# Motion Planning and Tracking for Tip Displacement and Deflection Angle for Flexible Beams

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*Explicit motion-planning reference solutions are presented for flexible beams with Kelvin–Voigt (KV) damping. The goal is to generate periodic reference signals for the displacement and deflection angle at the free-end of the beam using only actuation at the base. The explicit deflection angle reference solution is found as a result of writing the shear beam model in a strict-feedback form. Special "partial differential equation (PDE) backstepping" transformations relate the strict-feedback model to a "target system," governed by an exponentially stable wave equation with KV damping, whose displacement reference solution is relatively easy to find. The explicit beam displacement reference solution is found using the target system solution and an inverse backstepping transformation. The explicit reference solutions for the wave equation and shear beam with KV damping are novel results. State-feedback tracking boundary controllers are found by extending previous PDE backstepping stabilization results. Application of the shear beam results to the more complicated Timoshenko beam is discussed.* [DOI: 10.1115/1.3072152]

*Keywords: motion planning, trajectory generation, trajectory tracking, boundary control, PDE backstepping, wave equation, string, flexible beam, shear beam, Timoshenko beam, Kelvin–Voigt damping*

## 1 Introduction

Great strides have been made in the design and implementation of collocated boundary controllers—a control architecture with sensing and actuation implemented at the same boundary point—for flexible beams for vibration suppression and stabilization. A comprehensive monograph on collocated boundary control of flexible beams [1] presents the key approaches for studying stability and for imparting damping via boundary control. Work done in Refs. [2–6] has achieved analytical and experimental success in designing collocated boundary controllers for flexible beams. Recent work has also been done to extend vibration suppression and stabilizing controllers to noncollocated systems [7–10]—systems with actuation and sensing at different points. This work pursues the line of noncollocated boundary control, going beyond the problem of equilibrium stabilization to solve the problems of motion planning and reference tracking for flexible beams. Motion planning (trajectory generation) is the problem of finding the appropriate boundary input to produce a desired output. The full-state motion-planning reference solution can be used to find an open-loop boundary input, or combined with tracking boundary controllers to improve the rate of convergence to the reference solution.

Motion-planning results for strings and flexible structures without Kelvin–Voigt (KV) damping—internal/material damping—have been presented in Refs. [11–16]. This work considers systems with KV damping since they are more physically relevant, and note that the damping terms make finding the reference solution more difficult (though they make the stabilization problem slightly easier). The system models being considered are the wave equation (string and target system) and the shear and Timoshenko beams. A string is a single-input–single-output system, with the displacement at the base as the input, and the same quantity at the free-end as the output. A beam is a two-input–two-output system with the displacement and deflection angle at the base as the inputs, and the same quantities at the free-end as outputs. Figure 1 shows a diagram representing the problem setup. Motivation for this setup comes from a particular shake table control problem where the table provides boundary actuation to a structure, modeled here as a flexible beam, in order to impart a desired reference trajectory at its free-end. This setup also applies directly to the control of dynamic mode atomic force microscopy (AFM), where a cantilevered beam is actuated at the base to produce a sinusoidal output at the free-end.

The shear beam design begins by writing the model in a strict-feedback (spatially causal) form to which PDE backstepping techniques can be applied [17–20]. The deflection angle boundary controller is found as a result of this step, and a clever modification of that controller produces the explicit deflection angle reference solution. PDE backstepping transformations—state transformations that relate one PDE and boundary condition(s) to another—are used to relate the strict-feedback shear beam model to the target system—a reference model used in control design, governed by a wave equation with KV damping (see Sec. 2.2). The inverse state transformation is used to find the explicit beam displacement reference solution given the target system reference solution. The target system reference solution is found using the direct transformation between the target and string states and the explicit string reference solution. The explicit motion-planning reference solution for the string model is found by postulating the reference solution as a power series of the spatial variable with time dependent coefficients [11–16,21–23]. The advantage of employing PDE backstepping techniques is that they provide a means for the rather complicated shear beam reference solution to be found using the relatively simple reference solution for the string.

