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Compensating a String PDE in the Actuation or Sensing Path of an Unstable ODE

Miroslav Krstic

***Abstract*—How to control an unstable linear system with a long pure delay in the actuator path? This question was resolved using 'predictor' or 'finite spectrum assignment' designs in the 1970s. Here we address a more challenging question: How to control an unstable linear system with a wave partial differential equation (PDE) in the actuation path? Physically one can think of this problem as having to stabilize a system to whose input one has access through a string. The challenges of overcoming string/wave dynamics in the actuation path include their infinite dimension, finite propagation speed of the control signal, and the fact that all of their (infinitely many) eigenvalues are on the imaginary axis. In this technical note we provide an explicit feedback law that compensates the wave PDE dynamics at the input of an linear time-invariant ordinary differential equation and stabilizes the overall system. In addition, we prove robustness of the feedback to the error in *a priori* knowledge of the propagation speed in the wave PDE. Finally, we consider a dual problem where the wave PDE is in the sensing path and design an exponentially convergent observer.**

***Index Terms*—Linear time-invariant (LTI), ordinary differential equation (ODE), partial differential equation (PDE).**

I. Introduction

The "Smith predictor" and its extensions developed since the 1970s [1], [3]–[9], [13]–[19], [21]–[31] are important tools in several application areas. They allow to compensate a pure delay of arbitrary length in either the actuation or sensing path of a linear system, even when the system is unstable. Several results in adaptive control for unknown ODE parameters have been published [2], [20]. Extensions to nonlinear systems are also beginning to emerge [10].

In [11] we presented a first attempt of compensating infinite-dimensional actuator dynamics of more complex type than pure delay. We presented a design for diffusion-dominated partial differential equation (PDE) dynamics (such as the heat equation). While these dynamics do not have a finite speed of propagation, they are 'low-pass' and "phase-lag" to the extreme, as they have infinitely many (stable) poles and no zeros.

In this technical note we tackle a problem from a different class of PDE dynamics in the actuation or sensing path—the wave/string equation. The wave equation is challenging due to the fact that all of its (infinitely many) eigenvalues are on the imaginary axis, and due to the fact that it has a finite (limited) speed of propagation (large control doesn't help).

The problem studied here is more challenging than in [11] due to another difficulty—the PDE system is second order in time, which means that the state is 'doubly infinite dimensional' (distributed displacement and distributed velocity). This is not so much of a problem dimensionally, as it is a problem in constructing the state transformations for compensating the PDE dynamics. One has to deal with the coupling of two infinite-dimensional states.

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As in [13] for delay-ordinary differential equation (ODE) cascades, and in [11] for heat-PDE-ODE cascades, we design feedback laws that are given by explicit formulae. We start in Section II with an actuator compensation design with full state feedback. In Section III we approach the question of robustness of our infinite-dimensional feedback law with respect to small uncertainty in the wave propagation speed and provide an affirmative answer. Finally, in Section IV we develop a dual of our actuator dynamics compensator and design an infinite-dimensional observer which compensates the wave PDE dynamics of the sensor. Section V presents examples of controller and observer design.

II. Stabilization With Full-State Feedback

We consider the cascade of a wave (string) equation and an LTI finite-dimensional system given by

$$\dot{X}(t) = AX(t) + Bu(0,t) \quad (1)$$
$$u_{tt}(x,t) = u_{xx}(x,t) \quad (2)$$
$$u_x(0,t) = 0 \quad (3)$$
$$u_x(D,t) = U(t) \quad (4)$$

where $X \in \mathbb{R}^n$ is the ODE state, U is the scalar input to the entire system, and $u(x,t)$ is the state of the PDE dynamics of the actuator governed by a wave equation. The cascade system is depicted in Fig. 1. The Neumann actuation (4) physically amounts to force actuation. This is a common choice for wave equations as it allows an easy addition of boundary damping. We could also solve the problem with $u(D,t) = U(t)$, but with a different, novel approach for adding damping to wave equations. We don't pursue this problem as it would place more emphasis on the wave equation itself than on control of an ODE through a wave equation.

The length of the PDE domain, D, is arbitrary. Thus, we take the wave propagation speed to be unity without loss of generality. We assume that the pair (A,B) is stabilizable and take K to be a known vector such that $A+BK$ is Hurwitz.

