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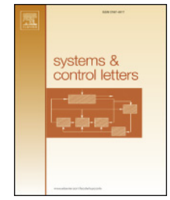

Compensating actuator and sensor dynamics governed by diffusion PDEs

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ABSTRACT

For (possibly unstable) ODE systems with actuator delay, predictor-based infinite-dimensional feedback can compensate for actuator delay of arbitrary length and achieve stabilization. We extend this concept to another class of PDE-ODE cascades, where the infinite-dimensional part of the plant is of diffusive, rather than convective type. We derive predictor-like feedback laws and observers, with explicit gain kernels. The gain kernels involve second-order matrix exponentials of the system matrix of the ODE plant, which is the result of the second-order-in-space character of the actuator/sensor dynamics. The construction of the kernel functions is performed using the continuum version of the backstepping method. Robustness to small perturbations in the diffusion coefficient is proved.

1. Introduction

For ODE systems with actuator and sensor delays, predictor-based control design and its extensions to observers, adaptive control, and even nonlinear system have been active areas of research over the last thirty years [1–27].

Though various finite-dimensional forms of actuator dynamics (consisting of linear and nonlinear integrators) have been successfully tackled in the context of the backstepping methods, realistic forms of infinite-dimensional actuator and sensor dynamics different than pure delays have not received attention.

In this note we address the problems of compensating the actuator and sensor dynamics dominated by diffusion, i.e., modeled by the heat equation. Purely convective/first-order hyperbolic PDE dynamics (i.e., transport equation, or, simply, delay) considered in [9], and diffusive/parabolic PDE dynamics (i.e., heat equation) considered here, introduce different problems with respect to controllability and stabilization. On the elementary level, the convective dynamics have constant magnitude response at all frequencies but are limited by a finite speed of propagation. The diffusive dynamics, when control enters through one boundary of a 1-D domain and exits (to feed the ODE) through the other, are not limited in the speed of propagation but introduce an infinite relative degree, with the associated significant roll-off of the magnitude response at high frequencies.

In this note we present an exact extension of the predictor feedback and observer design, from delay-ODE cascades [1,9] to diffusion PDE-ODE cascades. We apply the same ideas we employed in [9] to construct infinite-dimensional state transformations and Lyapunov–Krasovskii functionals. The key difference is in the transformation kernel functions (and the associated ODEs and PDEs which need to be solved). While in our work on delays [9] the kernel ODEs and PDEs were of first order, here they are of second-order. To be more precise, the design PDEs for control gains arising in delay problems were hyperbolic of first order, whereas with diffusion problems they are hyperbolic of second-order. As we did in [9], we solve them explicitly.

While the transfer function of the heat equation PDE dynamics is available—it is $\frac{1}{\cosh(D\sqrt{s})}$, where D is the delay value—our success in compensating these actuator dynamics is based on using the PDE representation, rather than working in the frequency domain. This was also the case with our past work with delay dynamics [9].

We start in Section 2 with an actuator compensation design with full state feedback. With a simple design we achieve closed-loop stability. We follow this with a more complex design which also endows the closed-loop system with an arbitrarily fast decay rate. In Section 3 we approach the question of robustness of our infinite-dimensional feedback law with respect to uncertainty in the diffusion coefficient. This question is rather nontrivial for actuator delays. We resolved it positively for small delay perturbations in [11] and we resolve it positively here for small perturbations in the diffusion coefficient. Finally, in Section 4 we develop a dual of our actuator dynamics compensator and design an infinite-dimensional observer which compensates the diffusion dynamics of the sensor.