The motion-planning results in the presence of Kelvin–Voigt damping for the string, target system, and especially for the shear beam models are novel and nontrivial results. The KV damping introduces a new complexity in propagating the control signal

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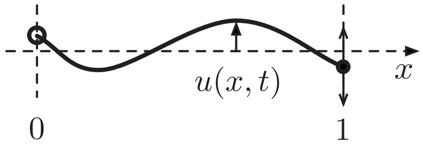


**Fig. 1 Diagram depicting a string/beam with transverse displacement $u(x,t)$: The goal is to generate and track a reference trajectory at $x=0$. The arrows at $x=1$ represent actuation, and the circle at $x=0$ represents the desired reference trajectory.**

from one boundary to the other boundary, and the shear beam requires a two-stage construction of the motion-planning solution, which does not arise with simple wave equations and Euler–Bernoulli beams. Furthermore, *explicit* motion-planning results are novel.

Aside from facilitating the motion-planning designs, the PDE backstepping approach is also used to combine the open-loop reference solutions with state-feedback boundary controllers to achieve exponential convergence to the reference trajectories. These results are extensions of the stabilizing boundary controller designs for the string model, and the shear and Timoshenko beams [17–20,24,25].

Section 2 presents the system models. Section 3 presents the reference solutions for the string, target system, and shear beam. Section 4 presents stabilizing tracking boundary controllers for the string and the shear beam. Section 5 presents simulation results for the generation and tracking of sinusoids for the string and the Timoshenko beam. The Appendix defines the key terms used in this work.

## 2 Plant Models

**2.1 String.** The string model is given by the wave equation

$$\varepsilon u_{tt} = (1 + d\partial_t)u_{xx} \tag{1}$$

$$u_x(0,t) = 0 \tag{2}$$

where $u(x,t)$ is the displacement of the string along $0 \le x \le 1$ at time $0 \le t < \infty$, with initial conditions $u_0(x)=u(x,0)$ and $\dot{u}_0(x)=u_t(x,0)$, $d$ is the KV damping coefficient, and $\varepsilon$ is the inverse of the nondimensional string stiffness. Partial derivatives with respect to space and time are denoted by subscripts $x$ and $t$, respectively. The boundary condition (2) at $x=0$ represents a free-end. The boundary input $u_x(1,t)$ is used as a control input.

**2.2 Target System.** The target system is an exponentially stable reference model used in PDE backstepping control design. The target system model used for string, shear beam, and Timoshenko beam designs [17–20,24,25], shown in Fig. 2 as a string with a spring at $x=0$ and a damper at $x=1$, is given by the wave equation

$$\varepsilon w_{tt} = (1 + d\partial_t)w_{xx} \tag{3}$$

$$w_x(0,t) = c_0 w(0,t) \tag{4}$$

$$w_x(1,t) = -c_1 w_t(1,t) \tag{5}$$

where $w(x,t)$ is the displacement of the target system, and has initial conditions $w_0(x)=w(x,0)$ and $\dot{w}_0(x)=w_t(x,0)$. The param-

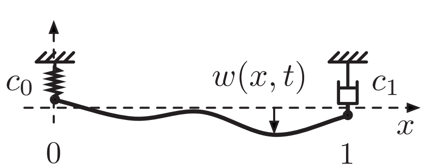


**Fig. 2 Diagram representing the target system**

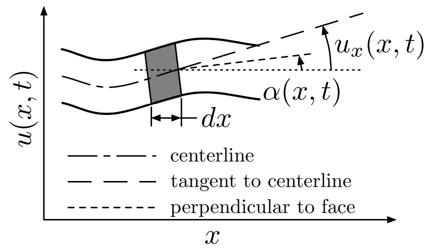


**Fig. 3 A differential element of length $dx$ in the Timoshenko beam: The diagram shows the relationship between the beam displacement $u(x,t)$, the slope $u_x(x,t)$, and the deflection angle $\alpha(x,t)$. This diagram has been adapted from a figure in Ref. [28].**

