Robust Control of Nonlinear Systems with Input Unmodeled Dynamics

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***Abstract*—In the rich repertoire of design methodologies for uncertain nonlinear systems, the problem of unmodeled dynamics has thus far received little attention. We initiate the investigation of nonlinear systems with *input* unmodeled dynamics. First we show that even in their simplest forms, they can result in dramatic shrinking of the region of attraction and in finite escape time. In this paper we introduce a *dynamic nonlinear damping* design which guarantees global boundedness in the presence of input unmodeled dynamics. With this design we achieve global asymptotic stability for strict-feedback systems and output-feedback systems.**

I. Introduction

Recent years have witnessed a rapid development of design methodologies for feedback control of nonlinear systems. In addition to the nonlinear geometric designs based on exact system models, a significant progress has been made in solving problems for nonlinear systems with two types of uncertainties. For nonlinear systems with unknown parameters, recent adaptive recursive designs achieve global boundedness and tracking [12]. For nonlinear systems with unknown static nonlinearities satisfying known functional bounds, several robust designs are available [1], [14], [15], [18], [3], [21], [4], [11], [2], [12].

In this rich repertoire of design methodologies the problem of unmodeled *dynamics* has received less attention. First results in this direction were presented in [6] and [7] using a control Lyapunov functions framework and in [17], [23], and [8] using a formulation with inaccessible input-to-state stable zero dynamics. These references do not address the important problem of unmodeled dynamics appearing at the system input. Thus far, this problem was only treated by Qu [20] who considered linear systems with linear input unmodeled dynamics and matched output-dependent nonlinear uncertainties.

A Motivating Example: In this paper we investigate the destabilizing effects of input unmodeled dynamics and develop a design method to guarantee boundedness and stability. Suppose that under the assumption $\mu = 0$, the feedback control law for the plant[1]

$$\dot{x} = 2x^3 + (1 + \mu\Delta(s))u \tag{1}$$

is designed as

$$u = -\tfrac{1}{2}x - 2x^3. \tag{2}$$

This design is based on the expectation that the resulting feedback system will be $\dot{x} = -\frac{1}{2}x$ which is true if $\mu = 0$. But how will the actual feedback system behave when $\mu \neq 0$? Let us examine the effects of two types of unmodeled dynamics: $\Delta_1(s) = -(1/s+1)$ and $\Delta_2(s) = -(s/s+1)$. For a qualitative analysis we represent the two resulting feedback systems in the form of second-order differential equations

$$\ddot{x}_1 + \frac{3}{2}\dot{x}_1 + \frac{1}{2}(1 - \mu - 4\mu x_1^2)x_1 = 0 \tag{3}$$

$$\ddot{x}_2 + \frac{3}{2}\left(1 - \frac{\mu}{3} - 4\mu x_2^2\right)\dot{x}_2 + \frac{1}{2}x_2 = 0. \tag{4}$$

In the first case two unstable equilibria of (3) appear, and the region of attraction of the stable equilibrium $x_1 = 0$ is a strip that shrinks with a rate $1/\sqrt{\mu}$. The second case is more interesting. The familiar Van der Pol-like form of (4) suggests the possibility of an unstable limit cycle. Indeed, Fig. 1 for $\mu = 1/5$ shows that the boundary of the region of attraction of $x_2 = 0$ is an unstable limit cycle. As it can be deduced from the term $1 - \mu/3 - 4\mu x_2^2$ in (4), this region shrinks with a rate of $1/\sqrt{\mu}$.

Organization of the Paper: In applications for which the above stability region is intolerably small, a new design is required to expand it. In Section II we introduce our *dynamic nonlinear damping* design which employs a filter inspired by the "normalizing signal" introduced to adaptive control by Praly [16] to bound uncertainties affecting the parameter update law. The dynamic nonlinear damping design, introduced in Section II for scalar systems, is extended in Sections III and IV to higher-order systems via integrator backstepping. In Section III we achieve global asymptotic stability for perturbed strict-feedback systems using full state feedback. In Section IV we consider output-feedback systems and achieve stabilization using output feedback only.

II. Dynamic Nonlinear Damping

The dynamic nonlinear damping design is first developed for scalar systems of the form

$$\dot{x} = f(x) + g(x)(1 + \mu\Delta(s))u, \quad x, u \in \mathbb{R} \tag{5}$$

where f and g are locally Lipschitz, $\mu \geq 0$, and $\Delta(s)$ satisfies the following assumption.

Assumption 2.1: The transfer function $\Delta(s)$ of the unmodeled dynamics is stable, proper, and analytic in $\Re e\{s\} \geq -\delta$, where $\delta > 0$ is known.

For a minimal realization of $\Delta(s)$

$$\dot{\xi} = A_\Delta \xi + b_\Delta u, \quad \xi \in \mathbb{R}^{n_\xi} \tag{6}$$

$$\Delta(s)u = c_\Delta \xi + qu \tag{7}$$

we have

$$|\Delta(s)u - qu| \leq k_0|\xi(0)|e^{-\delta t} + \|h_\delta\|_\infty \int_0^t e^{-\delta(t-\tau)}|u(\tau)|\, d\tau \tag{8}$$

where $k_0 \geq 0$ and $h_\delta(t) = c_\Delta e^{(A_\Delta + \delta I)t} b_\Delta$.

Lemma 2.1 (Dynamic Nonlinear Damping): Assume that a C^1 feedback control $u = \alpha(x)$ and a C^1 Lyapunov function V are known such that

$$\gamma_1(|x|) \leq V(x) \leq \gamma_2(|x|) \tag{9}$$

$$L_{f+g\alpha}V(x) \leq -\gamma_3(|x|), \quad \forall x \in \mathbb{R} \tag{10}$$

where γ_1, γ_2 are class $\mathcal{K}_\infty$ functions and γ_3 is a class $\mathcal{K}$ function. Assume, in addition, that there exists a class $\mathcal{K}_\infty$ function γ such that

$$|x| \leq \gamma(|L_gV(x)|), \quad \forall x \in \mathbb{R}. \tag{11}$$

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[1] For brevity, we perpetuate the customary abuse of notation by mixing nonlinear time-domain with linear frequency-domain models.

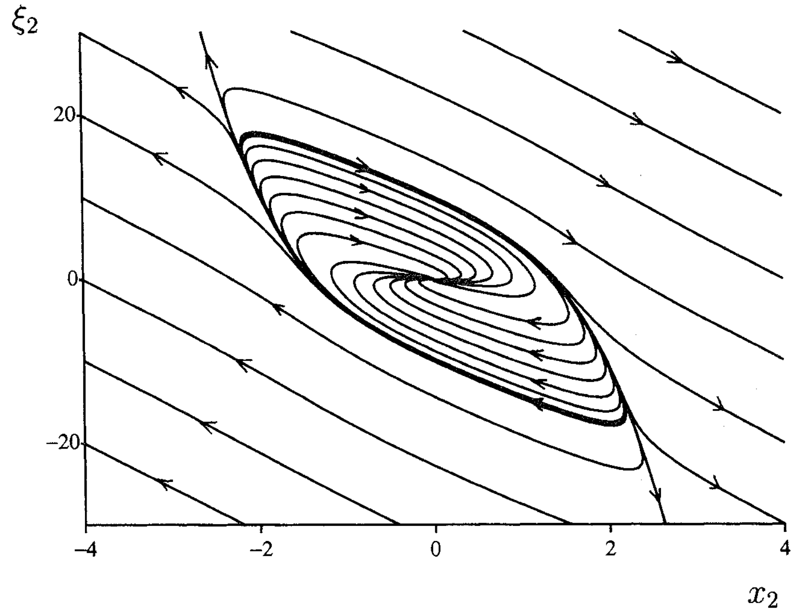

Fig. 1. Phase portrait of the perturbed closed-loop system with $\mu = 0.2$ and $\Delta_2(s) = -(s/s+1)$, where we denote $\xi_2 = -\Delta_2(s)u$. The unstable limit cycle is the boundary of the stability region. Its projection to the x-axis is approximately the interval $(-2.2, 2.2)$.

