Arbitrary Decay Rate for Euler-Bernoulli Beam by Backstepping Boundary Feedback

Andrey Smyshlyaev, *Member, IEEE*, Bao-Zhu Guo, and Miroslav Krstic, *Fellow, IEEE*

***Abstract*—We consider a problem of stabilization of the Euler-Bernoulli beam. The beam is controlled at one end (using position and moment actuators) and has the "sliding" boundary condition at the opposite end. We design the controllers that achieve any prescribed decay rate of the closed loop system, removing a long-standing limitation of classical "boundary damper" controllers. The idea of the control design is to use the well-known representation of the Euler-Bernoulli beam model through the Schrödinger equation, and then adapt recently developed backstepping designs for the latter in order to stabilize the beam. We derive the explicit integral transformation (and its inverse) of the closed-loop system into an exponentially stable target system. The transformation is of a novel Volterra/Fredholm type. The design is illustrated with simulations.**

***Index Terms*—Backstepping, boundary control, distributed parameter systems, Euler–Bernoulli beam, Riesz basis.**

I. Introduction

We consider a problem of stabilization of the Euler–Bernoulli beam by boundary feedback. The existing literature on control of the Euler–Bernoulli beam is extensive (see, e.g., [1]–[4], [6]–[8], [14]–[17] and references therein), however, unlike for the wave equation, "passive damper" controllers provide a limited damping for the beam. For most cases, they provide arbitrarily fast decay rate for the high modes but not for the low modes (or vise versa), see, e.g., [6] and numerical calculations in [11].

In this technical note we present a novel control design for the Euler-Bernoulli beam, which achieves any prescribed decay rate of the closed-loop system. The beam is controlled at one boundary using position and moment actuators and has the "sliding" boundary condition at the uncontrolled boundary (Fig. 1). We assume that the full state (displacement and velocity) measurements are available.

The idea of our method is to use the well-known representation of the Euler-Bernoulli beam model through the Schrödinger equation [18]. For the Schrödinger equation, recently developed controllers [11] based on the backstepping method [21] improved on the common "passive damper" controllers by moving all open-loop eigenvalues arbitrarily to the left in the complex plane. In this technical note we adapt the design from [11] to the Euler-Bernoulli beam. We should note that the design does not carry over trivially from one system to another because the boundary conditions of the beam do not directly correspond to the boundary conditions of the Schrödinger equation.

We derive the invertible integral transformation which, together with boundary feedbacks, converts the beam into an exponentially stable target system. The kernels of the transformation and control gains are given explicitly, expressed in terms of Kelvin functions. In contrast to other backstepping designs, that have been developed for more complicated models of the beams, such as the shear beam model [10] and the Timoshenko beam model [12], here the transformation is not of a strict-feedback form. Instead, it contains both Volterra- and Fredholm-type integrals.

Previously, two constructive approaches have been used to achieve an arbitrary decay rate for the beams. In [9], and later in [22], by choosing a special cost function the authors design the controllers that do not require a solution of the Riccati equation. In the above papers, the unbounded control operator is assumed to be admissible, which is not satisfied at least for the moment control for Euler-Bernoulli beam considered in this paper. A pole placement (or Riesz basis) approach was presented in [20], where an infinite number of eigenvalues is assigned by unbounded boundary feedback. In [23], a necessary and sufficient condition was given to assign the poles by bounded feedback. As shown in Example 2 in [23], this condition is verified for the moment control of Euler-Bernoulli beam. However, the resulting feedback is not as explicit as the controllers presented in this paper, as it is represented as an infinite sum of infinite products.

The technical note is organized as follows. We start by introducing the Euler-Bernoulli beam model in Section II. In Section III we summarize the backstepping design for the Schrödinger equation. We then present a target system for our control design in Section IV. In Section V we show that straightforward application of the control design for the Schrödinger equation results in the control laws that achieve stabilization to a constant profile of the beam. Building upon the results in Section V, we introduce the transformation and control laws that achieve stabilization to zero in Section VI. In Section VII we derive the explicit inverse transformation from the target system to the closed-loop system. Well-posedness of the closed-loop system is proved in Section VIII. The control design is illustrated with simulations in Section IX. Finally, we discuss possible extensions of the result in Section X.

Manuscript received September 01, 2008; revised November 04, 2008, December 09, 2008, and December 12, 2008. Current version published May 13, 2009. Recommended by Associate Editor K. A. Morris.

A. Smyshlyaev and M. Krstic are with the Department of Mechanical and Aerospace Engineering, University of California at San Diego, La Jolla CA 92093-0411 USA (e-mail: asmyshly@ucsd.edu; krstic@ucsd.edu).

B.-Z. Guo is with the Academy of Mathematics and Systems Science, Academia Sinica, Beijing 100080, China and also with the School of Computational and Applied Mathematics, University of the Witwatersrand, Johannesburg Wits 2050, South Africa (e-mail: bzguo@iss.ac.cn).

Digital Object Identifier 10.1109/TAC.2009.2013038

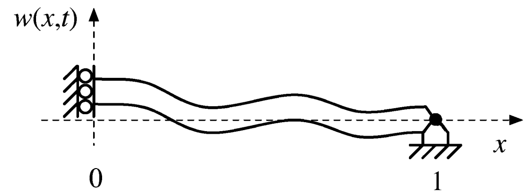

Fig. 1. Uncontrolled Euler-Bernoulli beam with the "sliding" boundary condition at $x = 0$ and the "hinged" boundary condition at $x = 1$.

II. Problem Formulation

Consider the Euler-Bernoulli beam model

$$\begin{cases} w_{tt}(x,t) + w_{xxxx}(x,t) = 0, & 0 < x < 1, t > 0 \\ w_x(0,t) = w_{xxx}(0,t) = 0, & t \geq 0 \\ w(1,t) = u_1(t), w_{xx}(1,t) = u_2(t), & t \geq 0. \end{cases} \tag{1}$$

Here w is a beam displacement and u_1, u_2 are the position and moment control inputs. The open-loop case $u_1 = u_2 \equiv 0$ corresponds to the beam with one "sliding" end and one "hinged" end (Fig. 1). The objective is to stabilize the zero equilibrium of the beam.