eters $c_0 > 0$ and $c_1 > 0$ are design gains representing the spring stiffness and damping coefficient of the spring and damper located at opposite ends of the string. The spring stiffness $c_0$ should be large to emulate a pinned boundary condition at $x=0$, and the damping coefficient $c_1$ should be chosen near $\sqrt{\varepsilon}$ to emulate a tuned damper at the end $x=1$.

**2.3 Flexible Beams.** The Timoshenko beam model is written as two coupled wave equations

$$\varepsilon u_{tt} = (1 + d\partial_t)(u_{xx} - \alpha_x) \tag{6}$$

$$\mu\varepsilon\alpha_{tt} = (1 + d\partial_t)(\varepsilon\alpha_{xx} + a(u_x - \alpha)) \tag{7}$$

$$u_x(0,t) = \alpha(0,t) \tag{8}$$

$$\alpha_x(0,t) = 0 \tag{9}$$

where $u(x,t)$ denotes the displacement and $\alpha(x,t)$ denotes the deflection angle, with initial conditions $u_0(x)=u(x,0)$, $\dot{u}_0(x)=u_t(x,0)$, $\alpha_0(x)=\alpha(x,0)$, and $\dot{\alpha}_0(x)=\alpha_t(x,0)$. The positive constants $a$, $\varepsilon$, and $\mu$ parametrize the appropriate nondimensional beam parameters as defined in Refs. [26,27]. The parameters $a$ and $\mu$ are proportional to the nondimensional cross-sectional area, and the nondimensional moment of inertia of the beam, respectively. The parameter $\varepsilon$ is inversely proportional to the nondimensional shear modulus of the beam. The beam has a free-end (8) and (9) at $x=0$, and is actuated at the end $x=1$ through the boundary inputs $u_x(1,t)$ and $\alpha(1,t)$. Figure 3 shows the relationships between the displacement $u(x,t)$, slope $u_x(x,t)$, and deflection angle $\alpha(x,t)$.

The shear beam model can be written as a singular perturbation ($\mu=0$) of the Timoshenko beam model, and is given by a wave equation coupled with a second-order-in-space ordinary differential equation (ODE)

$$\varepsilon u_{tt} = (1 + d\partial_t)(u_{xx} - \alpha_x) \tag{10}$$

$$0 = \varepsilon\alpha_{xx} + a(u_x - \alpha) \tag{11}$$

This model also has free-end boundary conditions (8) and (9), and boundary inputs $u_x(1,t)$ and $\alpha(1,t)$.

The shear and Timoshenko beam models are considered for this work since they are the more physically relevant and complete beam models. The Timoshenko beam model is the most accurate of the four, accounting for transverse displacement, bending moment, shear distortion, and rotary inertia [26].

## 3 Motion Planning

Motion planning for the displacement (string, target system, and shear beam) and deflection angle (shear beam) is done for sinusoidal reference trajectories since they are relevant functions in the fields of shake table control and AFM, where reference trajectories tend to be oscillatory, and since they can form a basis for more complicated functions.

**3.1 String.** The solution to the motion-planning problem for the string model (1) and (2) is found for the sinusoidal tip displacement reference trajectory

$$u^r(0,t) = A_u \sin(\omega_u t) \tag{12}$$

by postulating the reference solution $u^r(x,t)$ as a power series of the spatial variable with time dependent coefficients, i.e., $u^r(x,t) = \Sigma_{i=0}^{\infty} a_i(t) x^i / i!$. Examples of applications of this approach can be found in Refs. [11,16,22,23]. The string reference solution is