We recall from [13] that, if (2), (3) are replaced by the delay/transport equation,

$$u_t(x,t) = u_x(x,t) \quad (5)$$

then the predictor-based control law

$$U(t) = K\left[\mathrm{e}^{AD}X(t) + \int_0^D \mathrm{e}^{A(D-y)}Bu(y,t)dy\right] \quad (6)$$

achieves perfect compensation of the actuator delay and achieves exponential stability at $u \equiv 0$, $X = 0$. When the pure delay actuator dynamics are replaced by the wave equation dynamics, a much more involved feedback law is needed.

We seek an invertible transformation $(X,u,u_t) \mapsto (X,v,v_t)$ that converts (1)–(3) into

$$\dot{X}(t) = (A+BK)X(t) + Bv(0,t) \quad (7)$$
$$v_{tt}(x,t) = v_{xx}(x,t) \quad (8)$$
$$v_x(0,t) = 0 \quad (9)$$

and then another transformation $(X,v,v_t) \mapsto (X,w,w_t)$ that converts (7)–(9) into

$$\dot{X}(t) = (A+BK)X(t) + Bw(0,t) \quad (10)$$
$$w_{tt}(x,t) = w_{xx}(x,t) \quad (11)$$
$$w_x(0,t) = c_0 w(0,t), \qquad c_0 > 0. \quad (12)$$

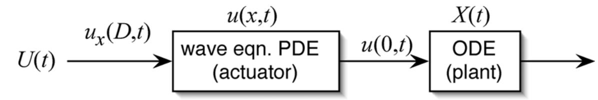

Fig. 1. The cascade of the heat equation PDE dynamics of the actuator with the ODE dynamics of the plant.

We also seek a feedback law that achieves

$$w_x(D,t) = -c_1 w_t(D,t), \qquad c_1 > 0. \quad (13)$$

The system (10)–(13) is exponentially stable, as we shall see. With the invertibility of the composite transformation $(X,u,u_t) \mapsto (X,w,w_t)$, we will achieve exponential stability of the closed-loop system in the original variables (X,u,u_t).

We postulate the transformation $(X,u,u_t) \mapsto (X,v,v_t)$ in the form

$$v(x,t) = u(x,t) - \int_0^x k(x,y)u(y,t)dy - \int_0^x l(x,y)u_t(y,t)dy - \gamma(x)X(t) \quad (14)$$

where the kernel functions $k(x,y)$, $l(x,y)$, and $\gamma(x)$ are to be found. By matching the systems (1)–(3) and (7)–(9), a lengthy but straightforward calculation leads to the following conditions on the kernels:

$$\gamma''(x) = \gamma(x)A^2 \quad (15)$$
$$\gamma(0) = K \quad (16)$$
$$\gamma'(0) = 0 \quad (17)$$
$$l_{xx}(x,y) = l_{yy}(x,y) \quad (18)$$
$$l(x,x) = 0 \quad (19)$$
$$l_y(x,0) = -\gamma(x)B \quad (20)$$
$$k_{xx}(x,y) = k_{yy}(x,y) \quad (21)$$
$$k(x,x) = 0 \quad (22)$$
$$k_y(x,0) = -\gamma(x)AB. \quad (23)$$

These differential equations can be solved explicitly. The solutions are

$$\gamma(x) = KM(x) \quad (24)$$
$$l(x,y) = m(x-y) \quad (25)$$
$$k(x,y) = \mu(x-y) \quad (26)$$
$$M(x) = [I \quad 0]\mathrm{e}^{\begin{bmatrix} 0 & A^2 \\ I & 0 \end{bmatrix}x}\begin{bmatrix} I \\ 0 \end{bmatrix} \quad (27)$$
$$m(s) = \int_0^s \gamma(\xi)Bd\xi \quad (28)$$
$$\mu(s) = \int_0^s \gamma(\xi)ABd\xi. \quad (29)$$

Thus the transformation $(X,u,u_t) \mapsto (X,v,v_t)$ is defined as

$$v(x,t) = u(x,t) - \int_0^x \mu(x-y)u(y,t)dy - \int_0^x m(x-y)u_t(y,t)dy - \gamma(x)X(t) \quad (30)$$

$$v_t(x,t)=u_t(x,t)-KBu(x,t)-\int_0^x \mu(x-y)u_t(y,t)dy - \int_0^x m''(x-y)u(y,t)dy-\gamma(x)AX(t). \tag{31}$$