Let us introduce a new complex variable

$$v(x,t) = w_t(x,t) - jw_{xx}(x,t) \tag{2}$$

where j is the imaginary unit. The direct substitution shows that v defined in this way satisfies the Schrödinger equation

$$\begin{cases} v_t(x,t) = -jv_{xx}(x,t), & 0 < x < 1, t > 0 \\ v_x(0,t) = 0, & t \geq 0 \\ v(1,t) = u(t), & t \geq 0. \end{cases} \quad (3)$$

In a recent paper [11], the backstepping controllers were designed that achieve an arbitrary decay rate for the system (3). This gives an idea to adapt the control design from [11] to the Euler-Bernoulli beam (1). However, the design is not going to be trivial because, as can be seen from (2), regulation of v to zero does not necessarily imply the regulation of w to zero. Before we proceed, we summarize the backstepping control design for (3).

III. Summary of Backstepping Design for the Schrödinger Equation

As shown in [11], the controller

$$v(1,t) = \int_0^1 k(1,y)v(y,t)dy \quad (4)$$

and the transformation

$$\psi(x,t) = v(x,t) - \int_0^x k(x,y)v(y,t)dy = (I - \mathbb{P})v \quad (5)$$

where $k(x,y)$ is a complex-valued function that satisfies the PDE

$$\begin{cases} k_{xx}(x,y) - k_{yy}(x,y) = cjk(x,y) \\ k_y(x,0) = 0 \\ k(x,x) = -\frac{cj}{2}x \end{cases} \quad (6)$$

on the domain $0 \leq y \leq x \leq 1$ with $c > 0$, map (3) into the following exponentially stable target system

$$\begin{cases} \psi_t(x,t) = -j\psi_{xx}(x,t) - c\psi(x,t) \\ \psi_x(0,t) = \psi(1,t) = 0. \end{cases} \quad (7)$$

The eigenvalues of this system are $\sigma = -c + j\pi^2(2n+1)^2/4$, $n = 0, 1, 2, \ldots$. Therefore, the design parameter c allows to move them arbitrarily to the left in the complex plane.

The solution to the PDE (6) is [21]

$$\begin{aligned} k(x,y) &= -cjx\frac{I_1\left(\sqrt{cj(x^2-y^2)}\right)}{\sqrt{cj(x^2-y^2)}} \\ &= x\sqrt{\frac{c}{2(x^2-y^2)}}\left[(j-1)\mathrm{ber}_1\left(\sqrt{c(x^2-y^2)}\right)\right. \\ &\qquad \left. -(1+j)\mathrm{bei}_1\left(\sqrt{c(x^2-y^2)}\right)\right]. \end{aligned} \quad (8)$$

Here $I_1(\cdot)$ is the modified Bessel function and $\mathrm{ber}_1(\cdot)$ and $\mathrm{bei}_1(\cdot)$ are the Kelvin functions, which are defined in terms of I_1 as

$$\mathrm{ber}_1(x) = -\mathrm{Im}\left\{I_1(\sqrt{j}x)\right\}, \quad \mathrm{bei}_1(x) = \mathrm{Re}\left\{I_1(\sqrt{j}x)\right\}. \quad (9)$$

The inverse transformation

$$v(x,t) = \psi(x,t) + \int_0^x l(x,y)\psi(y,t)dy = (I - \mathbb{P})^{-1}\psi \quad (10)$$

with

$$\begin{aligned} l(x,y) = x\sqrt{\frac{c}{2(x^2-y^2)}}&\left[(j+1)\mathrm{ber}_1\left(\sqrt{c(x^2-y^2)}\right)\right. \\ &\left. +(1-j)\mathrm{bei}_1\left(\sqrt{c(x^2-y^2)}\right)\right] \end{aligned} \quad (11)$$

maps (7) back into (3), (4).

IV. Target System

In this section we choose the target system which sets the desired behavior of the beam. Let us define

$$\alpha(x,t) = \int_x^1 \int_0^y \mathrm{Im}\left\{\psi(\xi,t)\right\} d\xi dy \quad (12)$$

where ψ is the solution of the target system (7) for the Schrödinger equation. It is straightforward to verify that α satisfies the following fourth-order PDE:

$$\begin{cases} \alpha_{tt} + 2c\alpha_t + c^2\alpha + \alpha_{xxxx} = 0 \\ \alpha_x(0,t) = \alpha_{xxx}(0,t) = 0 \\ \alpha(1,t) = \alpha_{xx}(1,t) = 0. \end{cases} \quad (13)$$

To establish stability and well posedness for this system, let us define the energy state space

$$H_\alpha = H_L^2(0,1) \times L^2(0,1)$$

where $H_L^2 = \{f \in H^2(0,1) | f'(0) = f(1) = 0\}$ and the following induced norm is used

$$\|(f,g)\|_{H_\alpha}^2 = \int_0^1 \left[\left|f''(x)\right|^2 + |g(x)|^2\right] dx, \quad \forall (f,g) \in H_\alpha.$$

System (13) can be written as

$$\frac{d}{dt}(\alpha, \alpha_t) = \mathbb{C}(\alpha, \alpha_t) + \mathbb{D}(\alpha, \alpha_t)$$

where $\mathbb{C}$ is a skew-adjoint operator in H_α defined by

$$\begin{cases} \mathbb{C}(f,g) = \left(g, -f^{(4)}\right), \ \forall (f,g) \in D(\mathbb{C}) \\ D(\mathbb{C}) = \left\{(f,g) \in H_\alpha | f \in H^4(0,1), \right. \\ \qquad \left. g \in H_L^2(0,1), f'''(0) = f''(1) = 0\right\} \end{cases} \quad (14)$$

and $\mathbb{D}$ is a bounded operator on H_α given by

$$\mathbb{D}(f,g) = (0, -2cg - c^2 f), \quad \forall (f,g) \in H_\alpha. \quad (15)$$

