\} w_x(x,t)dx$$

$$
\begin{aligned}
&= \int_0^D e^{gx}\delta'\left(|w(x,t)|\right)\operatorname{sgn}\left\{w(x,t)\right\} dw(x,t)\\
&= \int_0^D e^{gx}\delta'\left(|w(x,t)|\right) d\,|w(x,t)|\\
&= \int_0^D e^{gx} d\delta\left(|w(x,t)|\right)\\
&= e^{gx}\delta\left(|w(x,t)|\right)|_0^D - g\int_0^D e^{gx}\delta\left(|w(x,t)|\right)dx\\
&= -\delta\left(|w(x,0)|\right) - g\int_0^D e^{gx}\delta\left(|w(x,t)|\right)dx
\end{aligned}
\tag{32}
$$

which yields (30). Hence, we get $V(t) \leq V_0 e^{-gt}$. Next, we observe that $\int_0^D \delta(|w(x,t)|)dx \leq V(t) \leq e^{gD}\int_0^D \delta(|w(x,t)|)dx$. Combining the last two inequalities, we obtain (31). ■

Taking $\delta(r) = r^p$ and $g = bp$ for $p, b > 0$, we obtain the following corollary.

Corollary 4: The following holds for the system (28), (29):

$$
\|w(t)\|_{L_p[0,D]} \leq e^{b(D-t)}\|w_0\|_{L_p[0,D]}, \quad \forall t \geq 0 \tag{33}
$$

for any $b > 0$ and any $p \in [1, \infty)$.

This corollary does not cover the case $p = \infty$, which we are also interested in. This result is proved separately.

Theorem 5: Consider the functional $V(t) = \sup_{x\in[0,D]}|e^{cx}w(x,t)|$, where $w(x,t)$ is the classical solution of the system (28), (29), and c is any positive constant. Then, for all $t \geq 0$

$$
\dot V(t) \leq -cV(t) \tag{34}
$$

$$
\|w(t)\|_\infty \leq e^{c(D-t)}\|w_0\|_\infty. \tag{35}
$$

Proof: Let $\|w(t)\|_{c,\infty}$ denote the following "spatially weighted norm:"

$$
\begin{aligned}
\|w(t)\|_{c,\infty} &= \sup_{x\in[0,D]}|e^{cx}w(x,t)|\\
&= \lim_{n\to\infty}\left(\int_0^D e^{2ncx}w(x,t)^{2n}dx\right)^{\frac{1}{2n}}
\end{aligned}
\tag{36}
$$

where n is a positive integer. Then the derivative of $V(t)$ is given by

$$
\begin{aligned}
\dot V(t) &= \lim_{n\to\infty}\frac{d}{dt}\left(\int_0^D e^{2ncx}w(x,t)^{2n}dx\right)^{\frac{1}{2n}}\\
&= \lim_{n\to\infty}\frac{1}{2n}\left(\int_0^D e^{2ncx}w(x,t)^{2n}dx\right)^{\frac{1}{2n}-1}\\
&\quad\times\int_0^D e^{2ncx}\frac{d}{dt}w(x,t)^{2n}dx\\
&= \lim_{n\to\infty}\frac{1}{2n}\left(\int_0^D e^{2ncx}w(x,t)^{2n}dx\right)^{\frac{1}{2n}-1}\\
&\quad\times\int_0^D e^{2ncx}2nw(x,t)^{2n-1}w_t(x,t)dx\\
&= \lim_{n\to\infty}\frac{1}{2n}\left(\int_0^D e^{2ncx}w(x,t)^{2n}dx\right)^{\frac{1}{2n}-1}\\
&\quad\times\int_0^D e^{2ncx}2nw(x,t)^{2n-1}w_x(x,t)dx\\
&= \lim_{n\to\infty}\frac{1}{2n}\left(\int_0^D e^{2ncx}w(x,t)^{2n}dx\right)^{\frac{1}{2n}-1}\\
&\quad\times\int_0^D e^{2ncx}2nw(x,t)^{2n-1}dw(x,t)\\
&= \lim_{n\to\infty}\frac{1}{2n}\left(\int_0^D e^{2ncx}w(x,t)^{2n}dx\right)^{\frac{1}{2n}-1}\\
&\quad\times\int_0^D e^{2ncx}dw(x,t)^{2n}.
\end{aligned}
\tag{37}
$$

With integration by parts we get

$$
\begin{aligned}
\dot V(t) &= \lim_{n\to\infty}\frac{1}{2n}\left(\int_0^D e^{2ncx}w(x,t)^{2n}dx\right)^{\frac{1}{2n}-1}\\
&\quad\times\left(e^{2ncx}w(x,t)^{2n}|_0^D - 2nc\int_0^D e^{2ncx}dw(x,t)^{2n}\right)\\
&= \lim_{n\to\infty}\frac{1}{2n}\left(\int_0^D e^{2ncx}w(x,t)^{2n}dx\right)^{\frac{1}{2n}-1}\\
&\quad\times\left(-w(0,t)^{2n} - 2nc\int_0^D e^{2ncx}dw(x,t)^{2n}\right)\\
&= -\lim_{n\to\infty}w(0,t)^{2n}\frac{1}{2n}\left(\int_0^D e^{2ncx}w(x,t)^{2n}dx\right)^{\frac{1}{2n}-1}\\
&\quad - c\lim_{n\to\infty}\left(\int_0^D e^{2ncx}w(x,t)^{2n}dx\right)^{\frac{1}{2n}}\\
&= -\lim_{n\to\infty}w(0,t)^{2n}\frac{1}{2n}\left(\int_0^D e^{2ncx}w(x,t)^{2n}dx\right)^{\frac{1}{2n}-1}\\
&\quad - cV(t)
\end{aligned}
\tag{38}
$$

which yields (34) and finally $V(t) \leq V_0 e^{-ct}$. Then one gets (35) as follows:

$$\|w(t)\|_\infty \leq \|w(t)\|_{c,\infty} \leq \|w_0\|_{c,\infty} e^{-ct} \leq \|w_0\|_\infty e^{cD} e^{-ct}. \tag{39}$$

■

The following Lyapunov fact follows from Theorem 5.

Lemma 6: For any $h \in \mathcal{K}$ with $h'(0) < \infty$ and any $c > 0$, the derivative of the function $\breve{V}(t) = \int_0^{\|w(t)\|_{c,\infty}} (h(r)/r) dr$ along the classical solutions of the system (28), (29) is given by $\dot{\breve{V}}(t) \leq -ch(\|w(t)\|_{c,\infty})$.

VI. Stability Analysis for Forward-Complete Nonlinear Systems

The stability proof in Section III does not employ Lyapunov functionals but exploits properties of exact solutions to the closed-loop system. The absence of a Lyapunov functional would prevent a study of stability in the presence of disturbances and other uncertainties. Availability of a Lyapunov function is also important if one wants to conduct an inverse optimal re-design of the feedback law. In this section we construct a Lyapunov functional for the system and conduct a proof of global stability on the basis of this functional. A Lypunov construction requires that we somewhat strengthen the assumptions of forward completeness and global stabilizability.

Definition 6.1: System $\dot{Z} = f(Z, \omega)$ with $f(0,0) = 0$ is strongly forward complete if there exists a smooth function $R: \mathbb{R}^n \to \mathbb{R}_+$ and class $\mathcal{K}_\infty$ functions $\alpha_1, \alpha_2, \sigma$ such that

$$\alpha_1(|Z|) \leq R(Z) \leq \alpha_2(|Z|) \tag{40}$$

$$\frac{\partial R(Z)}{\partial Z} f(Z, \omega) \leq R(Z) + \sigma(|\omega|) \tag{41}$$

for all $Z \in \mathbb{R}^n$ and for all $\omega \in \mathbb{R}$.

This property differs from standard forward completeness [1] in the sense that we assume that $f(0,0) = 0$ and, in accordance with that, also assume that the function $R(\cdot)$ is positive definite.

Assumption 6.1: The system $\dot{Z} = f(Z, \omega)$ is strongly forward complete.

Assumption 6.2: The system $\dot{Z} = f(Z, \kappa(Z) + \omega)$ is input-to-state stable.

Before we proceed to our Lyapunov construction, we state a bound on the norm of the plant-predictor system.

Lemma 7: Let system (24) satisfy Assumption 6.1. There exists a function $\rho_1 \in \mathcal{K}_\infty$ such that

$$\|p(t)\|_{L_\infty[0,D]} \leq \rho_1 \left(|Z(t)| + \|u(t)\|_{L_\infty[0,D]} \right). \tag{42}$$

Proof: With the Lyapunov-like function $R(p(x,t))$ we get that

$$\frac{\partial R(p(x,t))}{\partial p} f(p(x,t), u(x,t)) \leq R(p(x,t)) + \sigma(|u(x,t)|) \tag{43}$$

from which it follows that:

$$\begin{aligned} R(p(x,t)) &\leq e^x R(p(0,t)) + \int_0^x e^{x-\xi} \sigma(|u(\xi,t)|) d\xi \\ &= e^x R(Z(t)) + \int_0^x e^{x-\xi} \sigma(|u(\xi,t)|) d\xi \\ &\leq e^x R(Z(t)) + (e^x - 1) \sup_{0 \leq \xi \leq x} \sigma(|u(\xi,t)|). \end{aligned} \tag{44}$$

Using (40), we get that $\alpha_1(|p(x,t)|) \leq e^x \alpha_2(|Z(t)|) + (e^x - 1) \sup_{0\leq\xi\leq x} \sigma(|u(\xi,t)|)$, which yields $|p(x,t)| \leq \alpha_1^{-1}(e^x \alpha_2(|Z(t)|) + (e^x - 1)\sup_{0\leq\xi\leq x} \sigma(|u(\xi,t)|)) \leq \alpha_1^{-1}(e^D \alpha_2(|Z(t)|) + (e^D - 1)\sigma(\sup_{0\leq\xi\leq x} |u(\xi,t)|))$. With standard properties of class $\mathcal{K}_\infty$ functions, we get the result of the lemma. ■

The next two lemmas relate the norms of the plant and of the target system.