2. Stabilization with full-state feedback

We consider the cascade of a heat equation and an LTI finite-dimensional system given by

$$\dot{X}(t) = AX(t) + Bu(0,t) \tag{1}$$

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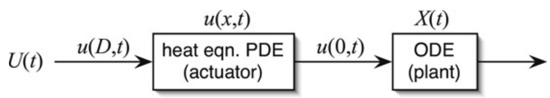

Fig. 1. The cascade of the heat equation PDE dynamics of the actuator with the ODE dynamics of the plant.

$$u_t(x,t) = u_{xx}(x,t) \tag{2}$$
$$u_x(0,t) = 0 \tag{3}$$
$$u(D,t) = U(t), \tag{4}$$

where $X \in \mathbb{R}^n$ is the ODE state, U is the scalar input to the entire system, and $u(x,t)$ is the state of the PDE dynamics of the diffusive actuator. The cascade system is depicted in Fig. 1.

The length of the PDE domain, D, is arbitrary. Thus, we take the diffusion coefficient to be unity without loss of generality. We assume that the pair (A,B) is stabilizable and take K to be a known vector such that $A+BK$ is Hurwitz.

We recall from [9] that, if (2), (3) are replaced by the delay/transport equation,

$$u_t(x,t) = u_x(x,t), \tag{5}$$

then the predictor-based control law

$$U(t) = K\left[\mathrm{e}^{AD}X(t) + \int_0^D \mathrm{e}^{A(D-y)}Bu(y,t)\mathrm{d}y\right] \tag{6}$$

achieves perfect compensation of the actuator delay and achieves exponential stability at $u \equiv 0, X = 0$.

The transfer function of the actuator dynamics (3)–(4) is given by

$$u(0,t) = \frac{1}{\cosh(D\sqrt{s})}[U(t)]. \tag{7}$$

This makes the task of compensating them rather non-routine as compared to compensating delay dynamics $u(0,t) = \mathrm{e}^{-Ds}[U(t)]$.

Next we state a new controller that compensates the *diffusive* actuator dynamics and prove exponential stability of the resulting closed-loop system.

Theorem 1 (*Stabilization*). *Consider a closed-loop system consisting of the plant (1)–(4) and the control law*

$$U(t) = K\begin{bmatrix} I & 0 \end{bmatrix}\left\{\mathrm{e}^{\begin{bmatrix} 0 & A \\ I & 0 \end{bmatrix}D}\begin{bmatrix} I \\ 0 \end{bmatrix}X(t) + \int_0^D \left(\int_0^{D-y}\mathrm{e}^{\begin{bmatrix} 0 & A \\ I & 0 \end{bmatrix}\xi}\mathrm{d}\xi\right)\begin{bmatrix} I \\ 0 \end{bmatrix}Bu(y,t)\mathrm{d}y\right\}. \tag{8}$$

For any initial condition such that $u(x,0)$ is square integrable in x and compatible with the control law (8), the closed-loop system has a unique classical solution and is exponentially stable in the sense of the norm

$$\left(|X(t)|^2 + \int_0^D u(x,t)^2\mathrm{d}x\right)^{1/2}. \tag{9}$$

Proof. We start by formulating an infinite-dimensional transformation of the form

$$w(x,t) = u(x,t) - \int_0^x q(x,y)u(y,t)\mathrm{d}y - \gamma(x)X(t), \tag{10}$$

with kernels $q(x,y)$ and $\gamma(x)$ to be derived, which should transform the plant (1)–(4), along with the control law (8), into the "target system"

$$\dot{X}(t) = (A+BK)X(t) + Bw(0,t) \tag{11}$$
$$w_t(x,t) = w_{xx}(x,t) \tag{12}$$
$$w_x(0,t) = 0 \tag{13}$$
$$w(D,t) = 0. \tag{14}$$

We first derive the kernels $q(x,y)$ and $\gamma(x)$ and then show that the target system is exponentially stable. The first two derivatives with respect to x of $w(x,t)$, as defined in (10), are given by

$$w_x(x,t) = u_x(x,t) - q(x,x)u(x,t) - \int_0^x q_x(x,y)u(y,t)\mathrm{d}y - \gamma'(x)X(t) \tag{15}$$

$$w_{xx}(x,t) = u_{xx}(x,t) - \big(q(x,x)\big)'u(x,t) - q(x,x)u_x(x,t) - q_x(x,x)u(x,t) - \int_0^x q_{xx}(x,y)u(y,t)\mathrm{d}y - \gamma''(x)X(t). \tag{16}$$