$$u^r(x,t) = -\frac{jA_u}{2}[\cosh(j\sigma x)e^{j\omega_u t} - \cosh(j\bar{\sigma}x)e^{-j\omega_u t}] \tag{13}$$

with the complex valued constant $\sigma = \omega_u \sqrt{\varepsilon} / \sqrt{1 + j\omega_u d}$. Equation (13) can be written as the purely real function

$$u^r(x,t) = \frac{A_u}{2}[e^{\hat{\beta}(\omega_u)x} \sin(\omega_u t + \beta(\omega_u)x) + e^{-\hat{\beta}(\omega_u)x} \sin(\omega_u t - \beta(\omega_u)x)] \tag{14}$$

where the real functions $\beta(n)$ and $\hat{\beta}(n)$ are defined as

$$\beta(n) = n\sqrt{\varepsilon}\sqrt{\frac{\sqrt{1+n^2d^2}+1}{2(1+n^2d^2)}} \tag{15}$$

$$\hat{\beta}(n) = n\sqrt{\varepsilon}\sqrt{\frac{\sqrt{1+n^2d^2}-1}{2(1+n^2d^2)}} \tag{16}$$

The open-loop displacement (Dirichlet) control $u^r(1,t)$ is found by evaluating Eq. (14) at $x=1$. The expression for the open-loop slope/force (Neumann) control input $u_x^r(1,t)$, found by evaluating the partial derivative with respect to $x$ of Eq. (14) at $x=1$, is

$$\begin{aligned} u_x^r(1,t) = \frac{A_u}{2}[&\hat{\beta}(\omega_u)e^{\hat{\beta}(\omega_u)} \sin(\omega_u t + \beta(\omega_u)) \\ &+ \beta(\omega_u)e^{\hat{\beta}(\omega_u)} \cos(\omega_u t + \beta(\omega_u)) \\ &- \hat{\beta}(\omega_u)e^{-\hat{\beta}(\omega_u)} \sin(\omega_u t - \beta(\omega_u)) \\ &- \beta(\omega_u)e^{-\hat{\beta}(\omega_u)} \cos(\omega_u t - \beta(\omega_u))] \end{aligned} \tag{17}$$

THEOREM 3.1. *The string model* (1) and (2) *is satisfied by the state reference trajectory* (14). *The output of the system satisfies the tip displacement reference trajectory* (12), *given the open-loop Neumann control input* (17).

*Proof.* The reference solution (14) evaluated at $x=0$ satisfies the desired reference trajectory (12). Equation (14) substituted into Eqs. (1) and (2) satisfies the string PDE and free-end boundary condition. ∎

**3.2 Target System.** The solution to the motion-planning problem for the target system model (3) and (4) is found using the reference solution for the string model and a PDE backstepping state transformation. The string model (1) and (2) with boundary actuation $u_x(1,t)$, and the target system (3)–(5) are related via the direct backstepping transformation $w(x,t) = u(x,t) + c_0 \int_0^x u(y,t)dy$ [24,25], which when substituted into Eqs. (3) and (4) satisfies Eqs. (1) and (2). Therefore, the target system reference solution $w^r(x,t)$, found as a function of the string reference solution $u^r(x,t)$, is

$$w^r(x,t) = u^r(x,t) + c_0 \int_0^x u^r(x,t)dy \tag{18}$$

Backstepping transformations preserve values at the boundary $x=0$, i.e., $w(0,t) = u(0,t)$, and so the target system tip displacement reference trajectory

$$w^r(0,t) = A_u \sin(\omega_u t) \tag{19}$$

is equivalent to the string reference trajectory $u^r(0,t)$. The reference solution, found by substituting Eq. (13) into Eq. (18) and evaluating the integral, is