Lemma 8: Let system (24) satisfy Assumption 6.1 and consider (26) as its output map. Then there exists a function $\rho_2 \in \mathcal{K}_\infty$ such that

$$|Z(t)| + \|w(t)\|_{L_\infty[0,D]} \leq \rho_2 \left(|Z(t)| + \|u(t)\|_{L_\infty[0,D]} \right). \tag{45}$$

Proof: With $|\kappa(p)| \leq \hat{\rho}(|p|)$, (26), and Lemma 7. ■

Lemma 9: Let system (25) satisfy Assumption 6.2 and consider (27) as its output map. Then there exists a function $\rho_3 \in \mathcal{K}_\infty$ such that

$$|Z(t)| + \|u(t)\|_{L_\infty[0,D]} \leq \rho_3 \left(|Z(t)| + \|w(t)\|_{L_\infty[0,D]} \right). \tag{46}$$

Proof: Under Assumption 6.2, there exists $\beta_1 \in \mathcal{KL}$ and $\gamma_1 \in \mathcal{K}$ such that $|\pi(x,t)| \leq \beta_1(|\pi(0,t)|, x) + \gamma_1(\sup_{0\leq\xi\leq x} |w(x,t)|) \leq \beta_1(|Z(t)|, x) + \gamma_1(\sup_{0\leq\xi\leq x} |w(x,t)|)$. Taking a supremum of both sides in x, we get that $\|\pi(t)\|_{L_\infty[0,D]} \leq \beta_1(|Z(t)|, 0) + \gamma_1(\|w(t)\|_{L_\infty[0,D]})$. With $|\kappa(p)| \leq \hat{\rho}(|p|)$ and (27) we obtain the result of the lemma. ■

Now we turn out attention to the full target system (19)–(21). Based on Assumption 6.2, there exists a smooth function $S: \mathbb{R}^n \to \mathbb{R}_+$ and class $\mathcal{K}_\infty$ functions $\alpha_3, \alpha_4, \alpha_5, \alpha_6$ such that

$$\alpha_3(|Z(t)|) \leq S(Z(t)) \leq \alpha_4(|Z(t)|) \tag{47}$$

$$\begin{aligned} &\frac{\partial S(Z(t))}{\partial Z} f(Z(t), \kappa(Z(t)) + w(0,t)) \\ &\quad \leq -\alpha_5(|Z(t)|) + \alpha_6(|w(0,t)|) \end{aligned} \tag{48}$$

for all $Z \in \mathbb{R}^n$ and for all $\omega \in \mathbb{R}$. Suppose that $\alpha_6(r)/r$ is a class $\mathcal{K}$ function, or α_6 has been appropriately majorized so this is true (with no generality lost). Take a Lyapunov function

$$V(t) = S(Z(t)) + \frac{2}{c} \int_0^{\|w(t)\|_{c,\infty}} \frac{\alpha_6(r)}{r} dr \tag{49}$$

where $c > 0$. This Lyapunov function is positive definite and radially unbounded (due to the assumption on α_6). With it we get the following result on stability in the norm of the target system.

Lemma 10: Let system (19)–(21) satisfy Assumption 6.2. There exists a function $\beta_4 \in \mathcal{KL}$ such that

$$|Z(t)| + \|w(t)\|_{L_\infty[0,D]} \leq \beta_4 \left(|Z(0)| + \|w(0)\|_{L_\infty[0,D]}, t \right). \tag{50}$$

Proof: From Lemma 6 we get that

$$\begin{aligned} \dot{V}(t) \leq &- \alpha_5(|Z(t)|) + \alpha_6(|w(0,t)|) - 2\alpha_6\left(\|w(t)\|_{c,\infty}\right) \\ \leq &- q\alpha_5(|Z(t)|) + \alpha_6\left(\sup_{x\in[0,D]} |w(x,t)|\right) \\ &- 2\alpha_6\left(\|w(t)\|_{c,\infty}\right) \end{aligned}$$

$$\leq -\alpha_5(|Z(t)|) + \alpha_6\left(\sup_{x\in[0,D]} |e^{cx}w(x,t)|\right) - 2\alpha_6\left(\|w(t)\|_{c,\infty}\right) \leq -\alpha_5(|Z(t)|) - \alpha_6\left(\|w(t)\|_{c,\infty}\right). \quad (51)$$

It follows, with the help of (47), that there exists $\alpha_7 \in \mathcal{K}$ so that $\dot{V}(t) \leq -\alpha_7(V(t))$ and then there exists a class $\mathcal{KL}$ function $\beta_2(\cdot,\cdot)$ such that $V(t) \leq \beta_2(V(0),t)$ for all $t \geq 0$. With additional routine class $\mathcal{K}$ calculations, using the definition (51), one can show that there exists a function $\beta_3 \in \mathcal{KL}$ such that $|Z(t)| + \|w(t)\|_{c,\infty} \leq \beta_3(|Z(0)| + \|w(0)\|_{c_\infty}, t)$. From (39) we get $\|w(t)\|_{L_\infty[0,D]} \leq \|w(t)\|_{c,\infty}$ and $\|w(0)\|_{c,\infty} \leq e^{cD}\|w(0)\|_{L_\infty[0,D]}$. Hence, $|Z(t)| + \|w(t)\|_{L_\infty[0,D]} \leq \beta_3(|Z(0)| + e^{cD}\|w(0)\|_{L_\infty[0,D]}, t)$, with which we arrive at the result of the lemma. ■

By combining Lemmas 8, 9, and 10, we get that $|Z(t)| + \|u(t)\|_{L_\infty[0,D]} \leq \rho_3(\beta_4(\rho_2(|Z(0)|+\|u(0)\|_{L_\infty[0,D]}), t))$. To summarize we obtain the following main result on closed-loop stability under predictor feedback in terms of the system norm of the original plant.

Theorem 11: Let system (19)–(21) satisfy Assumptions 6.1 and 6.2. Then there exists a function $\beta_5 \in \mathcal{KL}$ such that

$$|Z(t)| + \|u(t)\|_{L_\infty[0,D]} \leq \beta_5\left(|Z(0)| + \|u(0)\|_{L_\infty[0,D]}, t\right). \quad (52)$$

A slightly different and relevant way to state the same global asymptotic stability result is as follows.

Corollary 12: Let system (19)–(21) satisfy Assumptions 6.1 and 6.2. Then

$$|Z(t)| + \sup_{t-D\leq\theta\leq t} |U(\theta)| \leq \beta_5\left(|Z(0)| + \sup_{-D\leq\theta\leq 0} |U(\theta)|, t\right). \quad (53)$$

The norm on the delay state used in Theorem 11 and Corollary 12 is a somewhat nonstandard norm in the delay system literature. Stability in the sense of other norms also holds. To see this, take a Lyapunov functional $V(t) = S(Z(t)) + \int_0^D e^{gx}\delta(|w(x,t)|)dx$, where $g > 0$ and $\delta \in \mathcal{K}_\infty$. With the help of (30) and (48), its derivative is $\dot{V}(t) \leq -\alpha_5(|Z(t)|) + \alpha_6(|w(0,t)|) - \delta(|w(0,t)|) - g\int_0^D e^{gx}\delta(|w(x,t)|)dx$. With some routine class $\mathcal{K}$ majorizations, the following result is obtained.

Theorem 13: Let system (19)–(21) satisfy Assumptions 6.1 and 6.2. Then, for any class $\mathcal{K}_\infty$ function δ such that $\delta(r) \geq \alpha_6(r)$ for all $r \geq 0$, there exists a function $\beta_6 \in \mathcal{KL}$ such that $|Z(t)|^2 + \int_{t-D}^t \delta(|U(\theta)|)d\theta \leq \beta_6(|Z(0)|^2 + \int_{-D}^0 \delta(|U(\theta)|)d\theta, t)$.

Note that $\delta(\cdot)$ allows a significant degree of freedom in terms of relative (functional) weighting of the ODE state and the actuator state, however, this extra freedom is 'paid for' through $\beta_6(\cdot,\cdot)$.

The following example illustrates the nonlinear predictor feedback for a system that is forward complete.

Example 6.1: Consider the system $\dot{Z}_1(t) = Z_2(t)$, $\dot{Z}_2(t) = \sin(Z_1(t)) + U(t-D)$, which is motivated by the pendulum problem with torque control (one can view Z_1 as the angle relative to the upward equilibrium, and Z_2 as the angular velocity). A predictor-based feedback law for stabilization at the origin is given by $U(t) = -\sin(P_1(t)) - P_1(t) - P_2(t)$, $P_1(t) = \int_{t-D}^t P_2(\theta)d\theta + Z_1(t)$, $P_2(t) = \int_{t-D}^t [\sin(P_1(\theta)) + U(\theta)]d\theta + Z_2(t)$, with an appropriate initial condition on $P(\theta)$. The closed-loop system can be shown to be globally exponentially stable in terms of the norm $\left(Z_1^2(t) + Z_2^2(t) + \int_{t-D}^t U^2(\theta)d\theta\right)^{1/2}$ by employing quadratic choices for $S, R, \alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6, \delta$.

VII. Strict-Feedforward Systems

We now focus on a special subclass of the class of forward-complete systems—the strict-feedforward systems. The similarity in name is pure coincidence. For *forward*-complete systems, the forward refers to the direction of time. Such systems have finite solutions for all finite *positive* time. With feed*forward* systems, the word forward refers to the absence of feedback in the structure of the system. The system consists of a particular cascade of scalar systems.

Feedforward systems have received a large amount of attention since the early 1990s, starting with the introduction of this class and the first feedback laws in Teel's thesis [54], followed by the subsequent developments by Praly and Mazenc [33] and Jankovic, Sepulchre, and Kokotovic [13], and continuing with extensions and generalizations by many authors [2], [6], [11], [12], [14], [18], [26]–[29], [31], [32], [34]–[36], [48], [49], [51]–[53], [56], [57], [59].

While forward complete systems yield global stability when predictor feedback is applied to them, the strict-feedforward systems have an additional property that, despite being nonlinear, they can be solved explicitly. The consequence of this is that the predictor state can be defined explicitly. Related to this, the direct infinite-dimensional backstepping transformation can be explicitly constructed.

A special subclass of strict-feedforward systems exists, which are linearizable by coordinate change (Section VIII). For these systems, not only is the predictor state explicitly defined, but the closed-loop solutions can be found explicitly.

We introduce these ideas first through an example.