The first derivative of $w(x,t)$ with respect to t is

$$\begin{aligned} w_t(x,t) &= u_t(x,t) - \int_0^x q(x,y)u_t(y,t)\mathrm{d}y - \gamma(x)\left(AX(t) + Bu(0,t)\right) \\ &= u_{xx}(x,t) - \int_0^x q(x,y)u_{xx}(y,t)\mathrm{d}y - \gamma(x)\left(AX(t) + Bu(0,t)\right) \\ &= u_{xx}(x,t) - q(x,x)u_x(x,t) + q(x,0)u_x(0,t) + q_y(x,x)u(x,t) - q_y(x,0)u(0,t) \\ &\quad - \int_0^x q_{yy}(x,y)u(y,t)\mathrm{d}y - \gamma(x)\left(AX(t) + Bu(0,t)\right). \end{aligned} \tag{17}$$

Let us now examine the expressions

$$w(0,t) = u(0,t) - \gamma(0)X(t) \tag{18}$$
$$w_x(0,t) = -q(0,0)u(0,t) - \gamma'(0)X(t) \tag{19}$$
$$\begin{aligned} w_t(x,t) - w_{xx}(x,t) &= 2\big(q(x,x)\big)'u(x,t) + \big(\gamma''(x) - \gamma(x)A\big)X(t) \\ &\quad - \big(q_y(x,0) + \gamma(x)B\big)u(0,t) + \int_0^x \big(q_{xx}(x,y) - q_{yy}(x,y)\big)u(y,t)\mathrm{d}y, \end{aligned} \tag{20}$$

where we have employed the fact that $u_x(0,t) = 0$. A sufficient condition for (11)–(13) to hold for any continuous functions $u(x,t)$ and $X(t)$ is that $\gamma(x)$ and $q(x,y)$ satisfy

$$\gamma''(x) = A\gamma(x) \tag{21}$$
$$\gamma(0) = K \tag{22}$$
$$\gamma'(0) = 0, \tag{23}$$

which happens to represent a second order ODE in x, and

$$q_{xx}(x,y) = q_{yy}(x,y) \tag{24}$$
$$q(x,x) = 0 \tag{25}$$
$$q_y(x,0) = -\gamma(x)B, \tag{26}$$

which is a hyperbolic PDE of second order and of Goursat type. We then proceed to solve this cascade system explicitly. The explicit solution to the ODE (21)–(23) is readily found as

$$\gamma(x) = \begin{bmatrix} K & 0 \end{bmatrix}\mathrm{e}^{\begin{bmatrix} 0 & A \\ I & 0 \end{bmatrix}x}\begin{bmatrix} I \\ 0 \end{bmatrix}, \tag{27}$$

and the explicit solution to the PDE (24)–(26) is

$$q(x, y) = \int_0^{x-y} \gamma(\sigma) B \mathrm{d}\sigma. \tag{28}$$

In a similar manner, the inverse of the transformation (10) can be found. To summarize and to introduce compact notation for further use in the proof, the direct and inverse backstepping transformations are given by

$$w(x, t) = u(x, t) - \int_0^x m(x-y) u(y, t) \mathrm{d}y - KM(x)X(t) \tag{29}$$

$$u(x, t) = w(x, t) + \int_0^x n(x-y) u(y, t) \mathrm{d}y + KN(x)X(t), \tag{30}$$

where

$$m(s) = \int_0^s KM(\xi) B \mathrm{d}\xi \tag{31}$$

$$n(s) = \int_0^s Kn(\xi) B \mathrm{d}\xi \tag{32}$$

$$M(\xi) = \begin{bmatrix} I & 0 \end{bmatrix} \mathrm{e}^{\begin{bmatrix} 0 & A \\ I & 0 \end{bmatrix}\xi} \begin{bmatrix} I \\ 0 \end{bmatrix} \tag{33}$$

$$N(\xi) = \begin{bmatrix} I & 0 \end{bmatrix} \mathrm{e}^{\begin{bmatrix} 0 & A+BK \\ I & 0 \end{bmatrix}\xi} \begin{bmatrix} I \\ 0 \end{bmatrix}. \tag{34}$$