Then the dynamic feedback control law

$$u = \alpha(x) - \kappa(1 + |m| + |\alpha(x)|)L_gV(x), \quad \kappa > 0 \tag{12}$$

$$\dot{m} = -\delta m + |u| \tag{13}$$

guarantees that there exists $\mu^* > 0$ independent of initial conditions such that x, m, u, ξ are globally bounded for all $\mu \in [0, \mu^*)$. Furthermore, $x(t)$ converges to the interval

$$\left\{|x| \le \rho\left(\mu\frac{h + |q|\rho(\delta)}{\kappa}\right)\right\}, \quad h = \max\{\|h_\delta\|_\infty, |q|\} \tag{14}$$

where ρ is a class $\mathcal{K}$ function.

Remark 2.1: Each of the robustifying feedback terms in (12) has a separate role. While $-\kappa L_gV$ counteracts the exponentially decaying effect of initial conditions, $-\kappa|m|L_gV$ counteracts the strictly proper part of the perturbation $\Delta(s)$, and $-\kappa|\alpha|L_gV$ counteracts the throughput term q.

Proof of Lemma 2.1: The proof consists of two parts. In the first part we establish that $x \in \mathcal{L}^\infty$ and $m, \xi \in \mathcal{L}^{\infty,e}$. In the second part we show that $x(t)$ converges to an $O(\mu)$-interval which allows us to establish uniform boundedness of m, u, and ξ.

From (8) and (13) we first note that

$$|\Delta(s)u - qu| \le (k_0|\xi(0)| + \|h_\delta\|_\infty|m(0)|)e^{-\delta t} + \|h_\delta\|_\infty|m|. \tag{15}$$

By substituting (12) into (5), we obtain

$$\begin{aligned}\dot{x} &= f + g\alpha - \kappa(1 + \mu q)(1 + |m| + |\alpha|)gL_gV \\ &\quad + \mu g(\Delta u - qu + q\alpha).\end{aligned} \tag{16}$$

Since all the nonlinearities in (12), (13), and (16) are locally Lipschitz, the solution $x(t), m(t), \xi(t)$ exists and is unique on its maximal interval of existence $[0, t_f)$. By (10) and for $\mu|q| < 1$, the derivative of V along the solutions of (16) is

$$\begin{aligned}\dot{V} &= L_{f+g\alpha}V - \kappa(1 + \mu q)(1 + |m| + |\alpha|)(L_gV)^2 \\ &\quad + \mu(\Delta u - qu + q\alpha)L_gV \\ &\le -\gamma_3(|x|) - \kappa(1 - \mu|q|)(1 + |m| + |\alpha|)|L_gV| \\ &\quad \cdot \left[\gamma^{-1}(|x|) - \frac{\mu}{\kappa(1 - \mu|q|)} \cdot \frac{(k_0|\xi(0)| + \|h_\delta\|_\infty|m(0)|)e^{-\delta t} + \|h_\delta\|_\infty|m| + |q|\,|\alpha|}{1 + |m| + |\alpha|}\right].\end{aligned} \tag{17}$$

Hence, we have

$$\begin{aligned}|x| &\ge \gamma\left(\frac{\mu}{\kappa(1 - \mu|q|)}\max\{(k_0|\xi(0)| + \|h_\delta\|_\infty|m(0)|)\cdot e^{-\delta t},\right. \\ &\quad \left.\|h_\delta\|_\infty, |q|\}\right) \Rightarrow \dot{V} \le -\gamma_3(|x|).\end{aligned} \tag{18}$$

This in conjunction with (9) implies not only that x is bounded on $[0, t_f)$ but also that the bound is independent of t_f

$$\begin{aligned}|x(t)| &\le \gamma_1^{-1} \circ \gamma_2\left(\max\left\{\gamma\left(\frac{\mu}{\kappa(1 - \mu|q|)}\max\{k_0|\xi(0)|\right.\right.\right. \\ &\quad \left.\left.\left. + \|h_\delta\|_\infty|m(0)|, \|h_\delta\|_\infty, |q|\}\right), |x(0)|\right\}\right).\end{aligned} \tag{19}$$

Next, we show that $m(t)$ and $\xi(t)$ are bounded on $[0, t_f)$ by a positive function of t_f which is continuous on $[0, +\infty)$. This will enable us to establish that $t_f = +\infty$. From (12) and (13) we have

$$\frac{d}{dt}|m| \le (\kappa|L_gV| - \delta)|m| + |\alpha - \kappa(1 + |\alpha|)L_gV|. \tag{20}$$

In view of (19) and the continuity of α and L_gV, there exists a constant

$$G = \max\{\sup_{t\in[0,t_f)}\{\kappa|L_gV| - \delta\}, \sup_{t\in[0,t_f)}\{|\alpha - \kappa(1 + |\alpha|)L_gV|\}\} \tag{21}$$

which is independent of t_f, such that $(d/dt)|m| \le G|m| + G$. Thus, $|m(t)|$ is bounded by a function $M(t_f)$ which is continuous on $[0, \infty)$

$$|m(t)| \le [|m(0)| + 1]e^{Gt_f} \triangleq M(t_f), \quad \forall t \in [0, t_f). \tag{22}$$

Therefore, by (12) u is bounded on $[0, t_f)$ by a function of t_f which is continuous on $[0, \infty)$. The same is true for ξ because of (6) and the fact that A_Δ is Hurwitz. Thus the state $X = (x, m, \xi)$ of the feedback system (5), (6), (12), and (13) remains for all $t \in [0, t_f)$ inside a ball whose radius $R(t_f)$ is a continuous function on $[0, \infty)$. Therefore, $t_f = \infty$, since otherwise $X(t)$ would escape any compact set as $t \to t_f$, which would contradict the continuity of R. Hence $x \in \mathcal{L}^\infty$ and $m, \xi \in \mathcal{L}^{\infty,e}$. We have yet to prove that $m, \xi \in \mathcal{L}^\infty$.

To prove that $m \in \mathcal{L}^\infty$, we will show that for sufficiently small μ the term $\kappa|L_gV(x(t))| - \delta$ becomes negative in finite time and remains negative thereafter. In view of (18) and (9), by [22], there exists a class $\mathcal{KL}$ function β_1 and a class $\mathcal{K}$ function $\rho_1 = \gamma_1^{-1} \circ \gamma_2 \circ \gamma$ such that for all $0 \le s \le t$ we have

$$|x(t)| \le \beta_1(|x(s)|, t-s) + \rho_1(\sup_{s \le \tau \le t}\{|\zeta(\tau)|\}) \tag{23}$$

$$|\zeta(t)| \le \frac{\mu(k_0|\xi(0)| + \|h_\delta\|_\infty |m(0)|)}{\kappa(1-\mu|q|)} e^{-\delta(t-s)} + \frac{\mu h}{\kappa(1-|\mu|q|)}. \tag{24}$$