A. Example: A Second-Order Strict-Feedforward Nonlinear System

Consider the second order system

$$\dot{Z}_1(t) = Z_2(t) - Z_2^2(t)U(t-D), \quad \dot{Z}_2(t) = U(t-D). \quad (54)$$

This system is the simplest 'interesting' example of a strict-feedforward system. The nominal $(D = 0)$ controller is

$$U(t) = -Z_1(t) - 2Z_2(t) - \frac{1}{3}Z_2^3(t) \quad (55)$$

and it results in the closed loop system $\dot{\zeta}_1(t) = \zeta_2(t)$, $\dot{\zeta}_2(t) = -\zeta_1(t) - \zeta_2(t)$, where $\zeta(t)$ is defined by the diffeomorphic transformation

$$\zeta_1(t) = Z_1(t) + Z_2(t) + \frac{1}{3}Z_2^3(t), \quad \zeta_2(t) = Z_2(t). \quad (56)$$

The predictor is found by solving explicitly the nonlinear ODE $(\partial/\partial x)p_1(x,t) = p_2(x,t) - p_2^2(x,t)u(x,t)$, $(\partial/\partial x)p_2(x,t) = u(x,t)$ with initial conditions $p_1(0,t) = Z_1(t)$, $p_2(0,t) = Z_2(t)$. The control is given by

$$U(t) = -P_1(t) - 2P_2(t) - \frac{1}{3}P_2^3(t) \quad (57)$$

where $P_1(t) = p_1(D,t)$ and $P_2(t) = p_2(D,t)$ are given by

$$P_1(t) = Z_1(t) + DZ_2(t) + \int_{t-D}^{t} (t-\theta)U(\theta)d\theta - Z_2^2(t)\int_{t-D}^{t} U(\theta)d\theta - Z_2(t)\left(\int_{t-D}^{t} U(\theta)d\theta\right)^2 - \frac{1}{3}\left(\int_{t-D}^{t} U(\theta)d\theta\right)^3 \tag{58}$$

$$P_2(t) = Z_2(t) + \int_{t-D}^{t} U(\theta)d\theta. \tag{59}$$

The control law is a nonlinear infinite-dimensional operator, but it is given explicitly. The backstepping transformation is also given explicitly

$$w(x,t) = u(x,t) + p_1(x,t) + 2p_2(x,t) + \frac{1}{3}p_2^3(x,t) \tag{60}$$

$$p_1(x,t) = Z_1(t) + xZ_2(t) + \int_0^x (x-y)u(y,t)dy - Z_2^2(t)\int_0^x u(y,t)dy - Z_2(t)\left(\int_0^x u(y,t)dy\right)^2 - \frac{1}{3}\left(\int_0^x u(y,t)dy\right)^3 \tag{61}$$

$$p_2(x,t) = Z_2(t) + \int_0^x u(y,t)dy. \tag{62}$$

Now we derive the inverse transformation. This transformation is given by $u(x,t) = w(x,t) - \pi_1(x,t) - 2\pi_2(x,t) - (1/2)\pi_2^3$, where $\pi_1(x,t)$ and $\pi_2(x,t)$ are the solutions of the ODE $(\partial/\partial x)\pi_1(x,t) = \pi_2(x,t) + \pi_2^2(x,t)(\pi_1(x,t)+2\pi_2(x,t) + (1/2)\pi_2^3 - w(x,t))$, $(\partial/\partial x)\pi_2(x,t) - \pi_1(x,t) - 2\pi_2(x,t) - (1/2)\pi_2^3 + w(x,t)$, with initial condition $\pi_1(0,t) = Z_1(t)$, $\pi_2(0,t) = Z_2(t)$. It is hard to imagine that one could solve these ODEs for $\pi_1(x,t)$ and $\pi_2(x,t)$ directly. Indeed, for the general strict-feedback class, the π-system won't be solvable explicitly.

However, the present example is in the special subclass of linearizable strict-feedforward systems. Specifically, with a change of variable

$$h_1(t) = Z_1(t) + \frac{1}{3}Z_2^3(t), \quad h_2(t) = Z_2(t) \tag{63}$$

the plant (54), can be converted into

$$\dot{h}_1(t) = h_2(t), \quad \dot{h}_2(t) = U(t-D). \tag{64}$$

Then the inverse backstepping transformation is given by

$$u(x,t) = w(x,t) - \eta_1(x,t) - 2\eta_2(x,t) \tag{65}$$

where the functions $\eta_1(x,t)$ and $\eta_2(x,t)$ are defined through the ODE $(\partial/\partial x)\eta_1(x,t) = \eta_2(x,t)$, $(\partial/\partial x)\eta_2(x,t) - \eta_1(x,t) - 2\eta_2(x,t) + w(x,t)$ with initial conditions $\eta_1(0,t) = h_1(t) = Z_1(t) + (1/3)Z_2^3(t)$, $\eta_2(0,t) = h_2(t) = Z_2(t)$. This ODE is linear and we will solve it explicitly. For this, we need the matrix exponential $e^{\begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix}x} = e^{-x}\begin{bmatrix} 1+x & x \\ -x & 1-x \end{bmatrix}$. With the help of this matrix exponential, we find the solution

$$\eta_1(x,t) = e^{-x}\left[(1+x)Z_1(t) + (1+x)\frac{1}{3}Z_2^3(t) + xZ_2(t)\right] + \int_0^x (x-y)e^{-(x-y)}w(y,t)dy \tag{66}$$

$$\eta_2(x,t) = e^{-x}\left[-xZ_1(t) - x\frac{1}{3}Z_2^3(t) + (1-x)Z_2(t)\right] + \int_0^x (1-x+y)e^{-(x-y)}w(y,t)dy \tag{67}$$

with which we obtain an explicit definition of the inverse backstepping transformation (65). Like the direct one, this transformation is nonlinear and infinite-dimensional.

To summarize, for the example nonlinear plant (54), we have obtained both the transformation $(Z(t), u(x,t)) \mapsto (Z(t), w(x,t))$ and its inverse $(Z(t), w(x,t)) \mapsto (Z(t), u(x,t))$ explicitly.

Now we discuss the target system given by $\dot{\zeta}_1(t) = -\zeta_2(t) + w(0,t)$, $\dot{\zeta}_2(t) = -\zeta_1(t) - \zeta_2(t) + w(0,t)$, $w_t(x,t) = w_x(x,t)$, $w(D,t) = 0$, where the variables (ζ_1, ζ_2) are defined as in (56). This target system is a cascade of the exponentially stable transport PDE for $w(x,t)$ and the linear exponentially stable ODE for $\zeta(t)$, which allows us to establish the following stability result for the closed-loop system.

Proposition 14: Consider the plant (54), in closed loop with the controller (57)–(59). Its equilibrium at the origin $(Z,u) \equiv 0$ is globally asymptotically stable and locally exponentially stable in terms of the norm

$$\left(Z_1^2(t) + Z_2^2(t) + \int_0^D u^2(x,t)dx\right)^{1/2}. \tag{68}$$

Proof: We first perform a stability analysis of the (ζ, w) system using a standard Lyapunov functional as in [23], obtaining a stability estimate in terms of the norm

$$\left(\zeta_1^2(t) + \zeta_2^2(t) + \int_0^D w^2(x,t)dx\right)^{1/2}. \tag{69}$$

Then, we turn to the direct backstepping transformation (60)–(62) and to the forwarding transformation (56) to obtain a bound on the initial value of the norm (68) in terms of the initial value of the norm (69). Note that the relation between these norms would be nonlinear. Finally, we invoke the direct backstepping transformation (65) with (66), (67), as well as the inverse of the forwarding transformation (56), to bound (68) in terms of (69). This yields the result that there exists a class $\mathcal{KL}$ function $\beta(\cdot,\cdot)$ such that

$$Z_1^2(t)+Z_2^2(t)+\int_0^D u^2(x,t)dx \leq \beta\left(Z_1^2(0)+Z_2^2(0)+\int_0^D u^2(x,0)dx, t\right) \quad (70)$$

for all $t \geq 0$. In addition, due to the fact that the target system (ζ, w) is exponentially stable, and due to the fact that all the transformations and inverse transformations, though nonlinear, have a locally linear component, we obtain that $\beta(\cdot,\cdot)$ is locally linear in the first argument and exponentially decaying in t for sufficiently large t. ■

When we plug the predictor (58), (59) into the control law (57) we obtain the feedback

$$U(t)=-Z_1(t)-(2+D)Z_2(t)-\frac{1}{3}Z_2^3(t)-\int_{t-D}^{t}(2+t-\theta)U(\theta)d\theta. \quad (71)$$

Comparing with the nominal controller (55), this appears to be an amazing simplification of (57). This kind of a simplification won't be possible for strict-feedforward systems in general, but only for the subclass of strict-feedforward systems which are linearizable by a diffeomorphic change of coordinates. This simplified feedback could have been obtained by starting directly from the linearized system (64) and by applying linear predictor design from [23].

The linearizability of this example by coordinate change allows us to go beyond the result of Proposition 14. The open-loop system in variables (Z,u) is given by $\dot{Z}_1(t)=Z_2(t)-Z_2^2(t)u(0,t)$, $\dot{Z}_2(t)=u(0,t)$, $u_t(x,t)=u_x(x,t)$, $u(D,t)=U(t)$, whereas the closed-loop system in the variables (h,w) is $\dot{h}_1(t)=h_2(t)$, $\dot{h}_2(t)=-h_1(t)-2h_2(t)+w(0,t)$, $w_t(x,t)=w_x(x,t)$, $w(D,t)=0$. The transformations $(Z,u)\mapsto(h,w)$ and $(h,w)\mapsto(Z,u)$ employ the formulae $w(x,t)=u(x,t)+\int_0^x(2+x-y)u(y,t)dy+Z_1(t)+(2+x)Z_2(t)+(1/3)Z_2^3(t)$ and $u(x,t)=w(x,t)-\int_0^x(2-x+y)\mathrm{e}^{-(x-y)}w(y,t)dy-\mathrm{e}^{-x}[(1-x)h_1(t)+(2-x)h_2(t)]$. As we shall see in Example 8.1 in Section VIII, with these explicit transformations, one can not only get an easy estimate of the function $\beta(\cdot,\cdot)$ in (70), but one can find explicit solutions of the closed-loop system.