Now we proceed to prove exponential stability. Consider the Lyapunov function

$$V = X^T P X + \frac{a}{2}\|w\|^2, \tag{35}$$

where $\|w(t)\|^2$ is a compact notation for $\int_0^D w(x, t)^2 \,\mathrm{d}x$, the matrix $P = P^T > 0$ is the solution to the Lyapunov equation

$$P(A + BK) + (A + BK)^T P = -Q \tag{36}$$

for some $Q = Q^T > 0$, and the parameter $a > 0$ is to be chosen later. It is easy to show, using (29) and (30), that

$$\|w\|^2 \leq \alpha_1 \|u\|^2 + \alpha_2 |X|^2 \tag{37}$$

$$\|u\|^2 \leq \beta_1 \|w\|^2 + \beta_2 |X|^2, \tag{38}$$

where

$$\alpha_1 = 3\left(1 + D\|m\|^2\right) \tag{39}$$

$$\alpha_2 = 3\|KM\|^2 \tag{40}$$

$$\beta = 3\left(1 + D\|n\|^2\right) \tag{41}$$

$$\beta = 3\|KN\|^2. \tag{42}$$

Hence,

$$\underline{\delta}\left(|X|^2 + \|u\|^2\right) \leq V \leq \bar{\delta}\left(|X|^2 + \|u\|^2\right), \tag{43}$$

where

$$\underline{\delta} = \max\left\{\frac{a}{2}\alpha_1, \frac{a}{2}\alpha_2 + \lambda_{\max}(P)\right\} \tag{44}$$

$$\bar{\delta} = \frac{\max\{\beta_1, \beta_2 + 1\}}{\min\left\{\frac{a}{2}, \lambda_{\min}(P)\right\}}. \tag{45}$$

Taking a derivative of the Lyapunov function along the solutions of the PDE-ODE system (11)–(14), we get

$$\begin{aligned} \dot{V} &= -X^T Q X + 2X^T P B w(0, t) - a\|w_x\|^2 \\ &\leq -\frac{\lambda_{\min}(Q)}{2}|X|^2 + \frac{2|PB|^2}{\lambda_{\min}(Q)} w(0, t)^2 - a\|w_x\|^2 \\ &\leq -\frac{\lambda_{\min}(Q)}{2}|X|^2 - \left(a - \frac{8|PB|^2}{\lambda_{\min}(Q)}\right)\|w_x\|^2, \end{aligned} \tag{46}$$

where the last line is obtained by using Agmon's inequality. Taking

$$a > \frac{8|PB|^2}{\lambda_{\min}(Q)}, \tag{47}$$

and using Poincare's inequality, we get

$$\dot{V} \leq -bV, \tag{48}$$

where

$$b = \min\left\{\frac{\lambda_{\min}(Q)}{2\lambda_{\max}(P)}, \frac{1}{2} - \frac{4|PB|^2}{a\lambda_{\min}(Q)}\right\} > 0. \tag{49}$$

Hence,

$$|X(t)|^2 + \|u(t)\|^2 \leq \frac{\bar{\delta}}{\underline{\delta}} \mathrm{e}^{-bt}\left(|X_0|^2 + \|u_0\|^2\right) \tag{50}$$

for all $t \geq 0$, which completes the proof. □

The convergence rate to zero for the closed-loop system is determined by the eigenvalues of the PDE-ODE system (11)–(14). These eigenvalues are the union of the eigenvalues of $A+BK$, which are placed at desirable locations by the control vector K, and of the eigenvalues of the heat equation with a Neumann boundary condition on one end and a Dirichlet boundary condition on the other end. While exponentially stable, the heat equation PDE need not necessarily have fast decay. Its decay rate is limited by its first eigenvalue, $-\pi^2/\left(4D^2\right)$.