With similar derivations, one can show that the inverse of the transformation $(X,u,u_t)\mapsto(X,v,v_t)$ is defined as

$$u(x,t)=v(x,t)-\int_0^x \sigma(x-y)v(y,t)dy - \int_0^x n(x-y)v_t(y,t)dy-\rho(x)X(t) \tag{32}$$

$$u_t(x,t)=v_t(x,t)+KBv(x,t)-\int_0^x \sigma(x-y)v_t(y,t)dy - \int_0^x n''(x-y)v(y,t)dy-\rho(x)AX(t) \tag{33}$$

where

$$\rho(x)=-KN(x) \tag{34}$$

$$N(x)=[I\quad 0]\mathrm{e}^{\begin{bmatrix}0 & (A+BK)^2\\ I & 0\end{bmatrix}x}\begin{bmatrix}I\\0\end{bmatrix} \tag{35}$$

$$n(s)=\int_0^s \rho(\xi)Bd\xi \tag{36}$$

$$\sigma(s)=\int_0^s \rho(\xi)ABd\xi. \tag{37}$$

The transformation $(X,v,v_t)\mapsto(X,w,w_t)$ is simpler and given by

$$w(x,t)=v(x,t)+c_0\int_0^x v(y,t)dy \tag{38}$$

$$w_t(x,t)=v_t(x,t)+c_0\int_0^x v_t(y,t)dy \tag{39}$$

whereas its inverse is

$$v(x,t)=w(x,t)-c_0\int_0^x \mathrm{e}^{-c_0(x-y)}w(y,t)dy \tag{40}$$

$$v_t(x,t)=w_t(x,t)-c_0\int_0^x \mathrm{e}^{-c_0(x-y)}w_t(y,t)dy. \tag{41}$$

The composite transformation $(X,u,u_t)\mapsto(X,w,w_t)$ is

$$\begin{aligned} w(x,t)=u(x,t) &+\int_0^x\left(c_0-\mu(x-y)-c_0\int_0^{x-y}\mu(\xi)d\xi\right)u(y,t)dy\\ &-\int_0^x\left(m(x-y)+c_0\int_0^{x-y}m(\xi)d\xi\right)u_t(y,t)dy\\ &-\left(\gamma(x)+c_0\int_0^x\gamma(\xi)d\xi\right)X(t) \end{aligned} \tag{42}$$

$$\begin{aligned} w_t(x,t)=u_t(x,t)-KBu(x,t) &-\int_0^x\big(c_0m'(x-y)+m''(x-y)\big)u(y,t)dy\\ &+\int_0^x\left(c_0-\mu(x-y)-c_0\int_0^{x-y}\mu(\xi)d\xi\right)u_t(y,t)dy\\ &-\left(\gamma(x)+c_0\int_0^x\gamma(\xi)d\xi\right)AX(t) \end{aligned} \tag{43}$$

and its inverse is

$$\begin{aligned} u(x,t) &= w(x,t) - \int_0^x\Big(c_0\mathrm{e}^{-c_0(x-y)}+\sigma(x-y)\\ &\qquad -c_0\int_0^{x-y}\mathrm{e}^{-c_0(x-y-\xi)}\sigma(\xi)d\xi\Big)w(y,t)dy\\ &\quad-\int_0^x\left(n(x-y)-c_0\int_0^{x-y}\mathrm{e}^{-c_0(x-y-\xi)}n(\xi)d\xi\right)\\ &\qquad\times w_t(y,t)dy-\rho(x)X(t) \end{aligned} \tag{44}$$

$$\begin{aligned} u_t(x,t) &= w_t(x,t)+KBw(x,t)\\ &\quad-\int_0^x\Big(n''(x-y)-c_0n'(x-y)+c_0^2n(x-y)\\ &\qquad +c_0^3\int_0^{x-y}\mathrm{e}^{-c_0(x-y-\xi)}n(\xi)d\xi\Big)w(y,t)dy\\ &\quad-\int_0^x\Big(c_0\mathrm{e}^{-c_0(x-y)}+\sigma(x-y)\\ &\qquad -c_0\int_0^{x-y}\mathrm{e}^{-c_0(x-y-\xi)}\sigma(\xi)d\xi\Big)w_t(y,t)dy\\ &\quad-\rho(x)AX(t). \end{aligned} \tag{45}$$

Next, we design a controller that satisfies the boundary condition (13). First, from (42) we get

$$\begin{aligned} w_x(x,t)=u_x(x,t)+c_0u(x,t) &-\int_0^x\big(\mu'(x-y)+c_0\mu(x-y)\big)u(y,t)dy\\ &-\int_0^x\big(m'(x-y)+c_0m(x-y)\big)u_t(y,t)dy\\ &-\big(\gamma'(x)+c_0\gamma(x)\big)X(t). \end{aligned} \tag{46}$$