B. General Strict-Feedforward Nonlinear Systems: Integrator Forwarding

Consider the class of *strict-feedforward systems*

$$\dot{Z}_1(t)=Z_2(t)+\psi_1(Z_2(t),Z_3(t),\ldots,Z_n(t)) + \phi_1(Z_2(t),Z_3(t),\ldots,Z_n(t))u(0,t) \quad (72)$$

$$\vdots \quad (73)$$

$$\dot{Z}_{n-2}(t)=Z_{n-1}(t)+\psi_{n-2}(Z_{n-1}(t),Z_n(t)) + \phi_{n-2}(Z_{n-1}(t),Z_n(t))u(0,t) \quad (74)$$

$$\dot{Z}_{n-1}(t)=Z_n(t)+\phi_{n-1}(Z_n(t))u(0,t) \quad (75)$$

$$\dot{Z}_n(t)=u(0,t) \quad (76)$$

with input delay

$$u_t(x,t)=u_x(x,t), \quad u(D,t)=U(t) \quad (77)$$

or, for short

$$\dot{Z}_i(t)=Z_{i+1}(t)+\psi_i\left(\underline{Z}_{i+1}(t)\right)+\phi_i\left(\underline{Z}_{i+1}(t)\right)U(t-D) \quad (78)$$

where $i=1,2,\ldots,n$, $\underline{Z}_j=[Z_j,Z_{j+1},\ldots,Z_n]^T$, $Z_{n+1}(t)=U(t-D)$, $\phi_n=1$, $\phi_i(0)=0$, $\psi_i(Z_{i+1},0,\ldots,0)\equiv 0$, $(\partial\psi_i(0)/\partial Z_j)=0$ for $i=1,2,\ldots,n-1$, $j=i+1,\ldots,n$.

The nominal control design $(D=0)$ for the class of systems (78) is given by the following recursive procedure [18], [48]. Let

$$\vartheta_{n+1}=0, \quad \alpha_{n+1}=0. \quad (79)$$

For $i=n,n-1,\ldots,2,1$, the designer needs to symbolically (preferably) or numerically calculate

$$h_i(\underline{Z}_i)=Z_i-\vartheta_{i+1}(\underline{Z}_{i+1}) \quad (80)$$

$$\omega_i(\underline{Z}_{i+1})=\phi_i-\sum_{j=i+1}^{n-1}\frac{\partial\vartheta_{i+1}}{\partial Z_j}\phi_j-\frac{\partial\vartheta_{i+1}}{\partial Z_n} \quad (81)$$

$$\alpha_i(\underline{Z}_i)=\alpha_{i+1}-\omega_i h_i \quad (82)$$

$$\vartheta_i(\underline{Z}_i)=-\int_0^\infty\left[\xi_i^{[i]}(\tau,\underline{Z}_i)+\psi_{i-1}\left(\underline{\xi}_i^{[i]}(\tau,\underline{Z}_i)\right) +\phi_{i-1}\left(\underline{\xi}_i^{[i]}(\tau,\underline{Z}_i)\right)\alpha_i\left(\underline{\xi}_i^{[i]}(\tau,\underline{Z}_i)\right)\right]d\tau \quad (83)$$

where the notation in the integrand of (83) refers to the solutions of the (sub)system(s)

$$\frac{d}{d\tau}\xi_j^{[i]}=\xi_{j+1}^{[i]}+\psi_j\left(\underline{\xi}_{j+1}^{[i]}\right)+\phi_j\left(\underline{\xi}_{j+1}^{[i]}\right)\alpha_i\left(\underline{\xi}_i^{[i]}\right) \quad (84)$$

for $j=i,i+1,\ldots,n$, at time τ, starting from the initial condition $\underline{X}_i$. The control law for $D=0$ is given by

$$U(t)=\alpha_1(Z(t)). \quad (85)$$

It is important to understand the meaning of the integral in (83). Clearly, the solution $\underline{\xi}_i(\tau,\underline{Z}_i)$ is impossible to obtain analytically in general but, when possible, will lead to an implementable control law. Note that the last of the ϑ_i's that need tobe computed is ϑ_2 (ϑ_1 is not defined).

C. Predictor for Strict-Feedforward Systems

As in the case of general nonlinear systems (Section II), the predictor-based feedback law is obtained from (85) as

$$U(t)=\alpha_1(P(t))=\alpha_1(p(D,t)) \quad (86)$$

where the predictor variable $p(D,t)=P(t)$ is defined next. Consider the ODE (in x) given by

$$\frac{\partial}{\partial x}p_1(x,t)=p_2(x,t)+\psi_1(p_2(x,t),p_3(x,t),\ldots,p_n(x,t)) +\phi_1(p_2(x,t),p_3(x,t),\ldots,p_n(x,t))u(x,t) \quad (87)$$

$$\vdots$$

$$\frac{\partial}{\partial x}p_{n-2}(x,t)=p_{n-1}(x,t)+\psi_{n-2}(p_{n-1}(x,t),p_n(x,t)) + \phi_{n-2}(p_{n-1}(x,t),p_n(x,t))u(x,t) \quad (88)$$

$$\frac{\partial}{\partial x}p_{n-1}(x,t)=p_n(x,t)+\phi_{n-1}\left(p_n(x,t)\right)u(x,t) \quad (89)$$

$$\frac{\partial}{\partial x}p_n(x,t)=u(x,t) \quad (90)$$

with an initial condition

$$p_i(0,t)=Z_i(t), \quad i=1,\dots n. \quad (91)$$

The set of equations for $p_i(x,t)$ can be solved explicitly, starting from the bottom

$$p_n(x,t)=Z_n(t)+\int_0^x u(y,t)dy \quad (92)$$

continuing on to

$$\begin{aligned} p_{n-1}(x,t) &= Z_{n-1}(t) \\ &\quad+\int_0^x \left[p_n(y,t)+\phi_{n-1}\left(p_n(y,t)\right)u(y,t)\right]dy \\ &= Z_{n-1}(t)+xZ_n(t)+\int_0^x (x-y)u(y,t)dy \\ &\quad+\int_0^x \phi_{n-1}\left(Z_n(t)+\int_0^y u(\sigma,t)ds\right)u(y,t)dy \end{aligned} \quad (93)$$

and so on. For a general i, the predictor solution is given recursively as

$$\begin{aligned} p_i(x,t)=Z_i(t)+\int_0^x &\left[p_{i+1}(y,t)+\psi_i\left(p_{i+1}(y,t),\dots,p_n(y,t)\right)\right. \\ &\left.+\phi_i\left(p_{i+1}(y,t),\dots,p_n(y,t)\right)u(y,t)\right]dy. \end{aligned} \quad (94)$$

Clearly, this procedure involves only computation of integrals with no implicit problems to solve (such as differential or integral equations). Hence, the predictor state $p(D,t)=P(t)$, where its element $P_i(t)$ is defined in terms of $Z_i(t),\cdots,Z_n(t)$ and $U(\theta),\theta\in[0,D], i=n,n-1,\dots 1$, is obtainable explicitly, due to the strict-feedforward structure of the class of systems. An example of an explicit design of a nonlinear infinite-dimensional predictor for a third-order system (which is not linearizable) is presented in (115)–(117).

D. General Strict-Feedforward Nonlinear Systems: Stability Analysis

Before we start, we need to define the π-subsystem, which is used in the inverse backstepping transformation

$$\begin{aligned} \frac{\partial}{\partial x}\pi_1(x,t) &= \pi_2(x,t)+\psi_1\left(\pi_2(x,t),\dots,\pi_n(x,t)\right) \\ &\quad+\phi_1\left(\pi_2(x,t),\dots,\pi_n(x,t)\right) \\ &\quad\times\left(\alpha_1\left(\pi(x,t)\right)+w(x,t)\right) \end{aligned} \quad (95)$$

$$\vdots$$

$$\begin{aligned} \frac{\partial}{\partial x}\pi_{n-2}(x,t) &= \pi_{n-1}(x,t)+\psi_{n-2}\left(\pi_{n-1}(x,t),\pi_n(x,t)\right) \\ &\quad+\phi_{n-2}\left(\pi_{n-1}(x,t),\pi_n(x,t)\right) \\ &\quad\times\left(\alpha_1\left(\pi(x,t)\right)+w(x,t)\right) \end{aligned} \quad (96)$$

$$\begin{aligned} \frac{\partial}{\partial x}\pi_{n-1}(x,t) &= \pi_n(x,t)+\phi_{n-1}\left(\pi_n(x,t)\right) \\ &\quad\times\left(\alpha_1\left(\pi(x,t)\right)+w(x,t)\right) \end{aligned} \quad (97)$$

$$\frac{\partial}{\partial x}\pi_n(x,t)=\left(\alpha_1\left(\pi(x,t)\right)+w(x,t)\right) \quad (98)$$

with an initial condition

$$\pi_i(0,t)=Z_i(t), \quad i=1,\dots n. \quad (99)$$

The following result follows immediately from (26) and (27).

Lemma 15: If the mapping $\kappa:\mathbb{R}^n\to\mathbb{R}$ in (26), (27) is continuous and $\kappa(0)=0$, there exists a class $\mathcal{K}_\infty$ function ρ_1 such that

$$\|w(t)\|^2 \le 2\|u(t)\|^2+D\rho_1\left(\|p(t)\|_{L_\infty[0,D]}\right) \quad (100)$$

$$\|u(t)\|^2 \le 2\|w(t)\|^2+D\rho_1\left(\|\pi(t)\|_{L_\infty[0,D]}\right). \quad (101)$$

With Lemmas 16–18, presented next, we establish an estimate on the target system norm in terms of the plant norm.

Lemma 16: For the system

$$\begin{aligned} \frac{\partial}{\partial x}p_i(x,t) &= p_{i+1}(x,t)+\psi_i\left(p_{i+1}(x,t),\dots,p_n(x,t)\right) \\ &\quad+\phi_i\left(p_{i+1}(x,t),\dots,p_n(x,t)\right)u(x,t) \\ &\quad i=1,\dots,n-1 \end{aligned} \quad (102)$$

$$\frac{\partial}{\partial x}p_n(x,t)=u(x,t) \quad (103)$$

with initial condition $p(0,t)=Z(t)$, the following bounds hold:

$$|p_i(x,t)| \le |Z_i(t)|+\nu_i\left(\|p_{i+1}(t)\|_{L_\infty[0,D]}\right)\left(1+\|u(t)\|\right) \quad i=1,\dots,n-1 \quad (104)$$

$$|p_n(x,t)| = |Z_n(t)|+D\|u(t)\| \quad (105)$$

for all $x\in[0,D]$ and all $t\ge 0$, where $\nu_i(\cdot)$ are class $\mathcal{K}$ functions.