$$\begin{aligned} w^r(x,t) = &-\frac{jA_u}{2}[\cosh(j\sigma x)e^{j\omega_u t} - \cosh(j\bar{\sigma}x)e^{-j\omega_u t}] \\ &- \frac{c_0 A_u}{2}\left[\left(\frac{1}{\bar{\sigma}}\right)\sinh(j\bar{\sigma}x)e^{j\omega_u t} - \left(\frac{1}{\sigma}\right)\sinh(j\bar{\sigma}x)e^{-j\omega_u t}\right] \end{aligned}$$

with the complex valued constant $\sigma = \omega_u \sqrt{\varepsilon} / \sqrt{1 + j\omega_u d}$. The expression for $w^r(x,t)$ can be written as the purely real function

$$\begin{aligned} w^r(x,t) = \frac{A_u}{2}[&e^{\hat{\beta}(\omega_u)x} \sin(\omega_u t + \beta(\omega_u)x) \\ &+ e^{-\hat{\beta}(\omega_u)x} \sin(\omega_u t - \beta(\omega_u)x)] \\ &- \frac{c_0 A_u}{2}\{\gamma(\omega_u)[e^{\hat{\beta}(\omega_u)x} \cos(\omega_u t + \beta(\omega_u)x) \\ &- e^{-\hat{\beta}(\omega_u)x} \cos(\omega_u t - \beta(\omega_u)x)] \\ &- \hat{\gamma}(\omega_u)[e^{\hat{\beta}(\omega_u)x} \sin(\omega_u t + \beta(\omega_u)x) \\ &- e^{-\hat{\beta}(\omega_u)x} \sin(\omega_u t - \beta(\omega_u)x)]\} \end{aligned} \tag{20}$$

where the real valued functions $\beta(n)$ and $\hat{\beta}(n)$ are defined in Eqs. (15) and (16), respectively, and $\gamma(n)$ and $\hat{\gamma}(n)$ are defined as

$$\gamma(n) = \frac{1}{n\sqrt{\varepsilon}}\sqrt{\frac{\sqrt{1+n^2d^2}+1}{2}} \tag{21}$$

$$\hat{\gamma}(n) = \frac{1}{n\sqrt{\varepsilon}}\sqrt{\frac{\sqrt{1+n^2d^2}-1}{2}} \tag{22}$$

The open-loop displacement (Dirichlet) control $w^r(1,t)$ is found by evaluating Eq. (20) at $x=1$. The open-loop slope/force (Neumann) control $w_x^r(1,t)$ is found by evaluating the partial derivative with respect to $x$ of Eq. (20) at $x=1$.

THEOREM 3.2. *The target system* (3) and (4) *is satisfied by the state reference trajectory* (20). *The output of the system satisfies the tip displacement reference trajectory* (19), *given the open-loop Neumann control input* $w_x^r(1,t)$.

*Proof.* The reference solution (20) evaluated at $x=0$ satisfies the reference trajectory (19). Equation (20) substituted into Eqs. (3) and (4) satisfies the target system PDE and $x=0$ boundary condition. ∎

**3.3 Shear Beam.** The solution to the motion-planning problem for the shear beam model has two parts. Figure 4 shows a pictorial representation of the structure of the problem. A backstepping transformation is ultimately used to find the state reference trajectory; therefore the first part requires writing the shear beam model (10), (11), (8), and (9) in a strict-feedback (spatially causal) form that makes PDE backstepping tools applicable. The second part requires solving the simultaneous motion-planning problem by first finding the reference solution for the free-end deflection angle $\alpha(0,t)$, and then using a backstepping transformation to find the reference solution for the free-end displacement $u(0,t)$. Figure 5 shows the transformations relating the string, tar-