Then, the control law is

$$U(t) = (-c_0 + c_1 KB)u(D,t) - c_1 u_t(D,t) + \int_0^D p(D-y)u(y,t)dy + \int_0^D q(D-y)u_t(y,t)dy + \pi(D)X(t) \tag{47}$$

where

$$p(s) = \mu'(s) + c_0\mu(s) + c_1\left(m''(s) + c_0 m'(s)\right) \tag{48}$$

$$q(s) = m'(s) + c_0 m(s) + c_1\left(\mu(s) + c_0\int_0^s \mu(\xi)d\xi - c_0\right) \tag{49}$$

$$\pi(x) = \gamma'(x) + \gamma(x)(c_0 I + c_1 A) + c_1 c_0 \int_0^x \gamma(\xi)d\xi A. \tag{50}$$

Next we state a new controller that compensates the *wave PDE* actuator dynamics and prove exponential stability of the resulting closed-loop system.

Theorem 1: (Stabilization): Consider a closed-loop system consisting of the plant (1)–(4) and the control law (47). For any initial condition such that $u(\cdot,0) \in H^1(0,D)$ and $u_t(\cdot,0) \in L^2(0,D)$, the closed-loop system has a unique solution $(X(t), u(\cdot,t), u_t(\cdot,t)) \in C([0,\infty), \mathbb{R}^n \times H^1(0,D) \times L^2(0,D))$ and is exponentially stable in the sense of the norm

$$\left(|X(t)|^2 + u(0,t)^2 + \int_0^D u_x(x,t)^2 dx + \int_0^D u_t(x,t)^2 dx\right)^{1/2}. \tag{51}$$

Moreover, if the initial condition $(u(\cdot,0), u_t(\cdot,0))$ is compatible with the control law (47) and belongs to $H^2(0,D) \times H^1(0,D)$, then $(X(t), u(\cdot,t), u_t(\cdot,t)) \in C^1([0,\infty), \mathbb{R}^n \times H^1(0,D) \times L^2(0,D))$ is the classical solution of the closed-loop system.

Proof: We will use the system norms

$$\Omega(t) = u(0,t)^2 + \|u_x(t)\|^2 + \|u_t(t)\|^2 + |X(t)|^2 \tag{52}$$

$$\Xi(t) = w(0,t)^2 + \|w_x(t)\|^2 + \|w_t(t)\|^2 + |X(t)|^2 \tag{53}$$

where $\|u(t)\|^2$ is a compact notation for $\int_0^D u(x,t)^2 dx$. In addition, we employ a Lyapunov function

$$V(t) = X(t)^T P X(t) + aE(t) \tag{54}$$

where the matrix $P = P^T > 0$ is the solution to the Lyapunov equation $P(A+BK) + (A+BK)^T P = -Q$ for some $Q = Q^T > 0$, the parameter $a > 0$ is to be chosen later, and the function $E(t)$ is defined by

$$E(t) = \frac{1}{2}\left(c_0 w(0,t)^2 + \|w_x(t)\|^2 + \|w_t(t)\|^2\right) + \delta\int_0^D (1+y)w_x(y,t)w_t(y,t)dy \tag{55}$$

By using (43), (46), (45), and

$$\begin{aligned} u_x(x,t) = {} & w_x(x,t) - c_0 w(x,t) \\ & - \int_0^x \left(-c_0^2 e^{-c_0(x-y)} + \sigma'(x-y) - c_0\sigma(x-y) + c_0^2 \int_0^{x-y} e^{-c_0(x-y-\xi)}\sigma(\xi)d\xi\right) w(y,t)dy \\ & - \int_0^x \left(n'(x-y) - c_0 n(x-y) + c_0^2 \int_0^{x-y} e^{-c_0(x-y-\xi)} n(\xi)d\xi\right) w_t(y,t)dy \\ & - \rho'(x)X(t) \end{aligned} \tag{56}$$

$$u(0,t) = w(0,t) + KX(t) \tag{57}$$

and by using Poincare's inequality, for sufficiently small δ it is possible to show that there exist positive constants $\theta_1, \theta_2, \theta_3, \theta_4$ such that

$$\theta_1\Xi \le \Omega \le \theta_2\Xi \tag{58}$$

$$\theta_3\Xi \le V \le \theta_4\Xi. \tag{59}$$

Furthermore, it is readily shown that

$$\begin{aligned} \dot{E}(t) = & -\left(c_1 - \delta\frac{1+D}{2}\left(1+c_1^2\right)\right) w_t(D,t)^2 \\ & - \frac{\delta}{2}\left(w_t(0,t)^2 + c_0^2 w(0,t)^2\right) \\ & - \frac{\delta}{2}\left(\|w_x(t)\|^2 + \|w_t(t)\|^2\right). \end{aligned} \tag{60}$$