Theorem 1: Let $\mathbb{C}$, $\mathbb{D}$ be defined by (14), (15). Then:

i) There is a family of eigenfunctions of $\mathbb{C}+\mathbb{D}$, which form a Riesz basis for H_α. Hence, $\mathbb{C}+\mathbb{D}$ generates a C_0-semigroup $e^{(\mathbb{C}+\mathbb{D})t}$ on H_α and for any initial value $(\alpha(\cdot,0), \alpha_t(\cdot,0)) \in H_\alpha$, there exists a unique (mild) solution of (13):

$$(\alpha(\cdot,t), \alpha_t(\cdot,t)) = e^{(\mathbb{C}+\mathbb{D})t}(\alpha(\cdot,0), \alpha_t(\cdot,0)) \in C([0,\infty); H_\alpha)$$

and if, in addition, $(\alpha(\cdot,0), \alpha_t(\cdot,0)) \in D(\mathbb{C})$, then

$$(\alpha(\cdot,t), \alpha_t(\cdot,t)) \in C^1([0,\infty); D(\mathbb{C})).$$

ii) The spectrum-determined growth condition holds for the semigroup $e^{(\mathbb{C}+\mathbb{D})t}$: $\omega(\mathbb{C}+\mathbb{D}) = S(\mathbb{C}+\mathbb{D}) = -c$, where $\omega(\mathbb{C}+\mathbb{D})$

is the growth bound of $e^{(\mathbb{C}+\mathbb{D})t}$ and $S(\mathbb{C}+\mathbb{D})$ is the spectral bound of $\mathbb{C}+\mathbb{D}$.

iii) The system (13) is exponentially stable: there exists a constant $C_c > 0$ such that

$$E_\alpha(t) \leq C_c e^{(-c+\varepsilon)t} E_\alpha(0), \quad E_\alpha(t) = \frac{1}{2}\left(\|\alpha_{xx}\|^2 + \|\alpha_t\|^2\right).$$

Proof: It is easy to check that $\mathbb{C}^{-1}$ exists and is compact on H_α. By the general theory of functional analysis, there is a family of eigenfunctions of $\mathbb{C}$, which form an orthonormal basis for H_α. All linearly independent normalized eigenfunctions $\{(F_n(x), \varphi_n(x)), (\overline{F_n(x)}, \varphi_n(x))\}_{n=1}^{\infty}$ of $\mathbb{C}$ are given by

$$F_n(x) = \frac{\varphi_n(x)}{j\kappa_n^2}, \quad \kappa_n = \left(n + \frac{1}{2}\right)\pi, \quad \varphi_n = \cos\kappa_n x.$$

The eigenfunctions $\{(\widetilde{F}_n(x), \varphi_n(x)), (\overline{\widetilde{F}_n(x)}, \varphi_n(x))\}$ of $\mathbb{C}+\mathbb{D}$ can also be found explicitly as

$$\widetilde{F}_n(x) = \frac{\varphi_n(x)}{\mu_n}, \quad \mu_n = -c + j\kappa_n^2, \quad n = 0, 1, 2, \ldots.$$

It is easy to check that $\sum_{n=1}^{\infty} \|F_n - \widetilde{F}_n\|_{H_\alpha}^2 < \infty$, which is also true for the conjugates of F_n and $\widetilde{F}_n$. By classical Bari's theorem [5], we get that $\{(\widetilde{F}_n(x), \varphi_n(x)), (\overline{\widetilde{F}_n(x)}, \varphi_n(x))\}_{n=0}^{\infty}$ also forms a Riesz basis for H_α, which gives (i), and the eigenvalues of $\mathbb{C}+\mathbb{D}$ are $\mu_n = -c \pm j\pi^2(2n+1)^2/4$, which proves (ii) and (iii). The proof is completed.

As seen from the above proof, the eigenvalues of the target system are open-loop eigenvalues shifted to the left in the complex plane by the same distance. This adds both pure viscous damping and stiffness, as can be seen from (13).

V. Control Laws That Stabilize the Beam to a Constant Profile

From (12) and (7) it follows that the state ψ is expressed through α in the following way:

$$\psi = \alpha_t + c\alpha - j\alpha_{xx}. \tag{16}$$

Taking the real and imaginary parts of the transformation (5), we get

$$\alpha_t(x,t) + c\alpha(x,t) = w_t(x,t) - \int_0^x r(x,y) w_t(y,t) dy - \int_0^x s(x,y) w_{yy}(y,t) dy \tag{17}$$

$$\alpha_{xx}(x,t) = w_{xx}(x,t) + \int_0^x s(x,y) w_t(y,t) dy - \int_0^x r(x,y) w_{yy}(y,t) dy, \tag{18}$$

where the gains $r(x,y)$ and $s(x,y)$, defined correspondingly as the real and imaginary part of $k(x,y) = r(x,y) + js(x,y)$, satisfy two coupled PDEs on the domain $0 \leq y \leq x \leq 1$:

$$\begin{cases} r_{xx}(x,y) = r_{yy}(x,y) - cs(x,y) \\ r_y(x,0) = 0, \quad r(x,x) = 0 \end{cases} \tag{19}$$

and

$$\begin{cases} s_{xx}(x,y) = s_{yy}(x,y) + cr(x,y) \\ s_y(x,0) = 0, \quad s(x,x) = -\frac{c}{2}x. \end{cases} \tag{20}$$

Remark 1: Equations (19), (20) can be written as the one fourth order PDE:

$$\begin{cases} r_{xxxx} - 2r_{xxyy} + r_{yyyy} + c^2 r = 0 \\ r_y(x,0) = c, \quad r_{yyy}(x,0) = r_{xxy}(x,0) \\ r(x,x) = 0, \quad r_{xx}(x,x) - r_{yy}(x,x) = \frac{c^2}{2}x. \end{cases} \tag{21}$$

Given the solution of (21), we have $s = (r_{yy} - r_{xx})/c$. PDE (21) is the true control gain PDE and its 4th order is a consequence of the fact that the plant is inherently 4th order in space.