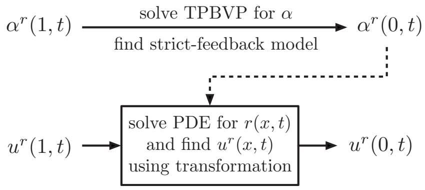


**Fig. 4 Pictorial representation of the structure of the input-output relationship $\{u^r(1,t),\alpha^r(1,t)\}\mapsto\{u^r(0,t),\alpha^r(0,t)\}$, and a description of the types of problems involved in solving the simultaneous motion-planning problem: Finding $\alpha^r(1,t)$ involves solving a two-point boundary-value problem (TPBVP) for $\alpha(x,t)$, then modifying the resulting boundary input $\alpha(1,t)$ to satisfy both spatial causality of the shear beam and motion planning. Finding $u^r(1,t)$ requires solving a PDE for the auxiliary system $r(x,t)$, then employing a direct transformation from $w^r(x,t)$ to $u^r(x,t)$.**

get system, and strict-feedback shear beam used to relate the string displacement reference solution (easiest to find) to the strict-feedback shear beam solution.

*3.3.1 Strict-Feedback Shear Beam Model*. The ODE in Eq. (11), with the boundary condition (9) and $\alpha(1,t)$ available as the control input, constitutes a two-point-boundary-value problem. The general solution [17–20] of that problem is $\alpha(x,t)=\cosh(bx)\alpha(0,t)+b\sinh(bx)u(0,t)-b^2\int_0^x\cosh(b(x-y))u(y,t)dy$, $b=\sqrt{a/\varepsilon}$, which can be evaluated at $x=1$ and written as

$$\alpha(0,t)=\frac{1}{\cosh(b)}\left(\alpha(1,t)-b\sinh(b)u(0,t)+b^2\int_0^1\cosh(b(1-y))u(y,t)dy\right) \tag{23}$$

Substituting the appropriate partial derivatives of the solution for $\alpha(x,t)$ into Eqs. (10) and (8), and using the expression for $\alpha(0,t)$ in Eq. (23), produces

$$\begin{aligned}\varepsilon u_{tt}=(1+d\partial_t)\Bigg[&u_{xx}+b^2u-b^2\cosh(bx)u(0,t)\\&+b^3\int_0^x\sinh(b(x-y))u(y,t)dy\\&-\frac{b\sinh(bx)}{\cosh(b)}\bigg(\alpha(1,t)-b\sinh(b)u(0,t)\\&+b\int_0^1\sinh(b(1-y))u(y,t)dy\bigg)\Bigg]\end{aligned} \tag{24}$$

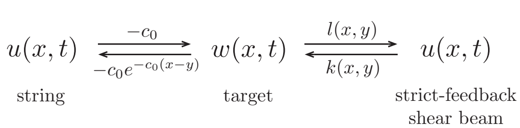


**Fig. 5 The string of invertible transformations involved in solving the shear beam motion-planning problem: The functions above and below the arrows represent the appropriate transformation gains**

$$\begin{aligned}u_x(0,t)=\frac{1}{\cosh(b)}\bigg(&\alpha(1,t)-b\sinh(b)u(0,t)\\&+b\int_0^1\sinh(b(1-y))u(y,t)dy\bigg)\end{aligned} \tag{25}$$

Backstepping tools require that the plant model be in a strict-feedback form; therefore Eqs. (24) and (25) cannot contain terms that violate spatial causality, for example, $\int_0^1\sinh(b(1-y))u(y)dy$. The boundary control $\alpha(1,t)$, present in Eqs. (24) and (25), is set to

$$\alpha(1,t)=b\sinh(b)u(0,t)-b\int_0^1\sinh(b(1-y))u(y,t)dy \tag{26}$$

simplifying Eqs. (24) and (25) into the strict-feedback shear beam model [17–20]

$$\begin{aligned}\varepsilon u_{tt}=(1+d\partial_t)\bigg(&u_{xx}+b^2u-b^2\cosh(bx)u(0,t)\\&+b^3\int_0^x\sinh(b(x-y))u(y,t)dy\bigg)\end{aligned} \tag{27}$$

$$u_x(0,t)=0 \tag{28}$$

to which backstepping tools can now be applied.