Proof: First, we note that (105) follows from (92). Then with the Cauchy–Schwartz inequality, from (94) we get

$$\begin{aligned} |p_i(x,t)| \le |Z_i(t)| &+\int_0^x \left|\psi_i\left(p_{i+1}(y,t),\dots,p_n(y,t)\right)\right|dy \\ &+\left(\int_0^D \phi_i^2\left(p_{i+1}(y,t),\dots,p_n(y,t)\right)dy\right)^{1/2}\|u(t)\|. \end{aligned} \quad (106)$$

With a suitably chosen class $\mathcal{K}$ function λ_i, we get

$$\begin{aligned} |p_i(x,t)| &\le |Z_i(t)|+\int_0^D \lambda_i\left(\left|\underline{p}_{i+1}(y,t)\right|\right)dy \\ &\quad+\left(\int_0^D \lambda_i\left(\left|\underline{p}_{i+1}(y,t)\right|\right)dy\right)^{1/2}\|u(t)\| \\ &\le |Z_i(t)|+\int_0^D \lambda_i\left(\left\|\underline{p}_{i+1}(t)\right\|_{L_\infty[0,D]}\right)dy \\ &\quad+\left(\int_0^D \lambda_i\left(\left\|\underline{p}_{i+1}(t)\right\|_{L_\infty[0,D]}\right)dy\right)^{1/2}\|u(t)\| \end{aligned}$$

$$\leq |Z_i(t)| + D\lambda_i\left(\left\|\underline{p}_{i+1}(t)\right\|_{L_\infty[0,D]}\right) + \left(D\lambda_i\left(\left\|\underline{p}_{i+1}(t)\right\|_{L_\infty[0,D]}\right)\right)^{1/2}\|u(t)\|. \quad (107)$$

Taking $\nu_i(\cdot) = \max\{D\lambda_i(\cdot), \sqrt{D\lambda_i(\cdot)}\}$, we complete the proof of the lemma. ■

By successive application of Lemma 16, in the order $i = n-1, n-2, \ldots, 2, 1$, we obtain that there exist class $\mathcal{K}$ functions $\sigma_i(\cdot)$ such that $|p_i(x,t)| \leq \sigma_i(|\underline{Z}_i(t)| + \|u(t)\|)$, $i = 1, \ldots, n$, for all $x \in [0, D]$ and all $t \geq 0$. Hence, the following result holds.

Lemma 17: There exists a class $\mathcal{K}$ function $\sigma^*(\cdot)$ such that $\|p(t)\|_{L_\infty[0,D]} \leq \sigma^*(|Z(t)| + \|u(t)\|)$ for all $t \geq 0$.

With Lemmas 15 and 17, we obtain the following result.

Lemma 18: There exists a class $\mathcal{K}$ function $\bar{\sigma}(\cdot)$ such that $|Z(t)| + \|w(t)\| \leq \bar{\sigma}(|Z(t)| + \|u(t)\|)$ for all $t \geq 0$.

This is an important upper bound on the transformation $(Z, u) \mapsto (Z, w)$, which we will use soon. However, we also need to derive a bound on the inverse of that transformation. Towards that end, we first prove the following result.

Lemma 19: There exists a class $\mathcal{K}$ function $\tau^*(\cdot)$ such that $\|\pi(t)\|_{L_\infty[0,D]} \leq \tau^*(|Z(t)| + \|w(t)\|)$ for all $t \geq 0$.

Proof: Consider the system (95)–(98), along with the diffeomorphic transformation $\zeta(t) = H(Z(t))$ defined by (79)–(84). Denote a transformed variable for the π-system, $\varepsilon(x,t) = H(\pi(x,t))$. With the observation that $Z_{i+1} + \psi_i + \phi_i\alpha_{i+1} = \sum_{j=i+1}^n (\partial\vartheta_{i+1}/\partial Z_j)(Z_{j+1} + \psi_j + \phi_j\alpha_{i+1})$, it is easy to verify that $(\partial/\partial x)\varepsilon_i(x,t) = \omega_i(\alpha_i + w(x,t) + \sum_{j=i+1}^n \omega_j\varepsilon_j)$. Noting from (85) and (82) that $\alpha_1 = -\sum_{i=1}^n \omega_i\varepsilon_i$, we get $(\partial/\partial x)\varepsilon_i = -\omega_i^2\varepsilon_i - \sum_{j=1}^{i-1}\omega_i\omega_j\varepsilon_j + \omega_i w(x,t)$ (note that this notation implies that $\partial\varepsilon_1/\partial x = -\omega_1^2\varepsilon_1 + \omega_1 w(x,t)$). Taking the Lyapunov function $\mathcal{S}(x,t) = (1/2)\sum_{i=1}^n \varepsilon_i^2(x,t)$, one obtains

$$\begin{aligned}\mathcal{S}_x(x,t) &= -\frac{1}{2}\sum_{i=1}^n \omega_i^2\varepsilon_i^2 - \frac{1}{2}\left(\sum_{i=1}^n \varepsilon_i\omega_i\right)^2 + w(x,t)\sum_{i=1}^n \omega_i\varepsilon_i \\ &\leq -\frac{1}{4}\sum_{i=1}^n \omega_i^2\varepsilon_i^2 - \frac{1}{2}\left(\sum_{i=1}^n \varepsilon_i\omega_i\right)^2 + nw^2(x,t) \\ &\leq -\frac{1}{4}\sum_{i=1}^n \omega_i^2\varepsilon_i^2 + nw^2(x,t). \end{aligned} \quad (108)$$

Noting that t is being treated only as a parameter here, we obtain, by integrating in x, the following bound $\mathcal{S}(x,t) \leq \mathcal{S}(0,t) + n\int_0^x w^2(y,t)dy - (1/4)\sum_{i=1}^n \int_0^x \omega_i^2\varepsilon_i^2 dy$. Since $\mathcal{S}(0,t) = (1/2)\sum_{i=1}^n \varepsilon_i^2(0,t) = (1/2)|\varepsilon(0,t)|^2 = (1/2)|H(\pi(0,t))|^2 = (1/2)|H(Z(t))|^2$, we get $(1/2)|H(\pi(x,t))|^2 \leq (1/2)|H(Z(t))|^2 + n\int_0^D w^2(y,t)dy$. Due to the fact that $H(\cdot)$ is a diffeomorphism, there exists a class $\mathcal{K}$ function $\tau^*(\cdot)$ such that $|\pi(x,t)| \leq \tau^*(|Z(t)| + \|w(t)\|)$ for all $x \in [0, D]$ and all $t \geq 0$, from which the result of the lemma follows by taking a supremum in x. ■

With Lemmas 15 and 19, we upper bound the plant norm in terms of the target system norm.

Lemma 20: There exists a class $\mathcal{K}_\infty$ function $\bar{\sigma}(\cdot)$ such that $\underline{\sigma}(|Z(t)| + \|u(t)\|) \leq |Z(t)| + \|w(t)\|$ for all $t \geq 0$.

Now we turn our attention to the target system (Z, w) and prove the following result on stability in the sense of its norm.

Lemma 21: There exists a function $\beta_1 \in \mathcal{KL}$ such that $|Z(t)| + \|w(t)\| \leq \beta_1(|Z(0)| + \|w(0)\|, t)$ for all $t \geq 0$.

Proof: Taking a Lyapunov function $S(t) = (1/2)\sum_{i=1}^n \zeta_i^2(t) = (1/2)|H(Z)|^2$, we have that $\dot{S}(t) \leq -(1/4)\sum_{i=1}^n \omega_i^2\zeta_i^2 + nw^2(0,t)$, which we proved in (108) using the 'predictor-equivalent' of the system model $\dot{\zeta}_i = -\omega_i^2\zeta_i - \sum_{j=1}^{i-1}\omega_i\omega_j\zeta_j + \omega_i w(0,t)$. Now we introduce an overall Lyapunov function $V(t) = S(t) + n\int_0^D e^{gx}w^2(x,t)dx$ where $g > 0$. Using (30), we get $\dot{V}(t) \leq -(1/4)\sum_{i=1}^n \omega_i^2\zeta_i^2 - ng\int_0^D e^{gx}w^2(x,t)dx$. Since the function $\sum_{i=1}^n \omega_i^2\zeta_i^2$ is positive definite (though not necessarily radially unbounded) in $Z(t)$, there exists a class $\mathcal{K}$ function $\alpha_1(\cdot)$ such that $\dot{V}(t) \leq -\alpha_1(V(t))$. Then there exists a class $\mathcal{KL}$ function $\beta_2(\cdot,\cdot)$ such that $V(t) \leq \beta_2(V(0), t)$ for all $t \geq 0$. With additional routine class $\mathcal{K}$ calculations one finds β_1 that completes the proof of the lemma. ■

By combining Lemmas 18, 20, and 21, we get the following main result on stability in the plant norm.

Theorem 22: Consider the closed-loop system consisting of the plant (72)–(77) and controller (86)–(91). There exists a function $\beta_3 \in \mathcal{KL}$ such that $|Z(t)| + \|u(t)\| \leq \beta_3(|Z(0)| + \|u(0)\|, t)$ for all $t \geq 0$.

A slightly different and relevant way to state the same global asymptotic stability result is as follows.

Corollary 23: Consider the closed-loop system consisting of the plant (72)–(77) and controller (86)–(91). Then

$$|Z(t)| + \left(\int_{t-D}^t U^2(\theta)d\theta\right)^{1/2} \leq \beta_3\left(|Z(0)| + \left(\int_{-D}^0 U^2(\theta)d\theta\right)^{1/2}, t\right), \quad \forall t \geq 0. \quad (109)$$

The following result is also true.

Theorem 24: The closed-loop system (72)–(77), (86)–(91) is locally exponentially stable in the norm $|Z(t)| + \|u(t)\|$.

This result, which we leave without a proof, is very much to be expected, since the linearized plant is a chain of integrators, with delay at the input, and the linearized feedback is predictor feedback of the standard form $U(t) = K(e^{AD}Z(t) + \int_{t-D}^t e^{A(t-\theta)}BU(\theta)d\theta)$, where $B = [0, \ldots, 0, 1]^t$, $K = [k_1, k_2, \ldots, k_n]$, $A = \{a_{i,j}\}$, and

$$k_i = -\binom{n}{i-1}, \quad a_{i,j} = \begin{cases} 1, & j = i+1 \\ 0, & \text{else.} \end{cases} \quad (110)$$

The spectrum of the nominal system matrix, $A + BK$, is $\{-1, -1, \ldots, -1\}$.