This shows that the set to which $x(t)$ converges is an $O(\mu)$-interval. However, it would appear from (24) that this interval depends on $\xi(0)$ and $m(0)$. With the help of [12, Lemma C.4] we now show that this interval is independent of initial conditions which will ultimately make the choice of μ^* independent of initial conditions. By [12, Lemma C.4] we have

$$|x(t)| \le \beta_2(|X(0)|, t) + \rho_2\left(\frac{\mu h}{\kappa(1-\mu|q|)}\right) \tag{25}$$

where the class $\mathcal{KL}$ function β_2 and the class $\mathcal{K}$ function ρ_2 are given by

$$\begin{aligned}\beta_2(r,t) = {}&\beta_1\left(2\beta_1(r, t/2) + 2\rho_1\left(2\frac{\mu(k_0 + \|h_\delta\|_\infty)r}{\kappa(1-\mu|q|)}\right), t/2\right)\\ &+ \rho_1\left(2\frac{\mu(k_0+\|h_\delta\|_\infty)r}{\kappa(1-\mu|q|)}e^{-\delta t/2}\right)\end{aligned} \tag{26}$$

$$\rho_2(r) = \beta_1(2\rho_1(2r), 0) + \rho_1(2r), \quad \forall r, t \ge 0. \tag{27}$$

Since V is C^1 and positive definite (and therefore has a minimum at $x = 0$), then $(\partial V/\partial x)(0) = 0$. Since, in addition, g is continuous, it follows that there exists a class $\mathcal{K}_\infty$ function ρ_3 such that

$$|L_gV(x)| \le \rho_3(|x|). \tag{28}$$

By substituting (25) into (28) and using the fact that if $a, b \ge 0$, then $\rho_3(a+b) \le \rho_3(2a) + \rho_3(2b)$ [22, (12)], we get

$$\begin{aligned}|L_gV(x(t))| &\le \rho_3(2\beta_2(|X(0)|, t)) + \rho_3 \circ 2\rho_2\left(\frac{\mu h}{\kappa(1-\mu|q|)}\right)\\ &\triangleq \beta_4(|X(0)|, t) + \rho_4\left(\frac{\mu h}{\kappa(1-\mu|q|)}\right).\end{aligned} \tag{29}$$

Since α and L_gV vanish at zero and are continuous, there exists a class $\mathcal{K}_\infty$ function ρ_5 such that

$$\begin{aligned}&|\alpha(x) - \kappa(1 + |\alpha(x)|)L_gV(x)|\\ &\quad \le \rho_5(|x|) \le \beta_6(|X(0)|, t) + \rho_6\left(\frac{\mu h}{\kappa(1-\mu|q|)}\right)\end{aligned} \tag{30}$$

where we have again used (25), and $\beta_6 \triangleq \rho_5 \circ 2\beta_2$ and $\rho_6 \triangleq \rho_5 \circ 2\rho_2$. Substitution of (29) and (30) into (20) yields

$$\begin{aligned}\frac{d}{dt}|m| \le {}& -\left[\delta - \kappa\rho_4\left(\frac{\mu h}{\kappa(1-\mu|q|)}\right) - \kappa\beta_4(|X(0)|, t)\right]|m|\\ &+ \beta_6(|X(0)|, t) + \rho_6\left(\frac{\mu h}{\kappa(1-\mu|q|)}\right).\end{aligned} \tag{31}$$

To prove boundedness of m we require that $\delta > \kappa\rho_4(\mu h/\kappa(1-\mu|q|))$, namely

$$\mu < \mu^* \triangleq \left(\frac{h}{\kappa\rho_4^{-1}(\delta/\kappa)} + |q|\right)^{-1} \le \frac{1}{|q|}. \tag{32}$$

By applying [12, Lemma B.8] to (31), we conclude that there exist $\beta_m \in \mathcal{KL}, \rho_m \in \mathcal{K}_\infty$ such that

$$|m(t)| \le \beta_m(|X(0)|, t) + \rho_m\left(\frac{\mu h}{\kappa(1-\mu|q|)}\right), \quad \forall t \ge 0. \tag{33}$$

Thus, m is globally bounded which, along with the boundedness of x, proves that u is bounded. Since $\Delta(s)$ is stable, ξ is bounded. Finally, we note from (25) that $x(t)$ converges to the interval $|x| \le \rho_2(\mu h/\kappa(1-\mu|q|)) \le \rho_2(\mu h/\kappa(1-\mu^*|q|))$. Substituting μ^* from (32) we establish that $x(t)$ converges to the interval $|x| \le \rho_2(\mu(h + |q|\kappa\rho_4^{-1}(\delta/\kappa)/\kappa))$. The choice $\rho(r) = \max\{\rho_2(r), \kappa\rho_4^{-1}(r/\kappa)\}$ proves (14). □

Remark 2.2: Even if we omit $-\kappa L_gV$ from the control law (12), the lemma still holds, but μ^* becomes dependent on initial conditions: μ^* decreases as $|\xi(0)|/m(0)$ increases. In this case we require $m(0) > 0$. □

The following example illustrates that the dynamic nonlinear damping design, in general, does *not* guarantee global asymptotic stability but only the convergence to a μ-small residual set around the origin which may contain, for example, other equilibria or periodic orbits. The example also illustrates that global asymptotic stability is achieved for smaller values of μ which we prove in the next section.

Example 2.1: Let us consider the perturbed nonlinear system

$$\dot{x} = 2x + 2x^3 + (1 + \mu\Delta(s))u \tag{34}$$

where $\Delta(s) = -(s/s+1)$, so that (34) differs from (1) only in the additional term $2x$. This term causes instability of the origin for larger values of μ. A control law of the form (12) is $u = -2.5x - 2x^3 - (m + |2.5x + 2x^3|)x$. It guarantees boundedness of all signals, and for smaller values of μ, exponential stability, as shown in Fig. 2(a) for $\mu = 0.5$. However, for a larger value $\mu = 0.7$, the origin becomes unstable and a stable three-dimensional periodic orbit (in the space (x, ξ, m)) appears. □

III. Strict-Feedback Systems with "Virtual Input" Unmodeled Dynamics

In the previous section we considered scalar systems. We now address higher-order systems in the strict-feedback form

$$\begin{aligned}\dot{x}_i &= (1 + \mu\Delta_i(s))x_{i+1} + \varphi_i(x_1, \cdots, x_i), \qquad 1 \le i \le n-1\\ \dot{x}_n &= (1 + \mu\Delta_n(s))u + \varphi_n(x).\end{aligned} \tag{35}$$

In this highly structured problem, we consider the variable x_{i+1} to be the input of the x_i-subsystem and apply the dynamic nonlinear damping design recursively.

We assume that φ_i are smooth, $\varphi_i(0) = 0$, and each of the transfer functions Δ_i satisfies Assumption 2.1 and has a minimal realization

$$\dot{\xi}_i = A_{\Delta i}\xi_i + b_{\Delta i}x_{i+1}, \quad \xi_i \in \mathbb{R}^{n_{\xi i}} \tag{36}$$

$$\Delta_i(s)x_{i+1} = c_{\Delta i}\xi_i + q_i x_{i+1} \tag{37}$$

where $x_{n+1} \triangleq u$.

In our backstepping design the control law will be built recursively, starting from the stabilizing function α_1 which robustly stabilizes the x_1-subsystem against the perturbation Δ_1 using dynamic nonlinear damping. Since the recursive design involves partial derivatives, we must ensure that our stabilizing functions are sufficiently differentiable. To this end, we will replace the absolute value function $|m|$ by a smooth approximation $|m|_\varepsilon \triangleq \sqrt{|m|^2 + \varepsilon^2} - \varepsilon$, where ε is a positive constant. (Note that $|\cdot|_\varepsilon$ is not a norm.)