Had we arrived at (21) by postulating the transformation (17), (18) and bypassing the connection with the Schrödinger equation, it would be extremely difficult to explicitly solve it. However, using the solution (8) it is easy to obtain the closed-form solution:

$$r(x,y) = x\sqrt{\frac{c}{2z}}\left[-\mathrm{ber}_1(\sqrt{cz}) - \mathrm{bei}_1(\sqrt{cz})\right], \tag{22}$$

$$s(x,y) = x\sqrt{\frac{c}{2z}}\left[\mathrm{ber}_1(\sqrt{cz}) - \mathrm{bei}_1(\sqrt{cz})\right], \tag{23}$$

where $z = x^2 - y^2$. The control laws are obtained by setting $x = 1$ in (17), (18):[1]

$$w_t(1) = \int_0^1 r(1,y) w_t(y) dy + \int_0^1 s(1,y) w_{yy}(y) dy \tag{24}$$

$$w_{xx}(1) = \int_0^1 r(1,y) w_{yy}(y) dy - \int_0^1 s(1,y) w_t(y) dy. \tag{25}$$

Note that the feedback (24) would be implemented as integral, not proportional, control.

The control laws (24), (25)stabilize the beam to a constant profile. To see this, we use the (2), (16), and the inverse transformation (10) to get

$$w_t(x) = \alpha_t(x) + c\alpha(x) + \int_0^x s(x,y)\alpha_{yy}(y) dy - \int_0^x r(x,y)\left(\alpha_t(y) + c\alpha(y)\right) dy, \tag{26}$$

$$w_{xx}(x) = \alpha_{xx}(x) - \int_0^x s(x,y)\left(\alpha_t(y) + c\alpha(y)\right) dy - \int_0^x r(x,y)\alpha_{yy}(y) dy. \tag{27}$$

In deriving (26), (27) we used the fact that $l(x,y) = -r(x,y) + js(x,y)$, which can be shown from (11). We can see that when α converges to zero, w_t and w_{xx} converge to zero. Since $w_x(0) = 0$, this implies that w converges to a constant. Therefore, the straightforward application of the control design for the Schrödinger equation to the

[1] From this point on, we suppress the dependence on time due to space constraints and for notational clarity, i.e. $w_t(1) \equiv w_t(1,t)$, etc.

Euler-Bernoulli beam equation results in controls that suppress oscillations without necessarily bringing the beam to the zero position.

VI. Control Laws That Guarantee Regulation to Zero

To achieve regulation to zero, we are going to modify the control law (24) to make it proportional, not integral, control. To this end, we want to express the second integral of (24) through the terms that contain only time derivatives w_t and w_{tt} and then integrate (24) with respect to time.

A. Control Laws

First, let us calculate (integrating by parts twice)

$$\int_0^1 s(1,y)w_{yy}(y)dy = -\int_0^1 \left[\int_y^1\int_0^z s(1,\xi)d\xi dz\right] w_{yyyy}(y)dy - \gamma_1 w_{xx}(1), \quad (28)$$

where

$$\gamma_1 = -\int_0^1 s(1,y)dy = \sinh\left(\sqrt{\frac{c}{2}}\right)\sin\left(\sqrt{\frac{c}{2}}\right). \quad (29)$$

Since

$$\int_y^1\int_0^z s(1,\xi)d\xi dz = (1-y)\int_0^1 s(1,\xi)d\xi - \int_y^1 s(1,\xi)(\xi-y)d\xi$$

and $w_{xxxx} = -w_{tt}$, from (28) and (24) we get

$$w_t(1) = \int_0^1 r(1,y)w_t(y)dy - \int_0^1 \gamma_1(1-y)w_{tt}(y)dy - \int_0^1 w_{tt}(y)\int_y^1 s(1,\xi)(\xi-y)d\xi dy - \gamma_1 w_{xx}(1). \quad (30)$$

In a similar way, we get

$$\int_0^1 r(1,y)w_{yy}(y)dy = -\int_0^1 w_{tt}(y)\int_y^1 r(1,\xi)(\xi-y)d\xi dy + (1-\gamma_2)w_{xx}(1) - \int_0^1(\gamma_2-1)(1-y)w_{tt}(y)dy, \quad (31)$$

where

$$\gamma_2 = 1 - \int_0^1 r(1,y)dy = \cosh\left(\sqrt{\frac{c}{2}}\right)\cos\left(\sqrt{\frac{c}{2}}\right). \quad (32)$$

Substituting (31) into (25) gives

$$w_{xx}(1) = -\frac{1}{\gamma_2}\int_0^1 s(1,y)w_t(y)dy - \frac{1}{\gamma_2}\int_0^1 w_{tt}(y)\left[(\gamma_2-1)(1-y) + \int_y^1 r(1,\xi)(\xi-y)d\xi\right]dy.$$

Using this equation with (30), after simplifications we get

$$w_t(1) = \int_0^1 \left[(r(1,y)+\gamma s(1,y))\,w_t(y) - \gamma(1-y)w_{tt}(y)\right]dy + \int_0^1 w_{tt}(y)\int_y^1 (\gamma r(1,\xi) - s(1,\xi))\,(\xi-y)d\xi dy, \quad (33)$$

where

$$\gamma = \frac{\gamma_1}{\gamma_2} = \tanh\left(\sqrt{\frac{c}{2}}\right)\tan\left(\sqrt{\frac{c}{2}}\right). \quad (34)$$

The control gains in (33) involve a division by γ_2, which may become zero for certain values of c. Therefore, c should satisfy the condition

$$c \neq \frac{\pi^2}{2}(2n+1)^2, \quad n = 0,1,2,\ldots \quad (35)$$

which is easily achievable because c is the designer's choice.