E. Example of Predictor Design for a Third-Order System That is Not Linearizable

To illustrate the construction of nominal forwarding design $(D = 0)$, we consider the following example:

$$\dot{Z}_1(t) = Z_2(t) + Z_3^2(t) \quad (111)$$

$$\dot{Z}_2(t) = Z_3(t) + Z_3(t)U(t-D) \quad (112)$$

$$\dot{Z}_3(t) = U(t-D). \quad (113)$$

The second order (Z_2, Z_3) subsystem is linearizable (and is of both Type I and Type II, as defined in [18]) when $D = 0$. Like Teel's "benchmark problem" [54], $\dot{Z}_1 = Z_2 + Z_3^2$, $\dot{Z}_2 = Z_3$, $\dot{Z}_3 = U$, the overall system (111)–(113) is *not* linearizable. While the benchmark system requires only two steps of 'forwarding' design because the (Z_2, Z_3) subsystem is linear, the system (111)–(113) requires three steps (for $D = 0$). The first two steps yield $\xi_3 = (Z_3 - \tau(Z_2 + Z_3 - (Z_3^2/2)))\mathrm{e}^{-\tau}$ and $\xi_2 = ((1+\tau)(Z_2 + Z_3 - (Z_3^2/2)) - Z_3)\mathrm{e}^{-\tau} + (1/2)(Z_3 - \tau(Z_2 + Z_3 - (Z_3^2/2)))^2\mathrm{e}^{-2\tau}$, which are then employed in $\alpha_2 = -\xi_2 - \xi_3 + (\xi_3^2/2)$. The third step of forwarding produces $\vartheta_2 = -2Z_2 - Z_3 + (5/8)Z_3^2 - (3/8)(Z_2 - (Z_3^2/2))^2$ and $\alpha_1 = -(1 + (3/4)Z_3)(Z_1 - \vartheta_2) - (Z_2 + Z_3 - (Z_3^2/2)) - Z_3$, namely, $U = \alpha_1(Z)$ when $D = 0$. The predictor feedback for system (111)–(113) is obtained as

$$\begin{aligned} U(t) =& \alpha_1(P(t)) \\ =& -P_1(t) - 3P_2(t) - 3P_3(t) - \frac{3}{8}P_2^2(t) \\ &+ \frac{3}{4}P_3(t)\Bigg(-P_1(t) - 2P_2(t) + \frac{1}{2}P_3(t) \\ &\quad + \frac{P_2(t)P_3(t)}{2} + \frac{5}{8}P_3^2(t) - \frac{1}{4}P_3^3(t) \\ &\quad - \frac{3}{8}\left(P_2(t) - \frac{P_3^2(t)}{2}\right)^2\Bigg) \end{aligned} \tag{114}$$

where the predictor $P(t) = p(D,t)$ is determined from the ODE system $(\partial/\partial x)p_1(x,t) = p_2(x,t) + p_3^2(x,t)$, $(\partial/\partial x)p_2(x,t) = p_3(x,t) + p_3(x,t)u(x,t)$, $(\partial/\partial x)p_3(x,t) = u(x,t)$ with initial condition $p_1(0,t) = Z_1(t)$, $p_2(0,t) = Z_2(t)$, $p_3(0,t) = Z_3(t)$. We start the solution process from $p_3(x,t)$, obtaining $p_3(x,t) = Z_3(t) + \int_0^x u(y,t)dy$. Then, substituting this solution into the ODE for $p_2(x,t)$, we obtain $p_2(x,t) = Z_2(t) + \int_0^x p_3(y,t)(1 + u(y,t))dy$, which yields $p_2(x,t) = Z_2(t) + xZ_3(t) + Z_3(t)\int_0^x u(y,t)dy + \int_0^x (x-y)u(y,t)dy + (1/2)(\int_0^x u(y,t)dy)^2$. In the last step of the predictor derivation we calculate $p_1(x,t) = Z_1(t) + \int_0^x p_2(y,t)dy + \int_0^x p_3^2(y,t)dy$, finally obtaining $p_1(x,t) = Z_1(t) + xZ_2(t) + (1/2)x^2Z_3(t) + xZ_3^2(t) + 3Z_3(t)\int_0^x (x-y)u(y,t)dy + (1/2)\int_0^x (x-y)^2u(y,t)dy + (3/2)\int_0^x (\int_0^y u(s,t)ds)^2 dy$.

From the explicit formulae for $p_1(x,t), p_2(x,t), p_3(x,t)$, we obtain explicit formulae for $P_1(t) = p_1(D,t)$, $P_2(t) = p_2(D,t)$, $P_3(t) = p_3(D,t)$ as

$$\begin{aligned} P_1(t) =& Z_1(t) + DZ_2(t) + \frac{1}{2}D^2Z_3(t) + DZ_3^2(t) \\ &+ 3Z_3(t)\int_{t-D}^{t}(t-\theta)U(\theta)d\theta + \frac{1}{2}\int_{t-D}^{t}(t-\theta)^2U(\theta)d\theta \\ &+ \frac{3}{2}\int_{t-D}^{t}\left(\int_{t-D}^{\theta}U(\sigma)d\sigma\right)^2 d\theta \end{aligned} \tag{115}$$

$$\begin{aligned} P_2(t) =& Z_2(t) + DZ_3(t) + Z_3(t)\int_{t-D}^{t}U(\theta)d\theta \\ &+ \int_{t-D}^{t}(t-\theta)U(\theta)d\theta + \frac{1}{2}\left(\int_{t-D}^{t}U(\theta)d\theta\right)^2 \end{aligned} \tag{116}$$

$$P_3(t) = Z_3(t) + \int_{t-D}^{t}U(\theta)d\theta. \tag{117}$$

Hence, the *explicit* infinite-dimensional nonlinear controller (114), (115)–(117) achieves global asymptotic stability of the non-linearizable strict-feedforward system (111)–(113).

F. An Alternative: A Design With Nested Saturations

The predictor given in Section VII-C can be combined with any stabilizing feedback for strict-feedforward systems. Thus, the nominal design based on integrator forwarding in Section VII-B is not the designer's only option. The design alternatives include the nested saturation designs [54], [56] and other designs for feedforward systems [33]. A nested saturation design in the absence of delay would have the form

$$U(t) = \mathcal{N}(Z(t)) \tag{118}$$

$$\begin{aligned} \mathcal{N}(Z(t)) =& -b_n\sigma(a_{nn}Z_n(t) \\ &+ b_{n+1}\sigma(a_{n-1,n-1}Z_{n-1}(t) + a_{n-1,n}Z_n(t) \\ &+ \cdots + b_2\sigma(a_{22}Z_2(t) + \cdots + a_{2n}Z_n(t) \\ &+ b_1\sigma(a_{11}Z_1(t) + \cdots + a_{1n}Z_n(t))))) \end{aligned} \tag{119}$$

where b_i and a_{ij} are positive constants and $\sigma(\cdot)$ is the standard unit saturation function. Sufficient conditions for b_i and a_{ij} are typically very conservative.

It is significant that the nested saturation designs guarantee not only stability but also a particular form of robustness to *input disturbances* which are L_2 bounded in time [57], as was the case with our design in this paper, which allowed it to be robust to the predictor error disturbance, $w(0,t)$. Since the predictor design is independent of the method employed to develop the nominal stabilizing controller, and it is given explicitly in Section VII-C, for the nested saturation approach, the predictor-compensated feedback would be $U(t) = \mathcal{N}(p(D,t))$.

A design based on nested saturations was developed by Mazenc *et al.* [31] for stabilization of strict-feedforward systems in the presence of input delay of arbitrary size. This design employs no predictor of any kind in the feedback law. Rather than performing a compensation of the delay, this design cleverly exploits the inherent robustness to delay in the particular structure of the feedback law and the plant. Only the strict-feedforward class with $\phi_i \equiv 0$ is considered in [31].

The nested saturation design in [31] has an advantage over the design in this paper in the sense of achieving robustness to input delay without any increase of the dynamic order of the controller, while our design clearly employs an infinite-dimensional compensator.

On the other hand, the advantage of our design is that its nominal design—integrator forwarding [48]—achieves quantifiable closed-loop performance, rather than just stability. The performance advantage of integrator forwarding over the nested saturation design was thoroughly illuminated in [49, Sec. 6.2.6], and it is reinforced by the proof of inverse optimality [48]. In the presence of delay, the predictor-based compensator maintains the performance of the nominal forwarding design, modulo the first D seconds. Thus, a clear complexity-versus-performance tradeoff exists between the design in this paper and the design in [31].

VIII. Linearizable Strict-Feedforward Systems

Most strict-feedforward systems are *not* feedback linearizable, however a small class of strict-feedforward systems is linearizable, and, in fact, it is linearizable by coordinate change alone, without the use of feedback. In this section we review the conditions for linearizability of strict-feedofrward systems, present a control algorithm which results in explicit formulae for control laws, present formulae for predictor feedbacks that compensate for actuator delays (which happen to be nonlinear in the ODE state but linear in the distributed actuator state), and derive the formulae for closed-loop solutions in the presence of actuator delay.

A. Integrator Forwarding (SJK) Algorithm Applied to Linearizable Strict-Feedforward Systems

In [18] it was shown that a strict-feedforward system (without delay)

$$\dot{Z}_i = Z_{i+1} + \psi_i(\underline{Z}_{i+1}) + \phi_i(\underline{Z}_{i+1})U, \quad i = 1, 2, \ldots, n \tag{120}$$

is linearizable provided the following assumption is satisfied (a systematic approach to satisfying this assumption was subsequently developed by Tall and Respondek [52], [53]).

Assumption 8.1: The functions $\psi_i(\underline{Z}_{i+1}), \phi_i(\underline{Z}_{i+1})$ can be written as $\phi_{n-1}(Z_n) = \theta_n'(Z_n)$, $\psi_{n-1}(Z_n) = 0$, and

$$\phi_i(\underline{Z}_{i+1}) = \sum_{j=i+1}^{n-1} \frac{\partial \theta_{i+1}(\underline{Z}_{i+1})}{\partial Z_j} \phi_j(\underline{Z}_{j+1}) + \frac{\partial \theta_{i+1}(\underline{Z}_{i+1})}{\partial Z_n} \tag{121}$$

$$\psi_i(\underline{Z}_{i+1}) = \sum_{j=i+1}^{n-1} \frac{\partial \theta_{i+1}(\underline{Z}_{i+1})}{\partial Z_j} \times \left(Z_{j+1} + \psi_j(\underline{Z}_{j+1})\right) - \theta_{i+2}(\underline{Z}_{i+2}) \tag{122}$$

for $i = n-2, \ldots, 1$, using some C^1 scalar-valued functions $\theta_i(\underline{Z}_i)$ satisfying $\theta_i(0) = (\partial \theta_i(0)/\partial Z_j) = 0$ for $i = 2, \ldots, n$, $j = i, \ldots, n$.