The strict-feedback shear beam model (27) and (28) with boundary actuation $u_x(1,t)$, and target system (3)–(5) are related through the *direct* backstepping transformation [17–20] $w(x,t)=u(x,t)-\int_0^xk(x,y)u(y,t)dy$, with $k(x,y)$ satisfying the partial integro-differential equation (PIDE)

$$k_{xx}=k_{yy}+b^2k-b^3\sinh(b(x-y))+b^3\int_y^xk(x,\xi)\sinh(b(\xi-y))d\xi \tag{29}$$

$$k(x,x)=-\frac{b^2}{2}x-c_0 \tag{30}$$

$$k_y(x,0)=-b^2\cosh(bx)+b^2\int_0^xk(x,y)\cosh(by)dy \tag{31}$$

which when substituted into Eqs. (3) and (4) satisfies Eqs. (27) and (28). The two systems are also related through the *inverse* backstepping transformation

$$u(x,t)=w(x,t)+\int_0^xl(x,y)w(y,t)dy \tag{32}$$

where $l(x,y)$ satisfies the PIDE

$$l_{xx}=l_{yy}-b^2l-b^3\sinh(b(x-y))-b^3\int_y^x\sinh(b(x-\xi))l(\xi,y)d\xi \tag{33}$$

$$l(x,x)=-\frac{b^2}{2}x-c_0 \tag{34}$$

$$l_y(x,0)=c_0l(x,0)-b^2\cosh(bx) \tag{35}$$

LEMMA 3.1. *The inverse backstepping transformation* (32)*, with* $l(x,y)$ *satisfying Eqs.* (33)–(35)*, substituted into Eqs.* (27) *and* (28) *satisfies Eqs.* (3) *and* (4).

*Proof.* Substituting Eq. (32) and its appropriate partial derivatives into Eqs. (27) and (28), and using the relationships in Eqs.

(3) and (4), shows that $l(x,y)$ must satisfy Eqs. (33)–(35) in order to satisfy the transformation from target to plant states. ■

*3.3.2 Simultaneous Motion Planning*. The boundary control signal (26) satisfies the need for the shear beam model to be spatially causal, but it also forces $\alpha(0,t)=0$ in Eq. (23) and eliminates the opportunity to do motion planning for the tip deflection angle. However, for a given reference signal $\alpha^r(0,t)$, augmenting the boundary control law (26) with the additive term $\cosh(b)\alpha^r(0,t)$ produces the boundary condition $\alpha(0,t)=\alpha^r(0,t)$ and satisfies the desired tip deflection angle reference trajectory. The strict-feedback shear beam model for motion planning is then

$$\varepsilon u_{tt} = (1+d\partial_t)\Bigg[ u_{xx} + b^2 u - b^2\cosh(bx)u(0,t) + b^3\int_0^x \sinh(b(x-y))u(y,t)dy - b\sinh(bx)\alpha^r(0,t)\Bigg] \tag{36}$$

$$u_x(0,t) = \alpha^r(0,t) \tag{37}$$

The explicit deflection angle reference solution is given by

$$\alpha^r(x,t) = \cosh(bx)\alpha^r(0,t) + b\sinh(bx)u^r(0,t) - b^2\int_0^x \cosh(b(x-y))u^r(y,t)dy \tag{38}$$

and the boundary control for motion planning is

$$\alpha^r(1,t) = \cosh(b)\alpha^r(0,t) + b\sinh(b)u^r(0,t) - b^2\int_0^1 \cosh(b(1-y))u^r(y,t)dy \tag{39}$$

where $u^r(x,t)$ is the state reference trajectory for the strict-feedback shear beam model (36) and (37), and can be found using a PDE backstepping transformation from the target system to the strict-feedback shear beam model.