We now integrate (33) with respect to time to get the controller

$$u_1(t) = \int_0^1 (r(1,y)+\gamma s(1,y))\,w(y,t)dy + \int_0^1 w_t(y,t)\int_y^1 (\gamma r(1,\xi) - s(1,\xi))\,(\xi-y)d\xi dy - \int_0^1 \gamma(1-y)w_t(y,t)dy, \quad (36)$$

where the constant of integration is chosen to be zero since this choice ensures the regulation of w to zero. To see this, note from the transformation (26), (27), and the boundary condition $w_x(0,t)=0$ that w converges to a constant. Suppose $w(x,\infty) \equiv A$, then passing to the limit $t \to \infty$ in (36) we get

$$A = A\int_0^1 (r(1,y)+\gamma s(1,y))\,dy = A\frac{\cosh(a)^2 - \sin(a)^2}{\cosh(a)\cos(a)} + A$$

where $a = \sqrt{c/2}$. Note that $\cosh(a)^2 - \sin(a)^2 > 1$ for all $c > 0$, and $\cos(a) \neq 0$ due to (35). Therefore, $A = 0$.

The other controller (18) can also be represented in terms of w and w_t as follows

$$u_2(t) = w_{xx}(1,t) = -\int_0^1 s(1,y)w_t(y,t)dy + \frac{c^2}{8}w(1,t) + \int_0^1 r_{yy}(1,y)w(y,t)dy. \quad (37)$$

B. Transformation

To find out what the transformation from w to α is, we start with the definition (12) and note from (5) and (2) that

$$\mathrm{Im}\,\{\psi(x)\} = -w_{xx}(x) + \int_0^x \left[r(x,y)w_{xx}(y) - s(x,y)w_t(y)\right]dy.$$

Substituting this into (12), we get

$$\alpha = w - w(1) + \int_x^1 \int_0^y \int_0^z \left[r(z,\xi)w_{\xi\xi}(\xi) - s(z,\xi)w_t(\xi)\right] d\xi dz dy.$$

Integrating by parts the first term in the integral, changing the order of integration in both integral terms, and substituting $w(1)$ from (36), we obtain the final form of the transformation:

$$\begin{aligned} \alpha(x) = w(x) &- \int_0^x \left(r(x,y) + cS(x,y)\right) w(y)dy \\ &+ \int_0^x S(x,y)w_t(y)dy \\ &+ \int_0^1 \left(cS(1,y) - \gamma s(1,y)\right) w(y)dy \\ &+ \int_0^1 w_t(y)\left(-S(1,y) + \gamma(1-y)\right)dy \\ &+ \int_0^1 w_t(y) \int_y^1 \left(s(1,\xi) - \gamma r(1,\xi)\right)(\xi - y)d\xi dy, \end{aligned} \quad (38)$$

where $S(x,y) = \int_y^x (x-\xi)s(\xi,y)d\xi$. Note that this integral transformation is not strict-feedback, it is of a mixed Volterra/Fredholm type.

VII. Inverse Transformation

To prove stability of the closed-loop system through the stability of the target system, we need the transformation which is inverse to (38). It is natural to assume that the inverse transformation has the same structure as the direct one, consisting of two Volterra and two Fredholm integrals of the state of the target system and its time derivative. Therefore, we look for it in the form

$$\begin{aligned} w(x) = \alpha(x) &+ \int_0^x A(x,y)\alpha(y)dy + \int_0^x B(x,y)\alpha_t(y)dy \\ &+ \int_0^1 C(y)\alpha(y)dy + \int_0^1 D(y)\alpha_t(y)dy, \end{aligned} \quad (39)$$

where A, B, C, D are the gains to be determined. Differentiating (39) w.r.t. time and space (twice) and matching the result to the (26) and (27) and to the boundary conditions $w_x(0,t) = w_{xxx}(0,t) = 0$, one can show that (due to lack of space we omit these straightforward calculations)

$$A = -r - 2cS, \quad B = -S, \quad C = 2cD \quad (40)$$

and $D(y)$ satisfies the ODE

$$D'''' + c^2 D = 0,\ D'(0) = D(1) = D''(1) = 0,\ D'''(0) = -c$$

which has the solution

$$\begin{aligned} D(y) = &\frac{\sinh(ay)\cos\left(a(y-2)\right) + \cos(ay)\sinh\left(a(y-2)\right)}{4a\left(\cosh(a)^2 - \sin(a)^2\right)} \\ &- \frac{\sin(ay)\cosh\left(a(y-2)\right) + \cosh(ay)\sin\left(a(y-2)\right)}{4a\left(\cosh(a)^2 - \sin(a)^2\right)}. \end{aligned}$$

Remark 2: The explicit form of the above transformation allows us to write the solution of the closed-loop system in closed form using the explicit solution of the target system. Starting with the explicit solution of (7) and using (12) and (16) we obtain the explicit solution of the target system (13). We then express the initial conditions of the target system through the initial conditions of the closed-loop system using (39). Finally, the closed-loop solution is obtained using the transformation (38). The result is

$$\begin{aligned} w(x,t) = &\sum_{n=0}^{\infty} \mathrm{e}^{-ct}\left(\kappa_n^2 \cos\left(\kappa_n^2 t\right) + c\sin\left(\kappa_n^2 t\right)\right) \\ &\times \left[P_n \cosh\left(\frac{cx}{2\beta_n}\right)\cos(\beta_n x) - Q_n \sinh\left(\frac{cx}{2\beta_n}\right)\sin(\beta_n x)\right] \\ &+ \sum_{n=0}^{\infty} \mathrm{e}^{-ct}\left(\kappa_n^2 \sin\left(\kappa_n^2 t\right) - c\cos\left(\kappa_n^2 t\right)\right) \\ &\times \left[Q_n \cosh\left(\frac{cx}{2\beta_n}\right)\cos(\beta_n x) + P_n \sinh\left(\frac{cx}{2\beta_n}\right)\sin(\beta_n x)\right], \end{aligned}$$