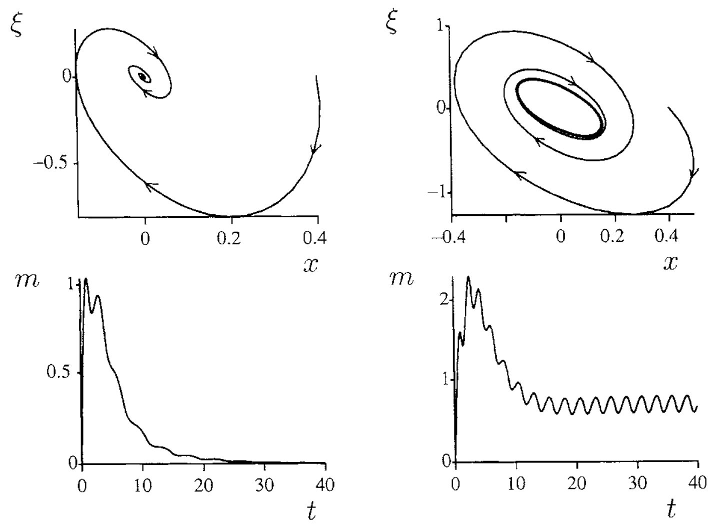

Fig. 2. Behavior inside the residual set. For $\mu = 0.5$, the origin is exponentially stable. For a larger $\mu = 0.7$, the origin becomes unstable, and a stable periodic orbit appears in the space (x, ξ, m).

Step 1: We define $z_1 = x_1$ and $z_2 = x_2 - \alpha_1$. As suggested by Lemma 2.1, we use

$$v_1 = -c_1 z_1 - \varphi_1 \tag{38}$$

$$w_1 = 1 + |m_1|_\varepsilon + |v_1|_\varepsilon \tag{39}$$

$$\alpha_1 = v_1 - \kappa_1 w_1 z_1 \tag{40}$$

$$\dot{m}_1 = -\delta m_1 + |x_2|_\varepsilon \tag{41}$$

where $c_1, \kappa_1, \varepsilon > 0$. The stabilizing function α_1 consists of two terms: v_1, which is the nominal part of α_1 for the case $\mu = 0$, and $-\kappa_1 w_1 z_1$ which is the dynamic nonlinear damping term for the case $\mu > 0$. Denoting the perturbation $\eta_1 \triangleq q_1 v_1 + (\Delta_1 - q_1)x_2$, the resulting z_1 subsystem is rewritten as

$$\dot{z}_1 = -[c_1 + \kappa_1(1 + \mu q_1)w_1]z_1 + (1 + \mu q_1)z_2 + \mu\eta_1. \tag{42}$$

Step $i(2 \le i \le n)$: Introducing $z_{i+1} = x_{i+1} - \alpha_i$, we have

$$\begin{aligned}\dot{z}_i =& (1 + \mu\Delta_i)x_{i+1} + \varphi_i \\ &- \sum_{j=1}^{i-1} \frac{\partial\alpha_{i-1}}{\partial x_j}((1 + \mu\Delta_j)x_{j+1} + \varphi_j) \\ &- \sum_{j=1}^{i-1} \frac{\partial\alpha_{i-1}}{\partial m_j}(-\delta m_j + |x_{j+1}|_\varepsilon).\end{aligned} \tag{43}$$

We see that the stabilizing function α_i has to account not only for Δ_i but also for $\Delta_1, \cdots, \Delta_{i-1}$. Therefore, we design

$$\begin{aligned}v_i =& -c_i z_i - \varphi_i + \sum_{j=1}^{i-1} \frac{\partial\alpha_{i-1}}{\partial x_j}(x_{j+1} + \varphi_j) \\ &+ \sum_{j=1}^{i-1} \frac{\partial\alpha_{i-1}}{\partial m_j}(-\delta m_j + |x_{j+1}|_\varepsilon)\end{aligned} \tag{44}$$

$$\begin{aligned}w_i =& 1 + |m_i|_\varepsilon + \sum_{j=1}^{i-1}(1 + |m_j|_\varepsilon)\left(\left|\frac{\partial\alpha_{i-1}}{\partial x_j}\right|_\varepsilon + \varepsilon\right) \\ &+ |v_i|_\varepsilon + \sum_{j=1}^{i-1}\left|\frac{\partial\alpha_{i-1}}{\partial x_j}x_{j+1}\right|_\varepsilon\end{aligned} \tag{45}$$

$$\alpha_i = v_i - \kappa_i w_i z_i \tag{46}$$

$$\dot{m}_i = -\delta m_i + |x_{i+1}|_\varepsilon. \tag{47}$$

The actual control law is

$$u = \alpha_n. \tag{48}$$

Denoting the perturbation terms by

$$\begin{aligned}\eta_i \triangleq& (\Delta_i - q_i)x_{i+1} - \sum_{j=1}^{i-1} \frac{\partial\alpha_{i-1}}{\partial x_j}(\Delta_j - q_j)x_{j+1} + q_i v_i \\ &- \sum_{j=1}^{i-1} \frac{\partial\alpha_{i-1}}{\partial x_j} q_j x_{j+1}\end{aligned} \tag{49}$$

we obtain the complete z-system in the bidiagonal form of (50) shown at the bottom of the page.

We denote $\xi = [\xi_1^T, \cdots, \xi_n^T]^T, m = [m_1, \cdots, m_n]^T, Z = (z, \xi, m)$, and $X = (x, \xi, m)$.

Theorem 3.1: For the closed-loop system (36), (41), (47), and (50), the following is true:

a) There exists $\mu^* > 0$ independent of initial conditions such that X and u are globally bounded for all $\mu \in [0, \mu^*)$. Furthermore, $x(t)$ converges to a ball whose radius is proportional to μ and independent of initial conditions.

b) There exists $\mu_1^* \le \mu^*$ independent of initial conditions such that the equilibrium $X = 0$ is globally asymptotically stable as

$$\dot{z} = \begin{bmatrix} -[c_1 + \kappa_1(1 + \mu q_1)w_1] & 1 + \mu q_1 & & 0 \\ 0 & \ddots & \ddots & \\ \vdots & \ddots & \ddots & 1 + \mu q_{n-1} \\ 0 & \cdots & 0 & -[c_n + \kappa_n(1 + \mu q_n)w_n] \end{bmatrix} z + \mu \begin{bmatrix} \eta_1 \\ \vdots \\ \eta_n \end{bmatrix}. \tag{50}$$

well as exponentially stable in any ball of finite radius around the origin, for all $\mu \in [0, \mu_1^*)$.