Fortunately, the compensated actuator dynamics, i.e., the w-dynamics in (13)–(14) can be sped up arbitrarily by a modified controller.

Theorem 2 (*Performance Improvement*). *Consider a closed-loop system consisting of the plant (1)–(4) and the control law*

$$U(t) = \phi(D)X(t) + \int_0^D \psi(D, y) u(y, t) \mathrm{d}y, \tag{51}$$

where

$$\phi(x) = KM(x) - \int_0^x \kappa(x, y) KM(y) \mathrm{d}y \tag{52}$$

$$\begin{aligned} \psi(x, y) &= \kappa(x, y) + \int_0^{x-y} KM(\xi) B \mathrm{d}\xi \\ &\quad - \int_y^x \kappa(x, \xi) \int_0^{\xi - y} KM(\eta) B \mathrm{d}\eta \mathrm{d}\xi \end{aligned} \tag{53}$$

$$\kappa(x, y) = -cx \frac{I_1\left(\sqrt{c\left(x^2 - y^2\right)}\right)}{\sqrt{c\left(x^2 - y^2\right)}}, \quad c > 0, \tag{54}$$

and I_1 denotes the appropriate Bessel function. For any initial condition such that $u(x, 0)$ is square integrable in x and compatible with the control law (51), the closed-loop system has a unique classical solution and its eigenvalues are given by the set

$$\operatorname{eig}\{A + BK\} \cup \left\{-c - \frac{\pi^2}{D^2}\left(n + \frac{1}{2}\right)^2, \ n = 0, 1, 2, \ldots\right\}. \tag{55}$$

Proof. Consider the new (invertible) state transformation,

$$z(x, t) = w(x, t) - \int_0^x \kappa(x, y) w(y, t) \mathrm{d}y. \tag{56}$$

By direct substitution of the transformation

$$w(x,t) = u(x,t) - \int_0^x \int_0^{x-y} KM(\xi)Bd\xi u(y,t)dy - KM(x)X(t) \quad (57)$$

into (56), and by changing the order of integration, one obtains that

$$z(x,t) = u(x,t) - \int_0^x \psi(x,y)u(y,t)dy - \phi(x)X(t), \quad (58)$$

where the functions $\psi(x,y)$ and $\phi(x)$ are as defined in the statement of the theorem. It was shown in [28, Sections VIII.A and VIII.B] that the function $\kappa(x,y)$ satisfies the PDE

$$\kappa_{xx}(x,y) = \kappa_{yy}(x,y) + c\kappa(x,y) \quad (59)$$

$$\kappa_y(x,0) = 0 \quad (60)$$

$$\kappa(x,x) = -\frac{c}{2}x. \quad (61)$$

Using these relations and (56), a direct verification yields the transformed closed-loop system

$$\dot{X}(t) = (A+BK)X(t) + Bz(0,t) \quad (62)$$

$$z_t(x,t) = z_{xx}(x,t) - cz(x,t) \quad (63)$$

$$z_x(0,t) = 0 \quad (64)$$

$$z(D,t) = 0. \quad (65)$$

With an elementary calculation of the eigenvalues of the z-system, the result of the theorem follows. □

3. Robustness to diffusion coefficient uncertainty

We now study robustness of the feedback law (8) to a perturbation in the diffusion coefficient of the actuator dynamics, i.e., we study stability robustness of the closed-loop system

$$\dot{X}(t) = AX(t) + Bu(0,t) \quad (66)$$

$$u_t(x,t) = (1+\varepsilon)u_{xx}(x,t) \quad (67)$$

$$u_x(0,t) = 0 \quad (68)$$

$$u(D,t) = \int_0^D m(D-y)u(y,t)dy + KM(D)X(t) \quad (69)$$

to the perturbation parameter ε, which we allow to be either positive or negative but small.