where $\kappa_n = \pi(n + 1/2)$, $\beta_n = \sqrt{(\kappa_n^2 + \sqrt{\kappa_n^4 + c^2})/2}$,

$$\begin{bmatrix} P_n \\ Q_n \end{bmatrix} = \int_0^1 \int_0^{1-y^2} \begin{bmatrix} \mathrm{bei}(\sqrt{c}z)w_1(y) - \mathrm{ber}(\sqrt{c}z)w_0''(y) \\ \mathrm{ber}(\sqrt{c}z)w_1(y) + \mathrm{bei}(\sqrt{c}z)w_0''(y) \end{bmatrix} \times \frac{2\kappa_n z \sin(\kappa_n \sqrt{z^2+y^2})}{(\kappa_n^4 + c^2)\sqrt{z^2+y^2}} dz dy$$

and $w_0 = w(x,0)$, $w_1 = w_t(x,0)$ are the initial conditions.

VIII. Main Result

The design procedure presented in previous sections makes it clear why the closed-loop system (1), (36), (37) is exponentially stable. In this section we give the precise statement of well-posedness and stability.

First, we define the state space

$$\begin{aligned} H = \Big\{(f,g) \in H^2(0,1) \times L^2(0,1) | f'(0) = 0, \\ f(1) = \int_0^1 \left(r(1,y) + \gamma s(1,y)\right) f(y)dy \\ -\gamma \int_0^1 (1-y)g(y)dy \\ + \int_0^1 g(y) \int_y^1 \left(\gamma r(1,\xi) - s(1,\xi)\right)(\xi - y)d\xi dy \Big\}. \end{aligned} \quad (41)$$

We define the inner product induced norm of H as the energy of the system:

$$\|(f,g)\|_H^2 = \int_0^1 \left[\left|f''(x)\right|^2 + |g(x)|^2\right] dx \quad (42)$$

for all $(f,g) \in H$. In fact, if $\|(f,g)\|_H^2 = 0$, then $g(x) = 0$ and $f''(x) = 0$ which, together with the boundary condition $f'(0) = 0$, shows that $f(x) = K = \mathrm{const}$. Substituting $(f, g) = (K, 0)$ into (41), we obtain $K = K\int_0^1 (r(1,y) + \gamma s(1,y))dy$. Since $\int_0^1 [r(1,y) + \gamma s(1,y)]dy \neq 1$, we get $K = 0$. Therefore, (42) defines a norm in H.

The closed-loop system can be written as

$$\frac{d}{dt}\left(w(\cdot,t), w_t(\cdot,t)\right) = \mathcal{A}\left(w(\cdot,t), w_t(\cdot,t)\right) \tag{43}$$

where $\mathcal{A}(f,g) = (g, -f^{(4)})$ for all $(f,g) \in D(\mathcal{A})$ and

$$D(\mathcal{A}) = \Big\{(f,g) \in H | \mathcal{A}(f,g) \in H, f'''(0) = 0, \\ f''(1) = \frac{c^2 f(1)}{8} + \int_0^1 \left[r_{yy}(1,y)f(y) - s(1,y)g(y)\right] dy\Big\}. \tag{44}$$

Next we state two results on the existence and boundedness of $\mathcal{A}^{-1}$ and the existence and uniqueness of a classical solution.

Lemma 2: Let $\mathcal{A}$ be defined by (44) and let the condition (35) hold. Then $\rho(\mathcal{A})$, the resolvent set of $\mathcal{A}$, is not empty. In fact, $0 \in \rho(\mathcal{A})$.

Proof: For any given $(p,q) \in H$, solving $\mathcal{A}(f,g) = (p,q)$ it is straightforward to obtain $g = p$ and

$$f(x) = c_2 + c_1\frac{x^2}{2} - \int_0^x \frac{(x-\tau)^2}{2} q(\tau)d\tau \tag{45}$$

where c_1 and c_2 are appropriate constants. Thus, $\mathcal{A}^{-1}$ exists and is bounded, that is to say, $0 \in \rho(\mathcal{A})$.

Lemma 3: Let $\mathcal{A}$ be defined by (44). Then eigenpairs of $\mathcal{A}$ are given by

$$\left\{(\mu_n, (f_n(x), \mu_n f_n(x))), \left(\overline{\mu_n}, \left(\overline{f_n(x)}, \overline{\mu_n}\overline{f_n(x)}\right)\right)\right\}_{n=1}^{\infty}$$

where

$$\mu_n = -c + j\frac{\pi^2(2n+1)^2}{4}, \; f_n(x) = \left[(I - \mathbb{P})^{-1}\varphi_n\right](x) \tag{46}$$

and $\varphi_n(x) = \cos(n - (1/2))\pi x$ and $\mathbb{P}$ is defined by (5).

Proof: Suppose $\mathcal{A}(f,g) = \lambda(f,g)$. Then we have $g = \lambda f$ with $\lambda \neq 0$ (guaranteed by Lemma 2). Let $p = \lambda f - jf''$. The pair (λ, p) solves the eigenvalue problem (see (3))

$$jp'' + \lambda p = 0, \; p'(0) = 0, \; p(1) = \int_0^1 k(1,y)p(y)dy. \tag{47}$$

There are two cases. When $p \equiv 0$, from the definition of p we have $\lambda f(x) = jf''(x)$. Together with the boundary condition $f'(0) = 0$ this gives $f(x) = \cosh \rho x$, where $\rho^2 = \lambda/j$. Since $\lambda \neq 0$, $\rho \neq 0$, the boundary conditions for $f(1)$ and $f''(1)$ with $f(x) = \cosh \rho x$ produce the same equality

$$f(1) = \int_0^1 \overline{k(1,y)} f(y)dy. \tag{48}$$

From this we get $\lambda = -c - j[(n - (1/2))\pi]^2$, $n = 1, 2, \ldots$, and the eigenfunctions are given by

$$f(x) = \overline{(I - \mathbb{P})^{-1}\varphi(x)}, \quad \varphi(x) = \cos\left(\frac{2n-1}{2}\pi x\right). \tag{49}$$