Then, by choosing

$$a \ge \frac{8|PB|^2}{\delta c_0^2 \lambda_{\min}(Q)} \tag{61}$$

we get

$$\dot{V} \le -\eta V \tag{62}$$

for some sufficiently small positive η. From (58), (59), (62), it follows that

$$\Omega(t) \le \frac{\theta_2\theta_4}{\theta_1\theta_3}\Omega(0)e^{-\eta t}. \tag{63}$$

The rest of the argument is almost identical to [12]. ■

III. Robustness to Uncertainty in the Wave Propagation Speed

We now study robustness of the feedback law (47) to a small perturbation of the propagation speed in the actuator dynamics, i.e., we study stability robustness of the closed-loop system

$$\dot{X}(t) = AX(t) + Bu(0,t) \tag{64}$$

$$u_{tt}(x,t) = (1+\varepsilon)u_{xx}(x,t) \tag{65}$$

$$u_x(0,t) = 0 \tag{66}$$

$$u_x(D,t) = (-c_0 + c_1 KB)u(D,t) - c_1 u_t(D,t) + \int_0^D p(D-y)u(y,t)dy + \int_0^D q(D-y)u_t(y,t)dy + \pi(D)X(t) \tag{67}$$

to the perturbation parameter ε, which we allow to be either positive or negative but small.

With a very long calculation, we arrive at the representation of the system (64)–(67) in the w-variable:

$$\dot{X}(t) = (A+BK)X(t) + Bw(0,t) \tag{68}$$

$$w_{tt}(x,t) = (1+\varepsilon)w_{xx}(x,t) + \varepsilon\Pi(x,t) \tag{69}$$

$$w_x(0,t) = c_0 w(0,t) \quad (70)$$
$$w_x(D,t) = -c_1 w_t(D,t) \quad (71)$$

where

$$\Pi(x,t) = \left(\gamma(x) + c_0 \int_0^x \gamma(\xi) d\xi\right) \times ((A+BK)X(t) + Bw(0,t)) + KBw_t(x,t) + (KB)^2 w(x,t) + \int_0^x g(x-y) w(y,t) dy + \int_0^x h(x-y) w_t(y,t) dy - \left(KB\rho(x) + \int_0^x \omega(x-\xi)\rho(\xi) d\xi\right) AX(t) \quad (72)$$

and where

$$\omega(x) = m''(x) + c_0 m'(x) \quad (73)$$
$$g(x) = KB\phi(x) + KB\omega(x) + \int_0^x \omega(x-y)\phi(y) dy \quad (74)$$
$$h(x) = KB\psi(x) + \omega(x) + \int_0^x \omega(x-y)\psi(y) dy \quad (75)$$
$$\phi(x) = -n''(x) + c_0 n'(x) - c_0^2 n(x) + c_0^3 \int_0^x \mathrm{e}^{-c_0(x-y)} n(y) dy \quad (76)$$
$$\psi(x) = -c_0 \mathrm{e}^{-c_0 x} - \sigma(x) + c_0 \int_0^x \mathrm{e}^{-c_0(x-y)} \sigma(y) dy. \quad (77)$$

The state perturbation $\Pi(x,t)$ is very complicated in appearance but $\int_0^D \Pi(x,t)^2 dx$ can be bounded in terms of $\Xi(t)$ as defined in (53), and hence also in terms of $V(t)$ as defined in (54). Consequently, the same kind of Lyapunov analysis can be conducted as in the proof of Theorem 1, with a slightly modified Lyapunov function

$$E(t) = \frac{1}{2}\left\{(1+\varepsilon)\left[c_0 w(0,t)^2 + \|w_x(t)\|^2\right] + \|w_t(t)\|^2\right\} + \delta \int_0^D (1+y) w_x(y,t) w_t(y,t) dy \quad (78)$$

and dominating the effect of $\Pi(x,t)$ for small ε, to prove the following robustness result.

Theorem: (Robustness to Small Error in Wave Propagation Speed): Consider the closed-loop system (64)–(67). There exists a sufficiently small $\varepsilon^* > 0$ such that for all $\varepsilon \in (-\varepsilon^*, \varepsilon^*)$ the closed-loop system is exponentially stable in the same sense as in Theorem 1.