If Assumpiton 8.1 is satisfied, then the functions $\theta_i(\underline{Z}_i)$ are used in the diffeomorphism

$$h_i = Z_i - \theta_{i+1}(\underline{Z}_{i+1}), \quad i = 1, \ldots, n-1 \tag{123}$$

$$h_n = Z_n \tag{124}$$

for transforming the strict-feedforward system (120) into a system of the "chain of integrators" form

$$\dot{h}_i = h_{i+1}, \quad i = 1, \ldots, n-1 \tag{125}$$

$$\dot{h}_n = U. \tag{126}$$

The general control design algorithm for linearizable strict-feedforward systems starts with $\vartheta_{n+1} = 0$, $\alpha_{n+1} = 0$, and continues recursively, for $i = n, n-1, \ldots, 2, 1$, as

$$\alpha_i(\underline{Z}_i) = -\sum_{j=i}^{n} \left(Z_j - \vartheta_{j+1}(\underline{Z}_{j+1})\right) \tag{127}$$

$$\xi_n^{[i]}(\tau, \underline{Z}_i) = \mathrm{e}^{-\tau} \sum_{k=0}^{n-i} \frac{(-\tau)^k}{k!} \left(Z_{n-k} - \vartheta_{n-k+1}(\underline{Z}_{n-k+1})\right) \tag{128}$$

$$\xi_j^{[i]}(\tau, \underline{Z}_i) = \mathrm{e}^{-\tau} \sum_{k=0}^{j-i} \frac{(-\tau)^k}{k!} \left(Z_{j-k} - \vartheta_{j-k+1}(\underline{Z}_{j-k+1})\right) + \vartheta_{j+1}\left(\xi_{j+1}^{[i]}(\tau, \underline{Z}_i)\right), \quad j = n-1, \ldots, i+1, i \tag{129}$$

$$\vartheta_i(\underline{Z}_i) = -\int_0^\infty \left[\xi_i^{[i]}(\tau, \underline{Z}_i) + \psi_{i-1}\left(\underline{\xi}_i^{[i]}(\tau, \underline{Z}_i)\right)\right. \tag{130}$$

$$\left. + \phi_{i-1}\left(\underline{\xi}_i^{[i]}(\tau, \underline{Z}_i)\right) \alpha_i\left(\underline{\xi}_i^{[i]}(\tau, \underline{Z}_i)\right)\right] d\tau. \tag{131}$$

The control law is $U = \alpha_1(Z)$.

There are two sets of linearizing coordinates, one given by $\zeta_i = Z_i - \vartheta_{i+1}(\underline{Z}_{i+1})$, which, with the control law $U = \alpha_1(Z) = -\zeta_1 - \zeta_2 - \cdots - \zeta_n$, yields the closed-loop system in the "Teel form" [55]

$$\dot{\zeta} = \begin{bmatrix} -1 & 0 & 0 & \cdots & 0 \\ -1 & -1 & 0 & & \vdots \\ \vdots & -1 & -1 & \ddots & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ -1 & \cdots & \cdots & -1 & -1 \end{bmatrix} \zeta \tag{132}$$

and the other set of coordinates given by $h_i = \sum_{j=i}^{n} \binom{n-i}{j-i} (-1)^{j-i} \zeta_j$, which, with the control law $U = \alpha_1(Z) = -\sum_{i=1}^{n} \binom{n}{i-1} h_i$ yields the closed-loop system in the companion form

$$\dot{h}_i = h_{i+1}, \quad i = 1, \ldots, n-1 \tag{133}$$

$$\dot{h}_n = -\sum_{i=1}^{n} \binom{n}{i-1} h_i. \tag{134}$$

Both the ζ-coordinates and the h-coordinates have a useful purpose, as we shall see when we study the system in the presence of actuator delay.

B. Predictor Feedback for Linearizable Strict-Feedforward Systems

Now we consider the system with actuator delay

$$\dot{Z}_i(t) = Z_{i+1}(t) + \psi_i\left(\underline{Z}_{i+1}(t)\right) + \phi_i\left(\underline{Z}_{i+1}(t)\right) u(0, t), \quad i = 1, \ldots, n \tag{135}$$

$$u_t(x, t) = u_x(x, t) \tag{136}$$

$$u(D, t) = U(t). \tag{137}$$

With the diffeomporphic transformation $G : Z \mapsto \zeta \mapsto h$, i.e., $h = G(Z)$, which is recursively defined by $h_n = Z_n$ and

$$h_i = \sum_{j=i}^{n} \binom{n-i}{j-i} (-1)^{j-i} \left[Z_j - \vartheta_{j+1}(\underline{Z}_{j+1})\right], \quad i = n-1, \ldots, 2, 1 \tag{138}$$

we get the system

$$\dot{h}_i(t) = h_{i+1}(t), \quad i = 1, \ldots, n-1 \tag{139}$$

$$\dot{h}_n(t) = u(0, t) \tag{140}$$

$$u_t(x,t) = u_x(x,t) \tag{141}$$
$$u(D,t) = U(t) \tag{142}$$

which is a cascade of a delay line and a chain of integrators. The predictor feedback design for this system is easy.

Denote by $\eta(x,t)$ the state of the system

$$\frac{\partial}{\partial x}\eta_i(x,t) = \eta_{i+1}(x,t), \quad i = 1,\ldots,n-1 \tag{143}$$
$$\frac{\partial}{\partial x}\eta_i(x,t) = u(x,t) \tag{144}$$

with initial condition $\eta(0,t) = h(t)$. The predictor feedback is given as

$$U(t) = \alpha_1\left(G^{-1}\left(\eta(D,t)\right)\right) = -\sum_{i=1}^{n}\binom{n}{i-1}\eta_i(D,t). \tag{145}$$

Fortunately, the η-system can be solved explicitly

$$\eta_i(x,t) = \sum_{j=i}^{n}\frac{x^{j-i}}{(j-i)!}h_j(t) + \int_0^x \frac{(x-y)^{n+1-i}}{(n+1-i)!}u(y,t)dy \tag{146}$$

so the "predictor" $\eta(D,t)$ is obtained as

$$\eta_i(D,t) = \sum_{j=i}^{n}\frac{D^{j-i}}{(j-i)!}h_j(t) + \int_0^D \frac{(D-y)^{n+1-i}}{(n+1-i)!}u(y,t)dy. \tag{147}$$

Substituting the transformation $G: Z \mapsto \zeta \mapsto h$, we get the predictor

$$\begin{aligned}\eta_i(D,t) = &\sum_{j=i}^{n}\frac{D^{j-i}}{(j-i)!}\sum_{l=j}^{n}\binom{n-j}{l-j}(-1)^{l-j}\\ &\times\left(Z_l(t) - \vartheta_{l+1}\left(\underline{Z}_{l+1}(t)\right)\right)\\ &+\int_0^D \frac{(D-y)^{n+1-i}}{(n+1-i)!}u(y,t)dy.\end{aligned} \tag{148}$$

Plugging this predictor into the predictor feedback law, we get the feedback law explicitly

$$\begin{aligned}U(t) &= \alpha_1\left(G^{-1}\left(\eta(D,t)\right)\right)\\ &= -\sum_{i=1}^{n}\binom{n}{i-1}\left[\sum_{j=i}^{n}\frac{D^{j-i}}{(j-i)!}\sum_{l=j}^{n}\binom{n-j}{l-j}(-1)^{l-j}\right.\\ &\qquad\times\left(Z_l(t) - \vartheta_{l+1}\left(\underline{Z}_{l+1}(t)\right)\right)\\ &\qquad\left.+\int_0^D \frac{(D-y)^{n+1-i}}{(n+1-i)!}u(y,t)dy\right].\end{aligned} \tag{149}$$

Replacing $u(y,t)$ by $U(t+x-D)$, we finally get

$$\begin{aligned}U(t) &= \alpha_1\left(G^{-1}\left(\eta(D,t)\right)\right)\\ &= -\sum_{i=1}^{n}\binom{n}{i-1}\left[\sum_{j=i}^{n}\frac{D^{j-i}}{(j-i)!}\sum_{l=j}^{n}\binom{n-j}{l-j}(-1)^{l-j}\right.\\ &\qquad\times\left(Z_l(t) - \vartheta_{l+1}\left(\underline{Z}_{l+1}(t)\right)\right)\\ &\qquad\left.+\int_{t-D}^{t} \frac{(t-\theta)^{n+1-i}}{(n+1-i)!}U(\theta)d\theta\right].\end{aligned} \tag{150}$$

This feedback law is linear in the infinite-dimensional delay state $U(\theta)$, but nonlinear in the ODE plant state $Z(t)$.

The infinite-dimensional backstepping transformation and its inverse are given by

$$w(x,t) = u(x,t) - \alpha_1\left(G^{-1}\left(\eta(x,t)\right)\right) \tag{151}$$
$$u(x,t) = w(x,t) + \alpha_1\left(G^{-1}\left(\varpi(x,t)\right)\right) \tag{152}$$

where

$$\frac{\partial}{\partial x}\varpi_1(x,t) = -\varpi_1(x,t) + w(x,t) \tag{153}$$
$$\frac{\partial}{\partial x}\varpi_i(x,t) = -\sum_{j=1}^{i}\varpi_j(x,t) + w(x,t), \quad i = 2,\ldots,n \tag{154}$$

with initial condition $\varpi(0,t) = \zeta(t)$.

The proof of stability for the general design in this section for linearizable strict-feedforward systems proceeds in a similar manner as for general strict-feedforward systems, except that a few of the steps can be completed explicitly or more directly by noting that, with the predictor feedback, the closed-loop system in the (ζ, w) variables is

$$\dot{\zeta}_1(t) = -\zeta_1(t) + w(0,t) \tag{155}$$
$$\dot{\zeta}_i(t) = -\sum_{j=1}^{i}\zeta_j(t) + w(0,t), \quad i = 2,\ldots,n \tag{156}$$
$$w_t(x,t) = w_x(x,t) \tag{157}$$
$$w(D,t) = 0. \tag{158}$$

In the end, the following result is obtained.