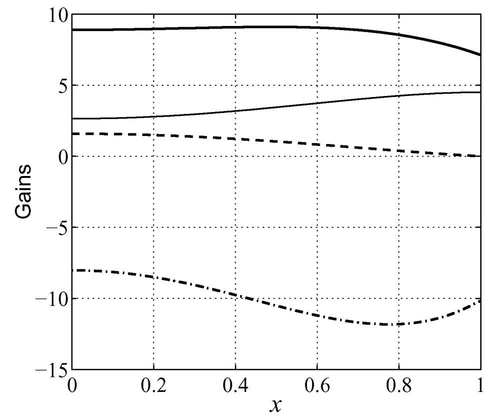

Fig. 2. Control gains for $c = 9$: from $w(x)$ to $w(1)$ (solid), from $w_t(x)$ to $w(1)$ (dashed), from $w(x)$ to $w_{xx}(1)$ (dash-dotted), from $w_t(x)$ to $w_{xx}(1)$ (thin solid).

When $p \neq 0$, (47) gives the solutions $\lambda = -c + j[(n - (1/2)\pi]^2$. Since the eigenvalues are symmetric about the real axis, another branch of eigenpairs is given by (46).

Lemma 4: Let $\mathcal{A}$ be defined by (44) and let the condition (35) hold. Then for any $(w(\cdot,0), w_t(\cdot,0)) \in D(A)$ there exists a unique classical solution to (43).

Proof: Suppose $(\alpha(x,0), \alpha_t(x,0)) \in D(\mathbb{C})$, then by Theorem 1 there exists a unique classical solution to (13). In Section VII we showed that at least one classical solution (39), (26) exists. To show uniqueness of this solution, it is enough to show that if w is a classical solution of (43) with $w(x,0) = w_t(x,0) \equiv 0$, then $w \equiv 0$. Given such w, from (2) we get that v is a solution of (3), (4) and using (5) we get that ψ is a solution of (7). From (12) and a series of calculations (VI-B)-(38), we get $\alpha(x,0) \equiv 0$. From (17) we get $\alpha_t(x,0) \equiv 0$. Therefore, by Theorem 1 $\alpha \equiv 0$. From (39), (26) we get $w \equiv 0$, which completes the proof.

Now we are ready to state the main result of the technical note.

Theorem 5: Let $\mathcal{A}$ be defined by (44) and let the condition (35) hold. Then:

i) $\mathcal{A}$ generates a C_0-semigroup on H. For any initial value $(w(\cdot,0), w_t(\cdot,0)) \in H$, there exists a unique (mild) solution to (43):

$$(w(\cdot,t), w_t(\cdot,t)) = e^{\mathcal{A}t}\left(w(\cdot,0), w_t(\cdot,0)\right) \in C\left([0,\infty); H\right)$$

ii) The spectrum-determined growth condition holds for the semigroup $e^{\mathcal{A}t}$: $\omega(\mathcal{A}) = S(\mathcal{A}) = -c$.

iii) The system (43) is exponentially stable at the origin: for any given $\varepsilon > 0$, there exists $M_\varepsilon > 0$, which depends only on ε, such that for all initial conditions $(w(\cdot,0), w_t(\cdot,0)) \in H$,

$$E(t) \leq M_\varepsilon e^{(-c+\varepsilon)t}E(0), \quad E(t) = \frac{1}{2}\left(\|w_t\|^2 + \|w_{xx}\|^2\right).$$

Proof: Statement (i) follows from Lemmas 2, 4 and Theorem 1.3 of [19] on p.102. Statement (iii) follows from the density of $D(\mathcal{A})$ in H and (26), (27) and Theorem 1. Statement (iii) implies $\omega(\mathcal{A}) \leq -c$. By Lemma 3, $S(\mathcal{A}) \geq -c$. Since one always has $\omega(\mathcal{A}) \geq S(\mathcal{A})$, we get (ii).

IX. Simulation Results

The results of simulation of the Euler-Bernoulli beam with the controllers (36), (37) are presented in Figs. 2, 3. In Fig. 2 the control gains

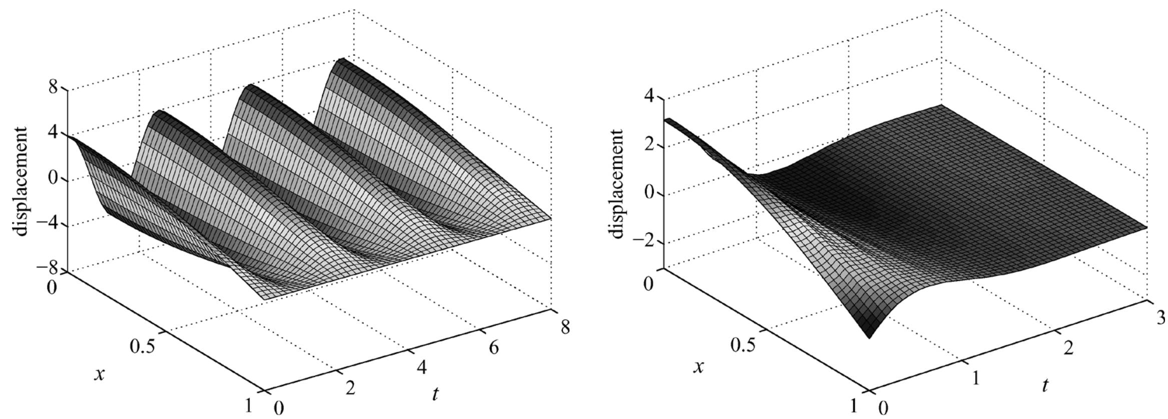

Fig. 3. Evolution of the Euler-Bernoulli beam. Left: open-loop response. Right: closed-loop response.

are shown for $c = 9$. Fig. 3 (left) shows the oscillations of the uncontrolled beam. With control, the beam is quckly brought to the zero equilibrium (Fig. 3, right).