IV. Observer Design

Consider the LTI ODE system in cascade with a wave PDE in the sensing path (as depicted in Fig. 2),

$$Y(t) = u(0,t) \quad (79)$$
$$u_{tt}(x,t) = u_{xx}(x,t) \quad (80)$$

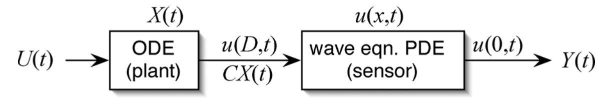

Fig. 2. The cascade of the ODE dynamics of the plant with the heat equation PDE dynamics of the sensor.

$$u_x(0,t) = 0 \quad (81)$$
$$u(D,t) = CX(t) \quad (82)$$
$$\dot{X}(t) = AX(t) + BU(t). \quad (83)$$

We recall from [13] that, if (80), (81) are replaced by the delay/transport equation, $u_t(x,t) = u_x(x,t)$ then the predictor-based observer

$$\hat{u}_t(x,t) = \hat{u}_x(x,t) + C\mathrm{e}^{Ax} L\left(Y(t) - \hat{u}(0,t)\right) \quad (84)$$
$$\hat{u}(D,t) = C\hat{X}(t) \quad (85)$$
$$\dot{\hat{X}}(t) = A\hat{X}(t) + BU(t) + \mathrm{e}^{AD} L\left(Y(t) - \hat{u}(0,t)\right) \quad (86)$$

achieves perfect compensation of the observer delay and achieves exponential stability at $u - \hat{u} \equiv 0, X - \hat{X} = 0$.

We are seeking an observer of the form

$$\hat{u}_{tt}(x,t) = \hat{u}_{xx}(x,t) + \alpha(x)\left(Y(t) - \hat{u}(0,t)\right) + \beta(x)\left(\dot{Y}(t) - \hat{u}_t(0,t)\right) \quad (87)$$
$$\hat{u}_x(0,t) = -a\left(Y(t) - \hat{u}(0,t)\right) - b\left(\dot{Y}(t) - \hat{u}_t(0,t)\right) \quad (88)$$
$$\hat{u}(D,t) = C\hat{X}(t) \quad (89)$$
$$\dot{\hat{X}}(t) = A\hat{X}(t) + BU(t) + \Lambda\left(Y(t) - \hat{u}(0,t)\right) \quad (90)$$

where the functions $\alpha(x), \beta(x)$, the scalars a, b, and the vector Λ are to be determined, to achieve exponential stability of the observer error system

$$\tilde{u}_{tt}(x,t) = \tilde{u}_{xx}(x,t) - \alpha(x)\tilde{u}(0,t) - \beta(x)\tilde{u}_t(0,t) \quad (91)$$
$$\tilde{u}_x(0,t) = a\tilde{u}(0,t) + b\tilde{u}_t(0,t) \quad (92)$$
$$\tilde{u}(D,t) = C\tilde{X}(t) \quad (93)$$
$$\dot{\tilde{X}}(t) = A\tilde{X}(t) - \Lambda\tilde{u}(0,t) \quad (94)$$

where

$$\tilde{u}(x,t) = u(x,t) - \hat{u}(x,t) \quad (95)$$
$$\tilde{X}(t) = X(t) - \hat{X}(t). \quad (96)$$

We consider the transformation

$$\tilde{w}(x) = \tilde{u}(x) - \Gamma(x)\tilde{X} \quad (97)$$

and try to find $\Gamma(x)$, along with $\alpha(x), \beta(x), a, b, \Lambda$, that convert (91)–(94) into the exponentially stable system

$$\tilde{w}_{tt}(x,t) = \tilde{w}_{xx}(x,t) \quad (98)$$
$$\tilde{w}_x(0,t) = c_0\tilde{w}_t(0,t) \quad (99)$$
$$\tilde{w}(D,t) = 0 \quad (100)$$
$$\dot{\tilde{X}}(t) = (A - \Lambda\Gamma(0))\tilde{X} - \Lambda\tilde{w}(0,t) \quad (101)$$

where $c_0 > 0$ and $A - \Lambda\Gamma(0)$ is a Hurwitz matrix.