Theorem 3 (*Robustness to Diffusion Uncertainty*). *Consider a closed-loop system (66)–(69). There exists a sufficiently small $\varepsilon^* > 0$ such that for all $\varepsilon \in (-\varepsilon^*, \varepsilon^*)$ the closed-loop system has a unique classical solution (under feedback-compatible initial data in L_2) and is exponentially stable in the sense of the norm $\left(|X(t)|^2 + \int_0^D u(x,t)^2 dx\right)^{1/2}$.*

Proof. It can be readily verified that

$$\dot{X}(t) = (A+BK)X(t) + Bw(0,t) \quad (70)$$

$$w_t(x,t) = (1+\varepsilon)w_{xx}(x,t) + \varepsilon KM(x)\left((A+BK)X(t) + Bw(0,t)\right) \quad (71)$$

$$w_x(0,t) = 0 \quad (72)$$

$$w(D,t) = 0. \quad (73)$$

Along the solutions of this system, the derivative of the Lyapunov function (35) is

$$\dot{V} \le -\frac{\lambda_{\min}(Q)}{2}|X|^2 - \left(a - \frac{8|PB|^2}{\lambda_{\min}(Q)} - |\varepsilon|a\right)\|w_x\|^2$$
$$+ a\varepsilon \int_0^D w(x)KM(x)dx\left((A+BK)X(t) + Bw(0,t)\right)$$
$$\le -\frac{\lambda_{\min}(Q)}{4}|X|^2 - \left(a - \frac{8|PB|^2}{\lambda_{\min}(Q)}\right)\|w_x\|^2$$
$$+ |\varepsilon|a\left(1 + 4\|\mu_1\| + |\varepsilon|a\frac{4\|\mu_2\|^2}{\lambda_{\min}(Q)}\right)\|w_x\|^2, \quad (74)$$

where $\mu_1(x) = KM(x)B$, $\mu_2(x) = |KM(x)|$. In the second inequality we have employed Young's and Agmon's inequalities. Choosing now, for example, $a = \frac{16|PB|^2}{\lambda_{\min}(Q)}$, it is possible to select $|\varepsilon|$ sufficiently small to achieve negative definiteness of $\dot{V}$. □

4. Observer design

Consider the LTI ODE system in cascade with diffusive sensor dynamics at the output (as depicted in Fig. 2),

$$Y(t) = u(0,t) \quad (75)$$

$$u_t(x,t) = u_{xx}(x,t) \quad (76)$$

$$u_x(0,t) = 0 \quad (77)$$

$$u(D,t) = CX(t) \quad (78)$$

$$\dot{X}(t) = AX(t) + BU(t). \quad (79)$$

The sensor dynamics are thus given by the transfer function

$$Y(t) = \frac{1}{\cosh(D\sqrt{s})}[CX(t)]. \quad (80)$$

We recall from [9] that, if (76), (77) are replaced by the delay/transport equation, $u_t(x,t) = u_x(x,t)$, then the predictor-based observer

$$\hat{u}_t(x,t) = \hat{u}_x(x,t) + Ce^{Ax}L\left(Y(t) - \hat{u}(0,t)\right) \quad (81)$$

$$\hat{u}(D,t) = C\hat{X}(t) \quad (82)$$

$$\dot{\hat{X}}(t) = A\hat{X}(t) + BU(t) + e^{AD}L\left(Y(t) - \hat{u}(0,t)\right) \quad (83)$$

achieves perfect compensation of the observer delay and achieves exponential stability at $u - \hat{u} \equiv 0, X - \hat{X} = 0$.

Next we state a new observer inspired by [30] that compensates the *diffusive* sensor dynamics and prove exponential convergence of the resulting observer error system.