X. Future Work

In future work there are several extensions of the result of the technical note to pursue. First, one would like to control beams with other types of boundary conditions. From the design procedure presented in the technical note it is clear that an extension to a hinged type of the uncontrolled end should not pose any difficulties. One would just change the type of the uncontrolled boundary condition in the Schrödinger equation from Neumann to Dirichlet. However, at present it is not clear yet how to exploit the connection of the Euler-Bernoulli beam with the Schrödinger equation in case of the beam with a free uncontrolled end, the most important case from practical point of view.

One would also like to extend the results of the technical note to the output-feedback case. For the Schrödinger equation, successful observer-based output-feedback design was developed in [11]. It seems that there are no conceptual obstacles in adapting this design to the Euler-Bernoulli beam.

Last but not least, the extension of our designs to 2D flexible structures, such as plates [13], is a promising research avenue.

References

[1] G. Chen, M. C. Delfour, A. M. Krall, and G. Payre, "Modeling, stabilization and control of serially connected beam," *SIAM J. Control Optim.*, vol. 25, pp. 526–546, 1987.

[2] A. Cheng and K. Morris, "Well-posedness of boundary control systems," *SIAM J. Control Optim.*, vol. 42, no. 4, pp. 1244–1265, 2003.

[3] J.-M. Coron and B. d'Andréa-Novel, "Stabilization of a rotating body beam without damping," *IEEE Trans. Automat. Control*, vol. 43, no. 5, pp. 608–618, May 1998.

[4] M. Dadfarnia, N. Jalili, B. Xian, and D. M. Dawson, "Lyapunov-based vibration control of translational Euler-Bernoulli beams using the stabilizing effect of beam damping mechanisms," *J. Vibration Control*, vol. 10, pp. 933–961, 2004.

[5] I. C. Gohberg and M. G. Krein, *Introduction to the Theory of Linear Nonselfadjoint Operators*. Providence, RI: American Math Society, 1969.

[6] B. Z. Guo and R. Yu, "The Riesz basis property of discrete operators and application to an Euler-Bernoulli beam equation with boundary linear feedback control," *IMA J. Math. Control Inf.*, vol. 18, pp. 241–251, 2001.

[7] B. Z. Guo, "Riesz basis approach to the stabilization of a flexible beam with a tip mass," *SIAM J. Control Optim.*, vol. 39, pp. 1736–1747, 2001.

[8] B. Z. Guo, "Riesz basis property and exponential stability of controlled Euler-Bernoulli beam equations with variable coefficients," *SIAM J. Control Optim.*, vol. 40, pp. 1905–1923, 2002.

[9] V. Komornik, "Rapid boundary stabilization of linear distributed systems," *SIAM J. Control Optim.*, vol. 35, pp. 1591–1613, 1997.

[10] M. Krstic, A. Balogh, and A. Smyshlyaev, "Backstepping boundary controller and observer for the undamped shear beam," in *Proc. 17th Int. Symp. Math. Theory Networks Syst.*, 2006, [CD ROM].

[11] M. Krstic, B. Z. Guo, and A. Smyshlyaev, "Boundary controllers and observers for Schrödinger equation," in *Proc. IEEE Conf. Decision Control*, 2007, pp. 4149–4154.

[12] M. Krstic, A. Siranosian, and A. Smyshlyaev, "Backstepping boundary controllers and observers for the slender Timoshenko beam: Part I—Design," in *Proc. IEEE Amer. Control Conf.*, 2006, pp. 2412–2417.

[13] J. Lagnese, *Boundary Stabilization of Thin Plates*. Philadelphia, PA: SIAM, 1989.

[14] I. Lasiecka and R. Triggiani, "Exact controllability of semilinear abstract systems with application to waves and plates boundary control problems," *Appl. Math. Optim.*, vol. 23, pp. 109–154, 1991.

[15] K. S. Liu and Z. Liu, "Exponential decay of energy of the Euler-Bernoulli beam with locally distributed Kelvin-Voigt damping," *SIAM J. Control Optim.*, vol. 36, pp. 1086–1098, 1998.

[16] Z. H. Luo, B. Z. Guo, and O. Morgul, *Stability and Stabilization of Infinite Dimensional Systems With Applications*. London, U.K.: Springer Verlag, 1999.

[17] *Control of Flexible Structures*, K. A. Morris, Ed. Providence, RI: AMS, 1993.

[18] E. Machtyngier and E. Zuazua, "Stabilization of the Schrödinger equation," *Portugaliae Mathematica*, vol. 51, pp. 243–256, 1994.

[19] A. Pazy, *Semigroups of Linear Operators and Applications to Partial Differential Equations*. Berlin, Germany: Springer-Verlag, 1983.

[20] R. Rebarber, "Spectral assignability for distributed parameter systems with unbounded scalar control," *SIAM J. Control Optim.*, vol. 27, pp. 148–169, 1989.

[21] A. Smyshlyaev and M. Krstic, "Closed form boundary state feedbacks for a class of 1D partial integro-differential equations," *IEEE Trans. Automat. Control*, vol. 49, no. 12, pp. 2185–2202, Dec. 2004.

[22] J. M. Urquiza, "Rapid exponential feedback stabilization with unbounded control operators," *SIAM J. Control Optim.*, vol. 43, pp. 2233–2244, 2005.

[23] C. Z. Xu and G. Sallet, "On spectrum and Riesz basis assignment of infinite-dimensional linear systems by bounded linear feedbacks," *SIAM J. Control Optim.*, vol. 34, pp. 521–541, 1996.