Theorem 25: Consider the closed-loop system consisting of the plant (135)–(137) under Assumption 8.1 and controller (150). There exists a class $\mathcal{KL}$ function $\beta_4(\cdot,\cdot)$ such that

$$|Z(t)|^2 + \int_{t-D}^{t} U^2(\theta)d\theta \le \beta_4\left(|Z(0)|^2 + \int_{-D}^{0} U^2(\theta)d\theta, t\right). \tag{159}$$

C. Explicit Closed-Loop Solutions for Linearizable Strict-Feedforward Systems

For linearizable strict-forward systems one can find the closed-loop solutions. Over the time interval $t \in [0,D]$ one uses the linear model

$$\dot{h}_i(t) = h_{i+1}(t), \quad i = 1,\ldots,n-1 \tag{160}$$
$$\dot{h}_n(t) = U(t-D) \tag{161}$$

whereas over the time interval $t \ge D$ one would use the model

$$\dot{h}_i(t) = h_{i+1}(t), \quad i = 1,\ldots,n-1 \tag{162}$$
$$\dot{h}_n(t) = -\sum_{i=1}^{n}\binom{n}{i-1}h_i(t) \tag{163}$$

where the delay has been completely compensated.

For the time period $t \in [0,D]$ we obtain

$$\begin{aligned}h(t) = &\sum_{j=i}^{n}\frac{t^{j-i}}{(j-i)!}\sum_{l=j}^{n}\binom{n-j}{l-j}(-1)^{l-j}\\ &\times\left(Z_l(0) - \vartheta_{l+1}\left(\underline{Z}_{l+1}(0)\right)\right)\end{aligned}$$

$$+\int_{-D}^{t-D} \frac{(t-D-\theta)^{n+1-i}}{(n+1-i)!} U(\theta)d\theta \quad (164)$$

whereas for the time period $t \geq D$ we get

$$h(t) = \mathrm{e}^{(\bar{A}+\bar{B}\bar{K})(t-D)} h(D) \quad (165)$$

where $\bar{B} = [0, \ldots, 0, 1]^T$, $\bar{K} = [\bar{k}_1, \bar{k}_2, \ldots, \bar{k}_n]$, $\bar{A} = \{a_{i,j}\}$,

$$\bar{k}_i = -\begin{pmatrix} n \\ i-1 \end{pmatrix}, \quad \bar{a}_{i,j} = \begin{cases} 1, & j = i+1 \\ 0, & \text{else} \end{cases} \quad (166)$$

and

$$h(D) = \sum_{j=i}^{n} \frac{D^{j-i}}{(j-i)!} \sum_{l=j}^{n} \begin{pmatrix} n-j \\ l-j \end{pmatrix} (-1)^{l-j}$$
$$\times \left(Z_l(0) - \vartheta_{l+1}\left(\underline{Z}_{l+1}(0)\right)\right) + \int_{-D}^{0} \frac{(-\theta)^{n+1-i}}{(n+1-i)!} U(\theta)d\theta. \quad (167)$$

Theorem 26: Consider the closed-loop system consisting of the plant (135)–(137) under Assumption 8.1 and controller (150). The closed-loop solution is given by $Z(t) = G^{-1}(h(t))$, where $h(t)$ is given by (164) for $t \in [0, D]$ and by (165), (167) for $t \geq D$.

Example 8.1: To illustrate this theorem, we return to the example plant (54), from Section VII-A. We will now calculate the explicit solution for this system in closed loop with feedback (71). For simplicity of calculations, we will assume that the initial actuator state is zero, namely, $U(\theta) = 0, \theta \in [-D, 0]$. Over the time interval $t \in [0, D]$ the solution $Z(t)$ is given by $Z_1(t) = Z_1(0) + tZ_2(0)$, $Z_2(t) = Z_2(0)$. To find the solution for $t \geq D$, we recall the linearizing transformation for this example from (63): $h_1(t) = Z_1(t) + (1/3)Z_2^3(t)$, $h_2(t) = Z_2(t)$. The resulting equations for $t \geq D$, $\dot{h}_1 = h_2$, $\dot{h}_2 = -h_1 - h_2$ can be solved as

$$h_1(t) = \mathrm{e}^{-(t-D)} \left[(1+t-D)h_1(D) + (t-D)h_2(D)\right] \quad (168)$$
$$h_2(t) = \mathrm{e}^{-(t-D)} \left[-(t-D)h_1(D) + (1-t+D)h_2(D)\right]. \quad (169)$$

Using the linearizing transformation, $h(D)$ is obtained as $h_1(D) = Z_1(0) + DZ_2(0) + (1/3)Z_2^3(0)$, $h_2(D) = Z_2(0)$. To find the solution $Z(t)$ for $t \geq 0$, we need the inverse of the linearizing transformation, (63): $Z_1(t) = h_1(t) - (1/3)h_2^3(t)$, $Z_2(t) = h_2(t)$. By substituting $h(D)$ into $h(t)$ and then into $Z(t)$, we obtain the closed-loop solutions explicitly as

$$Z_1(t) = \mathrm{e}^{-(t-D)} \Bigg[(1+t-D)\left(Z_1(0) + DZ_2(0) + \frac{1}{3}Z_2^3(0)\right)$$
$$+(t-D)Z_2(D)\Bigg] - \frac{1}{3}\mathrm{e}^{-3(t-D)}$$
$$\times \Bigg[-(t-D)\left(Z_1(0) + DZ_2(0) + \frac{1}{3}Z_2^3(0)\right)$$
$$+(1-t+D)Z_2(D)\Bigg]^3 \quad (170)$$

$$Z_2(t) = \mathrm{e}^{-(t-D)} \Bigg[-(t-D)\left(Z_1(0) + DZ_2(0) + \frac{1}{3}Z_2^3(0)\right)$$
$$+(1-t+D)Z_2(D)\Bigg] \quad (171)$$

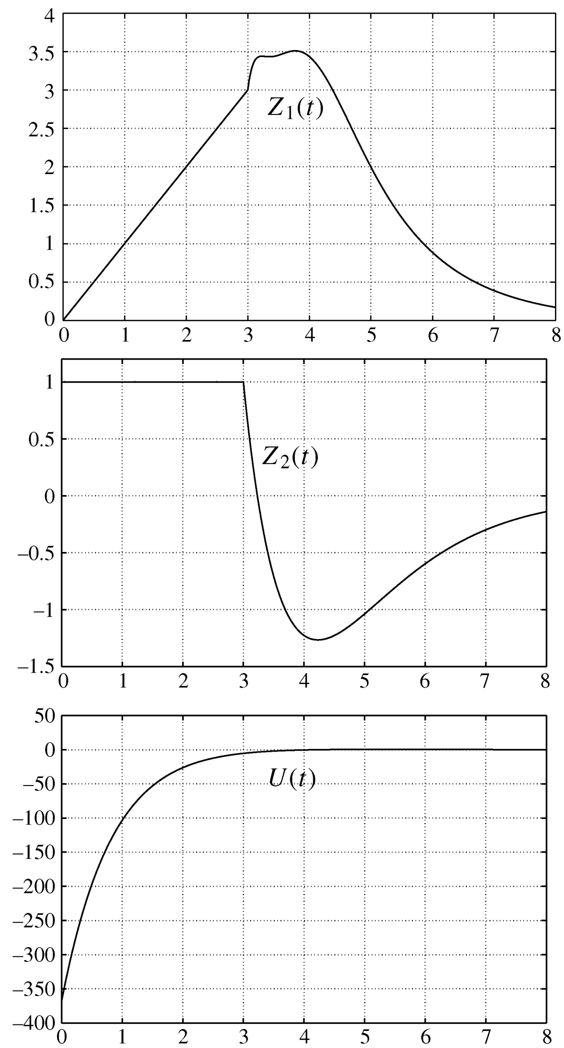

Fig. 3. Time responses from Example 8.1 for $D = 3$. Note the nonlinear transient of $Z_1(t)$ after $t = 3$ sec. The size of the control input $U(t)$ is due to the need to compensate for a long input delay, $D = 3$.

for $t \geq D$. The closed-loop control signal is $U(t) = -h_1(t+D) - 2h_2(t+D), t \geq 0$, which gives $U(t) = -\mathrm{e}^{-(t-D)}[(1-t+D)h_1(D) + (2-t+D)h_2(D)]$, and in its final form becomes

$$U(t) = -\mathrm{e}^{-(t-D)} \Bigg[\left(Z_1(0) + (2+D)Z_2(0) + \frac{1}{3}Z_2^3(0)\right)$$
$$-(t-D)\left(Z_1(0) + (1+D)Z_2(0) + \frac{1}{3}Z_2^3(0)\right)\Bigg] \quad (172)$$

for all $t \geq 0$. For example, if we take the initial conditions as $Z_1(0) = 0$ and $Z_2(0) = 1$, we obtain closed-loop solutions as given in Fig. 3.

IX. CONCLUSION

In this paper we presented a predictor-based design for compensating input delay of arbitrary length in nonlinear control systems. For the broad class of forward-complete systems, global stabilization is maintained but requires on-line solution of a nonlinear integral equation (whose solutions we guarantee to remain bounded in closed loop). For strict-feedback systems

TABLE I
PROPERTIES OF THE BACKSTEPPING TRANSFORMATION FOR DIFFERENT CLASSES OF SYSTEMS

	direct bkst. transf.		inverse bkst. transf.	
	global	explicit	global	explicit
all stabilizable syst.			✓	
fbk. lineariz. and strict-fbk			✓	✓
stabilizable fwd-complete	✓		✓	
strict-feedforward syst.	✓	✓	✓	
linearizable strict-ffwd	✓	✓	✓	✓

the predictor feedback is obtained explicitly. For the linearizable subclass of strict-feedforward systems the closed-loop solutions are obtained explicitly.

The infinite-dimensional nonlinear backstepping transformation (of the input delay state) is of key significance in the analysis of closed-loop stability. In Table I we summarize the properties of this transformation for various systems that we have considered in [19] and in the present paper. The explicit form of the direct backstepping transformation implies that the predictor feedback can be obtained explicitly, which is the case with strict-feedforward systems.

ACKNOWLEDGMENT

The author would like to thank I. Karafyllis and R. Vazquez for sharing with me some useful technical ideas, leading towards the current compact proof of Theorem 1.

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