By matching the systems (91)–(94) and (98)–(101), we obtain the conditions

$$\Gamma''(x) = \Gamma(x)A^2 \tag{102}$$
$$\Gamma'(0) = c_0\Gamma(0)A \tag{103}$$
$$\Gamma(D) = C \tag{104}$$

as well as

$$\alpha(x) = \Gamma(x)A\Lambda \tag{105}$$
$$\beta(x) = \Gamma(x)\Lambda \tag{106}$$
$$a = c_0\Gamma(0)\Lambda \tag{107}$$
$$b = c_0. \tag{108}$$

Solving the linear ODE two-point-boundary-value problem (102)–(104), we obtain

$$\Gamma(x) = \Gamma(0)G(x) \tag{109}$$

where

$$\Gamma(0) = CG(D)^{-1} \tag{110}$$
$$G(x) = [I \quad c_0A]\mathrm{e}^{\begin{bmatrix} 0 & A^2 \\ I & 0 \end{bmatrix} x} \begin{bmatrix} I \\ 0 \end{bmatrix}. \tag{111}$$

Thus, we have determined all the quantities needed to implement the observer (87)–(90) except Λ, which needs to be chosen so that the matrix $A - \Lambda\Gamma(0)$ is Hurwitz. We pick

$$\Lambda = G(D)L \tag{112}$$

where L is chosen so that the matrix $A - LC$ is Hurwitz. Since A and $G(D)$ commute, using $G(D)$ as a similarity transformation for the matrix $A - \Lambda\Gamma(0) = A - G(D)LCG(D)^{-1}$, we get that the matrices $A - LC$ and $A - \Lambda\Gamma(0)$ have the same eigenvalues, so the latter matrix is Hurwitz.

The system (98)–(101) is a cascade of a wave equation (98)–(100), which is exponentially stable due to the 'damping' boundary condition (99), and of the exponentially stable ODE (101). The entire observer error system is exponentially stable.

Theorem 3: (Observer Design and Convergence): Assume that $G(D)$ is non-singular. The observer (87)–(90), with gains defined through (105)–(112), guarantees that $\hat{X}$, $\hat{u}$ exponentially converge to X, u, i.e., more precisely, that the observer error system is exponentially stable in the sense of the norm

$$\left(\left|X(t) - \hat{X}(t)\right|^2 + \int_0^D \left[(u_x(x,t) - \hat{u}_x(x,t))^2 + (u_t(x,t) - \hat{u}_t(x,t))^2 \right] dx \right)^{1/2}. \tag{113}$$

Proof: Very similar to the proof of Theorem 1, with a Lyapunov function

$$V(t) = \tilde{X}(t)^T M(D)^{-T} P M(D)^{-1} \tilde{X}(t) + aE(t) \tag{114}$$

where $P = P^T > 0$ is the solution to $P(A - LC) + (A - LC)^T P = -Q$ for some $Q = Q^T > 0$, and with

$$E(t) = \frac{1}{2}\left(\|\tilde{w}_x(t)\|^2 + \|\tilde{w}_t(t)\|^2\right) + \delta \int_0^D (-1 - D + y)\tilde{w}_x(y,t)\tilde{w}_t(y,t)dy \tag{115}$$

The system norms are simpler

$$\Omega(t) = \|\tilde{u}_x(t)\|^2 + \|\tilde{u}_t(t)\|^2 + \left|\tilde{X}(t)\right|^2 \tag{116}$$
$$\Xi(t) = \|\tilde{w}_x(t)\|^2 + \|\tilde{w}_t(t)\|^2 + \left|\tilde{X}(t)\right|^2 \tag{117}$$

and the system transformations are much simpler

$$\tilde{w}_x(x,t) = \tilde{u}_x(x,t) - \Gamma'(x)\tilde{X}(t) \tag{118}$$
$$\tilde{w}_t(x,t) = \tilde{u}_t(x,t) - \Gamma(x)A\tilde{X}(t) + \Gamma(x)\Lambda\tilde{u}(0,t)$$
$$\tilde{u}_t(x,t) = \tilde{w}_t(x,t) + \Gamma(x)\left(A - \Lambda\Gamma(0)\right)\tilde{X}(t) - \Gamma(x)\Lambda\tilde{w}(0,t). \tag{119}$$

One obtains the inequalities (58), (59) with the help of Agmon's inequality, or, with the help of Poincare's inequality and the alternative representation of the state transformation

$$\tilde{w}_t(x,t) = \tilde{u}_t(x,t) + \Gamma(x)\Lambda\tilde{u}(x,t) - \Gamma(x)\Lambda \int_0^D \tilde{u}_x(y,t)dy - \Gamma(x)A\tilde{X}(t)$$
$$\tilde{u}_t(x,t) = \tilde{w}_t(x,t) - \Gamma(x)\Lambda\tilde{w}(x,t) + \Gamma(x)\Lambda \int_0^D \tilde{w}_x(y,t)dy + \Gamma(x)\left(A - \Lambda\Gamma(0)\right)\tilde{X}(t). \tag{120}$$