Proof: The proof is lengthy, and only its main points are given.

a) First we prove that z is bounded and converges to a compact set whose size is $O(\mu)$. From (36), (37) and (47), noting that $|x| \leq |x|_\varepsilon + \varepsilon$, we show that

$$|(\Delta_i(s) - q_i)x_{i+1}| \leq (k_{0i}|\xi_i(0)| + \|h_{\delta i}\|_\infty |m(0)|)e^{-\delta t} + \varepsilon\left(1 + \frac{1}{\delta}\right)\|h_{\delta i}\|_\infty + \|h_{\delta i}\|_\infty |m_i|_\varepsilon \tag{51}$$

where $k_{0i} \geq 0$ and $h_{\delta i}(t) = c_{\Delta i}e^{(A_{\Delta i}+\delta I)t}b_{\Delta i}$. To prove the boundedness of z and the convergence to an $O(\mu)$ set, we first show that each z_i-subsystem is ISpS (see Appendix for definition of ISpS) with z_{i+1} as input and an $O(\mu)$ ultimate bound. Along the solutions of (50), for $i \in \{1, \cdots, n-1\}$, we have

$$\begin{aligned}\frac{d}{dt}\left(\frac{z_i^2}{2}\right) &\leq -[c_i + \kappa_i(1 - \mu|q_i|)w_i]z_i^2 + \mu\eta_i z_i + (1 + \mu q_i)z_{i+1}z_i \\ &\leq -\frac{c_i}{2}z_i^2 - \kappa_i(1 - \mu|q_i|)w_i|z_i| \cdot \left[|z_i| - \frac{\mu}{\kappa_i(1 - \mu|q_i|)}\frac{|\eta_i|}{w_i}\right] \\ &\quad - \frac{c_i}{2}|z_i|\left[|z_i| - \frac{2(1 + \mu|q_i|)}{c_i}|z_{i+1}|\right].\end{aligned} \tag{52}$$

Using (45), (49), and (51), we prove that

$$\frac{|\eta_i|}{w_i} \leq (k_0|\xi(0)| + h_0|m(0)|)e^{-\delta t} + h \tag{53}$$

where $k_0 = \max_{1\leq j\leq n} k_{0j}, h_0 = \max_{1\leq j\leq n}\|h_{\delta j}\|_\infty, q = \max_{1\leq j\leq n}|q_j|$, and $h = \max\{h_0, q, \varepsilon[(1 + 1/\delta)h_0 + nq]\}$. Substituting (53) into (52), we get

$$|z_i| \geq \left\{\frac{\mu}{\kappa(1 - \mu q)}[(k_0|\xi(0)| + h_0|m(0)|)e^{-\delta t} + h] + \frac{2(1 + \mu q)}{c}|z_{i+1}|\right\} \Rightarrow \frac{d}{dt}(z_i^2) \leq -cz_i^2 \tag{54}$$

where $\kappa = \min_{1\leq j\leq n}\kappa_j$ and $c = \min_{1\leq j\leq n} c_j$. Therefore, by [22], for all $0 \leq s \leq t$

$$|z_i(t)| \leq |z_i(s)|e^{-ct/2} + \sup_{s\leq\tau\leq t}\left\{\frac{\mu}{\kappa(1 - \mu q)}[(k_0|\xi(0)| + h_0|m(0)|)e^{-\delta t} + h] + \frac{2(1 + \mu q)}{c}|z_{i+1}(\tau)|\right\}. \tag{55}$$

Then, applying [12, Lemma C.4], we obtain

$$|z_i(t)| \leq \left[|z_i(0)| + 2\frac{\mu(k_0|\xi(0)| + h_0|m(0)|)}{\kappa(1 - \mu q)}\right] \cdot e^{-\min\{(c/4),(\delta/2)\}t} + \frac{4(1 + \mu q)}{c}\sup_{0\leq\tau\leq t}|z_{i+1}(\tau)| + \frac{2\mu h}{\kappa(1 - \mu q)} \tag{56}$$

for $i \in \{1, \cdots, n-1\}$. Thus, we have made the ultimate bound $2\mu h/\kappa(1 - \mu q)$ independent of initial conditions. All the z_i-equations from (50) have two inputs, z_{i+1} and η_{i+1}, except for the z_n-equation which has only one input: η_n. When the same arguments as (52)–(56) are applied to the z_n-equation, we get

$$|z_n(t)| \leq \left(|z_n(0)| + 2\frac{\mu(k_0|\xi(0)| + h_0|m(0)|)}{\kappa(1 - \mu q)}\right) \cdot e^{-\min\{(c/2),(\delta/2)\}t} + \frac{2\mu h}{\kappa(1 - \mu q)}. \tag{57}$$

Inequalities (56) and (57) define a set of ISpS inequalities. By repeatedly applying [12, Lemma C.4], we show that there exist positive real numbers β_1, ρ_1, σ independent of initial conditions such that

$$\begin{aligned}|z(t)| &\leq \beta_1\left(1 + \frac{1 + \mu q}{c}\right) \cdot \left(|z(0)| + \frac{\mu(k_0|\xi(0)| + h_0|m(0)|)}{\kappa(1 - \mu q)}\right)e^{-\sigma t} \\ &\quad + \rho_1\left(1 + \frac{1 + \mu q}{c}\right)\frac{\mu h}{\kappa(1 - \mu q)}.\end{aligned} \tag{58}$$

By restricting μq to [0, 3/4], we arrive at a bound linear in μ

$$|z(t)| \leq \beta|Z(0)|e^{-\sigma t} + \gamma\mu \tag{59}$$

where β, γ are positive constants.

Thus z is bounded which implies that x_1 is bounded. Before we proceed, let us note that for any locally Lipschitz class $\mathcal{K}$ function ρ, the following inequality is readily shown to hold:

$$\rho(re^{-\sigma t}) \leq \rho^*(r)e^{-\sigma t} \tag{60}$$

where ρ^* is a locally Lipschitz class $\mathcal{K}_\infty$ function given by $\rho^*(r) \triangleq r\sup_{s\in[0,r]}\ \rho(s)/s$. In the rest of the proof, d will denote a generic positive finite constant (independent of initial conditions and μ), and $\rho(\cdot)$ will denote a generic locally Lipschitz class $\mathcal{K}_\infty$ function. We now prove that $x_2, \cdots, x_n, \xi, m, u$ are also bounded and converge to an $O(\mu)$ set. From (41), using (40), recalling that $v_1(0) = 0$, and noting that $|x|_\varepsilon \leq |x|$, we proceed as follows:

$$\begin{aligned}\frac{d}{dt}|m_1| &\leq -\delta|m_1| + |z_2| + |\alpha_1| \\ &\leq (-\delta + \kappa_1|z_1|)|m_1| + |z_2| + |v_1| + \kappa_1(1 + |v_1|_\varepsilon)|z_1| \\ &\leq -[\delta - \kappa(\beta|Z(0)|e^{-\sigma t} + \gamma\mu)]|m_1| + \rho(|Z(0)|)e^{-dt} \\ &\quad + \rho(\mu).\end{aligned} \tag{61}$$

If $\mu < \min\{(3/4q), (\delta/2\kappa\gamma)\}$, then by applying [12, Lemma B.8], we get

$$|m_1(t)| \leq \rho(|Z(0)|)e^{-dt} + \rho(\mu). \tag{62}$$

Recalling that $x_2 = z_2 + \alpha_1(z_1, m_1)$ and $\alpha_1(0, m_1) \equiv 0$, it follows that $|x_2(t)| \leq \rho(|Z(0)|)e^{-dt} + \rho(\mu)$, and hence $|\xi_1(t)| \leq \rho(|Z(0)|)e^{-dt} + \rho(\mu)$. The boundedness of $m_2, \cdots, m_n$ is established using an induction argument. Supposing that $|m_{i-1}(t)| \leq \rho(|Z(0)|)e^{-dt} + \rho(\mu)$, and noting that $v_i(z, m)|_{z=0} \equiv 0$, we have

$$\begin{aligned}\frac{d}{dt}|m_i| &\leq -\delta|m_i| + |z_{i+1}| + |\alpha_i| \\ &\leq -[\delta - \kappa(\beta|Z(0)|e^{-\sigma t} + \gamma\mu)]|m_i| + \rho(|Z(0)|)e^{-dt} \\ &\quad + \rho(\mu)\end{aligned} \tag{63}$$

and by [12, Lemma B.8] conclude that $\mu < \min\{(3/4q), (\delta/2\kappa\gamma)\}$ implies $|m_i(t)| \leq \rho(|Z(0)|)\ e^{-dt} + \rho(\mu)$. By examining (44)–(46), we observe that $\alpha_i(z, m)|_{z=0} \equiv 0$. Thus, from $x_{i+1} = z_{i+1} + \alpha_i$ it follows that $|x_{i+1}(t)|, |\xi_i(t)| \leq \rho(|Z(0)|)e^{-dt} + \rho(\mu)$. Thus, we prove that