Theorem 4 (*Observer Design and Convergence*). *The observer*

$$\hat{u}_t(x,t) = \hat{u}_{xx}(x,t) + CM(x)L\left(Y(t) - \hat{u}(0,t)\right) \quad (84)$$

$$\hat{u}_x(0,t) = 0 \quad (85)$$

$$\hat{u}(D,t) = C\hat{X}(t) \quad (86)$$

$$\dot{\hat{X}}(t) = A\hat{X}(t) + BU(t) + M(D)L\left(Y(t) - \hat{u}(0,t)\right). \quad (87)$$

where L is chosen such that $A - LC$ is Hurwitz, guarantees that $\hat{X}$, $\hat{u}$ exponentially converge to X, u, i.e., more specifically, that the observer error system is exponentially stable in the sense of the norm

$$\left(|X(t) - \hat{X}(t)|^2 + \int_0^D \left(u(x,t) - \hat{u}(x,t)\right)^2 dx\right)^{1/2}.$$

Proof. Introducing the error variables $\tilde{X} = X - \hat{X}$, $\tilde{u} = u - \hat{u}$, we obtain:

$$\tilde{u}_t(x,t) = \tilde{u}_{xx}(x,t) - CM(D)L\tilde{u}(0,t) \quad (88)$$

$$\tilde{u}_x(0,t) = 0 \quad (89)$$

$$\tilde{u}(D,t) = C\tilde{X}(t) \quad (90)$$

$$\dot{\tilde{X}}(t) = A\tilde{X}(t) - M(D)L\tilde{u}(0,t). \quad (91)$$

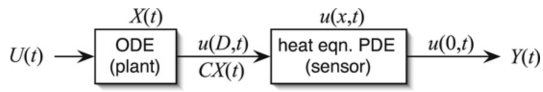

Fig. 2. The cascade of the ODE dynamics of the plant with the heat equation PDE dynamics of the sensor.

Consider the transformation

$$\tilde{w}(x) = \tilde{u}(x) - CM(x)M(D)^{-1}\tilde{X}. \quad (92)$$

Its derivatives in x and t are

$$\tilde{w}_x(x,t) = \tilde{u}_x(x,t) - CM'(x)M(D)^{-1}\tilde{X}(t) \quad (93)$$

$$\tilde{w}_{xx}(x,t) = \tilde{u}_{xx}(x,t) - CM''(x)M(D)^{-1}\tilde{X}(t) \quad (94)$$

$$\tilde{w}_t(x,t) = \tilde{u}_t(x,t) - CM(x)M(D)^{-1}\left(A\tilde{X}(t) - M(D)L\tilde{u}(0,t)\right), \quad (95)$$

and furthermore,

$$\tilde{w}(0,t) = \tilde{u}(0,t) - CM(D)^{-1}\tilde{X}(t), \quad (96)$$

where we have used the fact that $M(0) = I$. Then, using the fact that $M'(0) = 0$, that $M(D)^{-1}$ commutes with A (since $M(x)$ commutes with A for any x), that $M''(x) = M(x)A$, and that $\tilde{u}_x(0,t) = 0$, we obtain

$$\tilde{w}_t(x,t) = \tilde{w}_{xx}(x,t) \quad (97)$$

$$\tilde{w}_x(0,t) = 0 \quad (98)$$

$$\tilde{w}(D,t) = 0 \quad (99)$$

$$\dot{\tilde{X}}(t) = \left(A - M(D)LCM(D)^{-1}\right)\tilde{X} - M(D)L\tilde{w}(0,t). \quad (100)$$

The matrix $A - M(D)LCM(D)^{-1}$ is Hurwitz, which can be easily seen by using a similarity transformation $M(D)$, which commutes with A.