Then, one obtains (63), which completes the proof. ■

V. Example

We consider a second order plant $\dot{X}_1(t) = X_2(t)$, $\dot{X}_2(t) = -X_1(t) + u(0,t)$, $u_{tt} = u_{xx}$, $u_x(0,t) = 0$, $u_x(D,t) = U(t)$. The controller obtained from our procedure, using $u(0,t) = -hX_2(t)$ with $h > 0$ as the nominal controller whose purpose is to add damping to the resonant second order plant, is $U(t) = hc_1(\cos D + c_0 \sin D)X_1(t) + h(\sin D - c_0\cos D)X_2(t) - (c_0 + hc_1)u(D,t) - c_1u_t(D,t) + \int_0^D hc_1(\sin(D-y) - c_0\cos(D-y))u(y,t)dy - \int_0^D [h(\cos(D-y) + c_0\sin(D-y)) + c_0c_1]u_t(y,t)dy$.

We illustrate the observer design with a similar system, $\dot{X}_1(t) = X_2(t)$, $\dot{X}_2(t) = -X_1(t) + U(t)$, $u_{tt} = u_{xx}$, $u_x(0,t) = 0$, $u_x(D,t) = X_1(t)$, $Y(t) = u(0,t)$. With $g > 0$, the observer is obtained as

$$\hat{u}_{tt}(x,t) = \hat{u}_{xx}(x,t) - gc_0\sin(x)\left(Y(t) - \hat{u}(0,t)\right) + g\cos(x)\left(\dot{Y}(t) - \hat{u}_t(0,t)\right) \tag{121}$$
$$\hat{u}_x(0,t) = -gc_0\left(Y(t) - \hat{u}(0,t)\right) - c_0\left(\dot{Y}(t) - \hat{u}_t(0,t)\right) \tag{122}$$
$$\hat{u}(D,t) = \hat{X}_1(t) \tag{123}$$
$$\dot{\hat{X}}_1(t) = \hat{X}_2(t) + g\cos(D)\left(Y(t) - \hat{u}(0,t)\right) \tag{124}$$
$$\dot{\hat{X}}_2(t) = -\hat{X}_1(t) + U(t) - gc_0\sin(D)\left(Y(t) - \hat{u}(0,t)\right). \tag{125}$$

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Stability Analysis of Networked Sampled-Data Linear Systems With Markovian Packet Losses

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***Abstract*—We consider the stability properties of sampled-data networked linear systems with Markovian packet losses. A binary Markov chain is used to characterize the packet loss phenomenon of the network. We show that the sampled-data system under consideration can be considered as a randomly sampled system with an i.i.d. random sampling period. Necessary and sufficient conditions for the stochastic stability properties are established. Those conditions are based on the relationships of stability properties between the systems evolving in deterministic continuous time, deterministic discrete time, and random discrete time. In addition, the asymptotic stability of the system is also studied by using Lyapunov exponent method.**

***Index Terms*—I.i.d. random sampling, Markovian packet losses, sampled-data linear systems, stability.**

I. Introduction

Networked control has been a very hot research topic over the past decade. In networked control systems, there are wireless communication channels or networks between sensors, controllers, and actuators. Due to congestion and fading in communication channels, data losses may occur, which may result in system performance degradation or even instability. A sampled-data system is a networked control system if discrete-time signals of the sampled-data system are transmitted to the discrete-time controller via a digital communication channel. Traditionally, the communication link is assumed to be an ideal one which has infinite bandwidth and data packet dropout does not occur. In this technical note, we address the stability analysis of sampled-data networked linear systems with packet losses characterized by a Markov chain.

Recent work has advanced the research for networked control systems with packet losses. For example, [1] considered Kalman filtering for discrete-time linear systems with randomly intermittent observations, and the packet loss process is assumed to be an i.i.d. Bernoulli binary random sequence. Reference [2] considered estimation with lossy measurements in which the random packet loss is assumed to be governed by a two-state Markov chain. More recently, [3] studied the stability of Kalman filtering with Markovian packet losses by introducing stopping times to describe the transmission time or update time of measurements; see also [4], [5]. For sampled-data networked linear systems, [6] considered a stability problem of model-based networked sampled-data systems, where update time is defined as the time between two successful consecutive transmissions and considered as a bounded time-varying variable, an i.i.d. random variable, or a finite state Markov chain, respectively.

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