$$|X(t)| \leq \rho_0(|X(0)|)e^{-dt} + \rho_\mu(\mu) \tag{64}$$

where ρ_0 and ρ_μ are locally Lipschitz class $\mathcal{K}_\infty$ functions. Hence, all the states, as well as the control u, are globally bounded whenever

$$\mu < \mu^* = \min\left\{\frac{3}{4q}, \frac{\delta}{2\kappa\gamma}\right\}. \tag{65}$$

b) Now we prove the stability part. First, given a ball of radius R around the origin, we show that the solution $X(t) = (x(t), \xi(t), m(t))$ enters the ball in finite time provided μ is sufficiently small. Second, we show that the equilibrium at the origin of the system with $\mu = 0$ is exponentially stable, and its region of attraction contains any ball of finite radius around the origin. Third, we prove that, for sufficiently small μ, the perturbed system also has an exponentially stable equilibrium at the origin whose region of attraction contains the ball B_R.

We first rewrite the closed-loop system

$$\begin{aligned}\dot{x}_i &= x_{i+1} + \varphi_i(x_1, \cdots, x_i) + \mu(c_{\Delta i}\xi_i + q_i x_{i+1}),\\ &\qquad 1 \le i \le n-1\\ \dot{x}_n &= u(x,m) + \varphi_n(x) + \mu(c_{\Delta n}\xi_n + q_n u(x,m))\\ \dot{\xi} &= A_\Delta \xi + b_\Delta \begin{bmatrix} x_2 \\ \vdots \\ x_n \\ u(x,m)\end{bmatrix}, \quad \dot{m} = -\delta m + \begin{bmatrix} |x_2|_\varepsilon \\ \vdots \\ |x_n|_\varepsilon \\ |u(x,m)|_\varepsilon\end{bmatrix}\end{aligned}$$

where $A_\Delta = \operatorname{diag}\{A_{\Delta 1}, \cdots, A_{\Delta n}\}, b_\Delta = \operatorname{diag}\{b_{\Delta 1}, \cdots, b_{\Delta n}\}$ in the following compact form:

$$\dot{X} = F_0(X) + \mu F_1(X). \tag{66}$$

From the bound (64) on the state of this system, we draw the following two conclusions.

1) Given any positive constant $R \ge 2\rho_\mu(\mu)$, there exists a finite time $T = \max\{0, 1/d \ln 2\rho_0(|X(0)|)/R\}$ such that the solution $X(t)$ of the system (66) enters the ball B_R at $t \le T$. (Note that μ is chosen so that $B_{\rho_\mu(\mu)} \subseteq B_{R/2}$.) This implies, in particular, that $|X(t)| \le 2\rho_0(|X(0)|)e^{-dt}$, $\forall t \le T$.
2) The unperturbed system $\dot{X} = F_0(X)$ is globally asymptotically stable as well as exponentially stable in any ball of finite radius around the origin.

Because of the latter conclusion, by the standard converse Lyapunov theorem for exponential stability (see, e.g., [10, Th. 4.5]), there exists a C^1 function $V\colon B_{\sqrt{(\alpha_2/\alpha_1)}R} \to \mathbb{R}_+$ and positive constants $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ such that

$$\begin{aligned}&\alpha_1|X|^2 \le V(x) \le \alpha_2|X|^2, \quad \frac{\partial V}{\partial X}F_0(X) \le -\alpha_3|X|^2\\ &\left|\frac{\partial V}{\partial X}\right| \le \alpha_4|X|.\end{aligned} \tag{67}$$

Since the perturbation F_1 is locally Lipschitz and vanishes at the origin, there exists a positive constant L such that $|F_1(X)| \le L|X|$, $\forall X \in B_R$. In view of (67), the derivative of V along the solutions of (66) is

$$\dot{V} \le -(\alpha_3 - \mu L\alpha_4)|X|^2, \quad \forall t \ge T \tag{68}$$

where it is important to note that α_3, α_4 and L depend only on R but not on μ. Hence the equilibrium at the origin of the system (66) is exponentially stable provided $\mu < \alpha_3/L\alpha_4$. Its region of attraction contains the positively invariant set $\{V(X) \le \alpha_2 R^2\} \supset B_R$. Thus we have proven the following: if $|X(0)| \le \rho_0^{-1}(R/2)$, in which case $T = 0$, then

$$|X(t)| \le \sqrt{\frac{\alpha_2}{\alpha_1}}|X(0)|e^{-(\alpha_3 - \mu L\alpha_4/2\alpha_2)t}, \quad \forall t \ge 0. \tag{69}$$

If $|X(0)| \ge \rho_0^{-1}(R/2)$, in which case $T = 1/d \ln 2\rho_0(|X(0)|)/R > 0$, then

$$\begin{aligned}|X(t)| &\le \begin{cases} 2\rho_0(|X(0)|)e^{-dt} & t \le T\\ \sqrt{\frac{\alpha_2}{\alpha_1}}Re^{-(\alpha_3 - \mu L\alpha_4/2\alpha_2)(t-T)} & t \ge T\end{cases}\\ &\le 2\rho_0(|X(0)|)e^{-dt} + \sqrt{\frac{\alpha_2}{\alpha_1}}2\rho_0(|X(0)|)\\ &\quad \cdot \left(\frac{2\rho_0(|X(0)|)}{R}\right)^{\alpha_3 - \mu L\alpha_4/2d\alpha_2} e^{-(\alpha_3 - \mu L\alpha_4/2\alpha_2)t}.\end{aligned} \tag{70}$$

From the last two inequalities it follows that

$$|X(t)| \le \rho(|X(0)|)e^{-dt}. \tag{71}$$

Hence, if

$$\mu < \mu_1^* \triangleq \min\left\{\mu^*, \rho_\mu^{-1}\left(\frac{R}{2}\right), \frac{\alpha_3}{L\alpha_4}\right\} \tag{72}$$

then the equilibrium at the origin is globally asymptotically stable as well as exponentially stable in any ball of finite radius around the origin.[2] □

The design presented in this section applies, in particular, to the system

$$\dot{x} = f(x) + g(x)(1 + \mu\Delta(s))u, \qquad x \in \mathbb{R}^n \tag{73}$$

which is assumed to be feedback linearizable for $\mu = 0$ (see [13] for details).

IV. Output Feedback with Input Unmodeled Dynamics

When only the output y of a nonlinear system is available for measurement, the input unmodeled dynamics $\Delta(s)$ are more difficult to deal with because they interfere, not only with the control, but also with the state estimation. We restrict our attention to systems in the output-feedback form with input unmodeled dynamics

$$\begin{aligned}\dot{x} &= Ax + \varphi(y) + b(1 + \mu\Delta(s))[\sigma(y)u]\\ y &= e_1^T x = x_1\end{aligned} \tag{74}$$

$$A = \begin{bmatrix} 0 & & \\ \vdots & I_{n-1} & \\ 0 & \cdots & 0\end{bmatrix}, \qquad b = [0, \cdots, 0, b_m, \cdots, b_0]^T. \tag{75}$$

We assume that $\Delta(s)$ satisfies Assumption 2.1 and has a state-space representation (6) and (7) with u replaced by $\sigma(y)u, B(s) = b_m s^m + \cdots + b_1 s + b_0$ is a Hurwitz polynomial, σ and the components $\varphi_1, \cdots, \varphi_n$ of φ are smooth functions, $\varphi_1(0), \cdots, \varphi_n(0) = 0$, and $\sigma(y) \ne 0$ $\forall y \in \mathbb{R}$.