With a Lyapunov function

$$V = \tilde{X}^T M(D)^{-T} P M(D)^{-1}\tilde{X} + \frac{a}{2}\int_0^D \tilde{w}(x)^2\,dx, \quad (101)$$

where $P = P^T > 0$ is the solution to the Lyapunov equation $P(A - LC) + (A - LC)^T P = -Q$ for some $Q = Q^T > 0$, one gets

$$\dot{V} = -\tilde{X}^T M(D)^{-T} Q M(D)^{-1}\tilde{X} - 2\tilde{X}^T M(D)^{-T} PL\tilde{w}(0,t) - \frac{a}{2}\|\tilde{w}_x\|^2. \quad (102)$$

Applying Young's and Agmon's inequalities, taking a is sufficiently large, and then applying Poincare's inequality, one can show that $\dot{V} \le -\mu V$ for some $\mu > 0$, i.e., the $(\tilde{X}, \tilde{w})$ system is exponentially stable at the origin. From (92) we get exponential stability in the sense of $\left(|\tilde{X}(t)|^2 + \int_0^D \tilde{u}(x,t)^2 dx\right)^{1/2}$. □

The convergence rate of the observer is limited by the first eigenvalue of the heat Eqs. (97)–(99), i.e., by $-\pi^2/(4D^2)$. A similar observer re-design, as applied for the full-state control design in Theorem 2, can be applied to speed up the observer convergence.

5. Conclusions

In this note we developed explicit formulae for full-state control laws and observers in the presence of diffusion-governed actuator and sensor dynamics.

Since using the control law (8) we have established the stabilizability of system (1)–(4), other control designs should be possible—both heuristic control designs for *some* pairs (A, B) and other systematic PDE-based control designs for all stabilizable pairs (A, B). For example, an optimal control problem could be formulated, with quadratic penalties on $X(t)$ and $U(t)$, as well as an L_2 penalty (in x) on $u(x,t)$, yielding an operator Riccati-equation based control law. This alternative would lack the explicit character of our control law (8).

While the approach we use in this paper is the same as in [9] – PDE backstepping – there is a difference in the end result (8) as opposed to (6). The controller (6) is 'obvious' in retrospect, it is based on D-seconds-ahead 'prediction.' In contrast the controller (8) is a rather non-obvious choice. The controller (51)–(54) with an arbitrarily fast decay rate is an even less obvious choice.

Though we focused on purely diffusion-based actuator dynamics $u_t(x,t) = u_{xx}(x,t)$, there is no obstacle to extending the present results to diffusion-advection actuator dynamics $u_t(x,t) = u_{xx}(x,t) + bu_x(x,t)$, where b can have any value, or to reaction-diffusion dynamics $u_t(x,t) = u_{xx}(x,t) + \lambda u_x(x,t)$ which can have many unstable eigenvalues, or to much more complex dynamics governed by parabolic partial integro-differential equations.

It is reasonable to ask many questions regarding the possibility of extension of these results to other types of cascades. For example, can these results be extended to actuators and sensors which are of wave equation (second order hyperbolic) type? This is the subject of our companion paper [29].

How about an extension to other types of cascades? For example, an unstable reaction-diffusion (parabolic) PDE with boundary control entering through a delay? Our design works in this case to the extent that a feedback transformation can be constructed to convert the closed-loop system into a cascade of two exponentially stable systems, a transport equation feeding into a heat equation. However, difficulties arise when trying to construct a composite Lyapunov–Krasovskii functional for the two PDEs because they are connected through a Dirichlet type of boundary condition (which is a fundamental problem—PDEs from different classes interacting through boundary conditions). In this case one must resort to higher order norms to characterize stability. This is the subject of our ongoing research, both for parabolic and second-order hyperbolic PDEs with input delays.

We have also studied other cascade combinations of PDEs, such as heat-wave and wave-heat cascades, connected through Dirichlet or Neumann variables. Parts of the PDE control community consider these coupled problems to be representative of PDE problems modeling fluid-structure interactions. The heat-wave and wave-heat cascades give rise to more serious challenges than delay-heat and delay-wave cascades. After a rather major effort to identify conditions on the backstepping transformation kernels, one is faced with formidable, uncommon PDEs that contain fourth-order derivatives in time or space, plus additional effects. These are also subjects of our ongoing research.

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