For the case $\mu = 0$, an observer-based controller has been designed in [9]. We employ the same observer

$$\dot{\hat{x}} = A_0\hat{x} + K_0 y + \varphi(y) + b\sigma(y)u \tag{76}$$

[2] It is important to note that (71) does not imply global exponential stability because for a class $\mathcal{K}_\infty$ function ρ, which is not globally Lipschitz, there does not exist a globally valid Lipschitz constant such that the standard definition of exponential stability [5, Def.26.2] which assumes linearity in the initial condition is satisfied.

where K_0 is chosen so that $A_0 = A - K_0 e_1^T$ is Hurwitz, but we also require that all of the eigenvalues of A_0 have real parts smaller than $-\delta$. Along with the observer, let us introduce its input-free replica

$$\dot{\zeta} = A_0\zeta + K_0 y + \varphi(y), \quad \zeta(0) = \hat{x}(0). \tag{77}$$

Subtracting (76) from (74) we obtain the observer error system

$$\dot{\tilde{x}} = A_0\tilde{x} + b\mu\Delta(s)[\sigma(y)u], \quad \tilde{x} = x - \hat{x}. \tag{78}$$

The states of the three systems, (76)–(78), are related by the following equality:

$$\tilde{x} = \mu\Delta(s)(\hat{x} - \zeta) + \epsilon_t \tag{79}$$

where ϵ_t is the exponentially decaying signal due to initial conditions

$$\epsilon_t = e^{A_0 t}\tilde{x}(0) + \mu e^{A_0 t} * (bc_\Delta e^{A_\Delta t})\xi(0) \tag{80}$$

and $*$ denotes convolution.

As in [9], the backstepping procedure is performed through the observer in $\rho = n - m$ steps. We present only the case $\rho > 1$.

Step 1: We define $z_1 = x_1 = y$ and $z_2 = \hat{x}_2 - \alpha_1$ and consider

$$\begin{aligned}\dot{z}_1 &= \hat{x}_2 + \varphi_1 + \tilde{x}_2 \\ &= (1+\mu q)\alpha_1 + (1+\mu q)z_2 + \varphi_1 - \mu q\zeta_2 \\ &\quad + \mu(\Delta - q)(\hat{x}_2 - \zeta_2) + \epsilon_{t2}.\end{aligned} \tag{81}$$

Led by the experience with previous solutions, we design

$$v_1 = -c_1 z_1 - \varepsilon_1 \tag{82}$$

$$w_1 = 1 + |m|_\varepsilon + |v_1 - \zeta_2|_\varepsilon \tag{83}$$

$$\alpha_1 = v_1 - \kappa_1 w_1 z_1 \tag{84}$$

where the m filter is given by

$$\dot{m} = -\delta m + |\hat{x}_2 - \zeta_2|_\varepsilon. \tag{85}$$

The resulting z_1 subsystem is rewritten as

$$\begin{aligned}\dot{z}_1 &= -[c_1 + \kappa_1(1+\mu q)w_1]z_1 + (1+\mu q)z_2 \\ &\quad + \mu q(v_1 - \zeta_2) + \mu(\Delta - q)(\hat{x}_2 - \zeta_2) + \epsilon_{t2}.\end{aligned} \tag{86}$$

Step $i(2 \le i \le \rho)$: Introducing $z_{i+1} = \hat{x}_{i+1} - \alpha_i$, we have

$$\begin{aligned}\dot{z}_i &= \hat{x}_{i+1} + K_{0i}(y - \hat{x}_1) + \varphi_i - \frac{\partial\alpha_{i-1}}{\partial y}(\hat{x}_2 + \varphi_1 + \tilde{x}_2) \\ &\quad - \frac{\partial\alpha_{i-1}}{\partial\zeta}(A_0\zeta + K_0 y + \varphi) \\ &\quad - \sum_{j=1}^{i-1}\frac{\partial\alpha_{i-1}}{\partial\hat{x}_j}[\hat{x}_{j+1} + K_{0j}(y - \hat{x}_1) + \varphi_j] \\ &\quad - \frac{\partial\alpha_{i-1}}{\partial m}(-\delta m + |\hat{x}_2 - \zeta_2|_\varepsilon).\end{aligned} \tag{87}$$

The dynamic perturbation acts through the term $-(\partial\alpha_{i-1}/\partial y)\tilde{x}_2$. With (79) in mind, we design

$$\begin{aligned}v_i &= -c_i z_i - K_{0i}(y - \hat{x}_1) - \varphi_i + \frac{\partial\alpha_{i-1}}{\partial y}(\hat{x}_2 + \varphi_1) \\ &\quad + \frac{\partial\alpha_{i-1}}{\partial\zeta}(A_0\zeta + K_0 y + \varphi) \\ &\quad + \sum_{j=1}^{i-1}\frac{\partial\alpha_{i-1}}{\partial\hat{x}_j}[\hat{x}_{j+1} + K_{0j}(y - \hat{x}_1) + \varphi_j] \\ &\quad + \frac{\partial\alpha_{i-1}}{\partial m}(-\delta m + |\hat{x}_2 - \zeta_2|_\varepsilon)\end{aligned} \tag{88}$$

$$w_i = (1 + |m|_\varepsilon + |\hat{x}_2 - \zeta_2|_\varepsilon)\left(\left|\frac{\partial\alpha_{i-1}}{\partial y}\right|_\varepsilon + \varepsilon\right) \tag{89}$$

$$\alpha_i = v_i - \kappa_i w_i z_i. \tag{90}$$

The actual control law is

$$u = \frac{1}{b_m\sigma(y)}(\alpha_\rho - \hat{x}_{\rho+1}). \tag{91}$$

Substituting (79) and (88)–(90) into (87), we write

$$\begin{aligned}\dot{z}_i &= -(c_i + \kappa_i w_i)z_i + z_{i+1} - \mu\frac{\partial\alpha_{i-1}}{\partial y}q(\hat{x}_2 - \zeta_2) \\ &\quad - \mu\frac{\partial\alpha_{i-1}}{\partial y}(\Delta - q)(\hat{x}_2 - \zeta_2) - \frac{\partial\alpha_{i-1}}{\partial y}\epsilon_{t2}\end{aligned} \tag{92}$$

where $z_{\rho+1} \triangleq 0$.

Theorem 4.1: Consider the closed-loop system (6), (74), (76), (77), and (85) and denote $X = (x, \hat{x}, \zeta, \xi, m)$. Then, the same statements are true as in Theorem 3.1.

Proof: The proof is similar to the proof of Theorem 3.1 and we will only outline the points that are different. We denote $Z = (z, \tilde{x}, \zeta, \xi, m)$. Let us start by noting that (80) implies that there exists a positive real number k_ϵ such that $|\epsilon_t| \le k_\epsilon(|\tilde{x}(0)| + |\xi(0)|)e^{-\delta t}$. First, we show that

$$\begin{aligned}&|\mu q(v_1 - \zeta_2) + \mu(\Delta - q)(\hat{x}_2 - \zeta_2) + \epsilon_{t2}|/w_1 \\ &\quad \le \mu h + R_0 e^{-\delta t}\end{aligned} \tag{93}$$

$$\begin{aligned}&\left|\frac{\partial\alpha_{i-1}}{\partial y}[\mu q(\hat{x}_2 - \zeta_2) + \mu(\Delta - q)(\hat{x}_2 - \zeta_2) + \epsilon_{t2}]\right|/w_i \\ &\quad \le \mu h + R_0 e^{-\delta t}\end{aligned} \tag{94}$$

where $h = \max\{\|h_\delta\|_\infty, |q|, \varepsilon[(1 + (1/\delta))h_0 + nq]\}$ and $R_0 = k_\epsilon(|\tilde{x}(0)| + |\xi(0)|) + \mu(k_0|\xi(0)| + \|h_\delta\|_\infty|m(0)|)$. By proceeding as in the proof of Theorem 3.1, we arrive at

$$|z(t)| \le \beta|Z(0)|e^{-\sigma t} + \gamma\mu \tag{95}$$

where $\beta, \sigma, \gamma > 0$. This implies that $|y(t)| \le \beta|Z(0)|e^{-\sigma t} + \gamma\mu$. In the sequel we use the following notation: $\mathcal{M}(r, \mu) = \{f \in C^0 | f(t) \le \rho(r)e^{-dt} + \rho(\mu)\}$.[3] Thus, (77) implies that $\zeta \in \mathcal{M}(|Z(0)|, \mu)$. Now from (85) and using (84), we proceed as follows:

$$\begin{aligned}\frac{d}{dt}|m| &\le -\delta|m| + |\alpha_1 + z_2 - \zeta_2| \\ &\le -(\delta - \kappa_1|z_1|)|m| + |-\kappa_1(1 + |v_1 - \zeta_2|_\varepsilon)z_1 \\ &\quad + v_1 - \zeta_2 + z_2|\end{aligned} \tag{96}$$

and using [12, Lemma C.4], we conclude that there exists $\mu^* > 0$ independent of initial conditions such that $m \in \mathcal{M}(|Z(0)|, \mu)$ for all $\mu \in [0, \mu^*)$. From (78) we obtain

$$\begin{aligned}\tilde{x}_1 &= \mu\frac{B(s)}{K(s)}\Delta(s)[\sigma(y)u] \\ &= \mu\Delta(s)\frac{1}{K(s)}\left[y^{(n)} - \sum_{i=1}^{n}\frac{d^{n-i}\varphi_j}{dy^{n-i}}y^{(n-i)}\right]\end{aligned} \tag{97}$$

where $K(s) = \det(sI - A_0)$ which implies that $\tilde{x}_1 \in \mathcal{M}(|Z(0)|, \mu)$. Hence $\hat{x}_1 \in \mathcal{M}(|Z(0)|, \mu)$. From $\hat{x}_i = z_i + \alpha_{i-1}, i = 2, \cdots, \rho$, we recursively establish that $\hat{x}_2, \cdots, \hat{x}_\rho \in \mathcal{M}(|Z(0)|, \mu)$. Therefore, from (79) it follows that $\tilde{x}_2, \cdots, \tilde{x}_\rho \in \mathcal{M}(|Z(0)|, \mu)$ which, in turn, implies that $x_2, \cdots, x_\rho \in \mathcal{M}(|Z(0)|, \mu)$. Now we set out to prove the same for $x_{\rho+1}, \cdots, x_n$ and $\tilde{x}_{\rho+1}, \cdots, \tilde{x}_n$, and therefore, for $\hat{x}_{\rho+1}, \cdots, \hat{x}_n$. Let us consider the similarity transformations

$$\begin{bmatrix} x_1 \\ \vdots \\ x_\rho \\ \bar{x}\end{bmatrix} = \begin{bmatrix} I_\rho & 0_{\rho\times m} \\ & T\end{bmatrix}x, \quad \begin{bmatrix}\tilde{x}_1 \\ \vdots \\ \tilde{x}_\rho \\ \breve{x}\end{bmatrix} = \begin{bmatrix} I_\rho & 0_{\rho\times m} \\ & T\end{bmatrix}\tilde{x} \tag{98}$$

[3] As in the previous proofs, d denotes a generic positive finite constant (independent of initial conditions and μ), and $\rho(\cdot)$ denotes a generic locally Lipschitz class $\mathcal{K}_\infty$ function.

where

$$T = [A_b^\rho e_1, \cdots, A_b e_1, I_m]$$
$$A_b = \begin{bmatrix} -b_{m-1}/b_m & \vdots & I_{m-1} & \\ -b_0/b_m & 0 & \cdots & 0 \end{bmatrix}. \tag{99}$$

It is readily verified that

$$Tb = 0, \quad TA = A_b T + TA^\rho b e_1^T. \tag{100}$$

Therefore, from (74) and (78) we obtain

$$\dot{\overline{x}} = A_b \overline{x} + T[A^\rho by + \varphi(y)] \tag{101}$$
$$\dot{\check{x}} = A_b \check{x} + T(A^\rho b - K_0)\tilde{x}_1. \tag{102}$$

These two systems have asymptotically stable homogeneous parts with inputs $y, \tilde{x}_1 \in \mathcal{M}(|X(0)|, \mu)$. This proves that $\overline{x}, \check{x} \in \mathcal{M}(|X(0)|, \mu)$. So, from (98), we have $x, \tilde{x} \in \mathcal{M}(|X(0)|, \mu)$, and therefore, $\hat{x} \in \mathcal{M}(|X(0)|, \mu)$. Thus, $u \in \mathcal{M}(|X(0)|, \mu)$, and, consequently, $\xi \in \mathcal{M}(|X(0)|, \mu)$. Hence

$$|X(t)| \leq \rho_0(|X(0)|)e^{-dt} + \rho_\mu(\mu) \tag{103}$$

where ρ_0 and ρ_μ are locally Lipschitz class $\mathcal{K}_\infty$ functions. The proof of global asymptotic stability is the same as in Theorem 3.1. We first rewrite the closed-loop system

$$\begin{aligned}
\dot{x} &= Ax + \varphi(x_1) + b\sigma(x_1)u(x_1, \hat{x}, \zeta, m) \\
&\quad + \mu b[c_\Delta \xi + q\sigma(x_1)u(x_1, \hat{x}, \zeta, m)] \\
\dot{\hat{x}} &= A_0\hat{x} + K_0 x_1 + \varphi(x_1) + b\sigma(x_1)u(x_1, \hat{x}, \zeta, m) \\
\dot{\zeta} &= A_0\zeta + K_0 x_1 + \varphi(x_1) \\
\dot{\xi} &= A_\Delta \xi + b_\Delta \sigma(x_1)u(x_1, \hat{x}, \zeta, m) \\
\dot{m} &= -\delta m + |\hat{x}_2 - \zeta_2|_\varepsilon
\end{aligned}$$

in the form (66) and then proceed as in the proof of Theorem 3.1. □

APPENDIX

Definition A.1: The system $\dot{x} = f(t, x, u)$ is said to be input-to-state practically stable (ISpS) if there exist a class $\mathcal{KL}$ function β, a class $\mathcal{K}$ function γ, and a positive real number d such that for any $x(0)$ and for any input $u(\cdot)$ continuous on $[0, \infty)$, the solution exists for all $t \geq 0$ and satisfies $|x(t)| \leq \beta(|x(s)|, t - s) + \gamma(\sup_{s \leq \tau \leq t} |u(\tau)|) + d$ for all $0 \leq s \leq t$.

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Correction to "Adaptive Control of Robot Manipulators with Flexible Joints"

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I. INTRODUCTION

The above paper[1] contains two flaws that were recently brought to our attention by Hsu [1]. 1) The first mistake concerns the definition of the signal q_{2d} in (6) and (C.20): At time $t = 0$, the term $q_{2d}(0)$ appears on both sides of the equalities in (6) and (C.20). Hence the initial conditions on the state and on the desired trajectory q_{1d} and its

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