The strict-feedback shear beam model for motion planning (36) and (37) with boundary actuation $u_x(1,t)$, and the target system (3)–(5) are related through the *direct* backstepping transformation

$$w(x,t) = u(x,t) - \int_0^x k(x,y)u(y,t)dy + r(x,t) \tag{40}$$

where $k(x,y)$ satisfies Eqs. (29)–(31), and $r(x,t)$ is the state of an auxiliary system satisfying the PDE

$$\varepsilon r_{tt} = (1+d\partial_t)\Bigg[ r_{xx} + \bigg(-b\sinh(bx) + b\int_0^x k(x,y)\sinh(by)dy\bigg)\alpha^r(0,t)\Bigg] \tag{41}$$

$$r(0,t) = 0 \tag{42}$$

$$r_x(0,t) = \alpha^r(0,t) \tag{43}$$

The auxiliary state $r(x,t)$ is required to satisfy the transformation from target to plant when the reference solution for the tip deflection angle is introduced into the design. The two systems are also related through the *inverse* backstepping transformation

$$u(x,t) = w(x,t) - r(x,t) + \int_0^x l(x,y)[w(y,t) - r(y,t)]dy \tag{44}$$

where $l(x,y)$ and $r(x,t)$ satisfy Eqs. (33)–(35) and (41)–(43).

LEMMA 3.2. *The direct backstepping transformation* (40)*, with* $k(x,y)$ *satisfying Eqs.* (29)–(31) *and* $r(x,t)$ *satisfying Eqs.* (41)–(43)*, substituted into Eqs.* (3) *and* (4) *satisfies Eqs.* (36) *and* (37).

*Proof.* Substituting Eq. (40) and its partial derivatives into Eqs. (3) and (4), using the relationships in Eqs. (36) and (37), shows that $k(x,y)$ and $r(x,t)$ must satisfy Eqs. (29)–(31) and (41)–(43). ■

LEMMA 3.3. *The inverse backstepping transformation* (44)*, with* $l(x,y)$ *satisfying Eqs.* (33)–(35) *and* $r(x,t)$ *satisfying Eqs.* (41)–(43)*, substituted into Eqs.* (36) *and* (37) *satisfies Eqs.* (3) *and* (4).

*Proof.* Substituting Eq. (44) and its appropriate partial derivatives into Eqs. (36) and (37), and using the relationships in Eqs. (3) and (4), shows that $l(x,y)$ and $r(x,t)$ must satisfy Eqs. (33)–(35) and (41)–(43), respectively. ■

The explicit displacement reference solution for the strict-feedback shear beam model for motion planning, found using the inverse transformation (44), is

$$u^r(x,t) = w^r(x,t) - r(x,t) + \int_0^x l(x,y)[w^r(y,t) - r(y,t)]dy \tag{45}$$

where $w^r(x,t)$ is given in Eq. (20), and $r(x,t)$ must be found for a particular tip deflection angle reference trajectory $\alpha^r(0,t)$. The shear beam tip displacement reference trajectory, found by evaluating Eq. (45) at $x=0$, is given in Eq. (12).

The open-loop displacement (Dirichlet) boundary control is found by evaluating Eq. (45) at $x=1$. The open-loop slope/force (Neumann) boundary control, found by evaluating the partial derivative with respect to $x$ of Eq. (45) at $x=1$, is

$$u_x^r(1,t) = w_x^r(1,t) - r_x(1,t) + l(1,1)[w^r(1,t) - r(1,t)] + \int_0^1 l_x(1,y)[w^r(y,t) - r(y,t)]dy \tag{46}$$

where $w^r(1,t)$ is given by Eq. (20) evaluated at $x=1$, and $w_x^r(1,t)$ is given by the partial derivative with respect to $x$ of Eq. (20) evaluated at $x=1$. The expressions $r(1,t)$ and $r_x(1,t)$ can be derived from the solution for $r(x,t)$.

To that end consider the sinusoidal tip deflection angle reference trajectory given by

$$\alpha^r(0,t) = A_\alpha \sin(\omega_\alpha t) \tag{47}$$

where $A_\alpha$ and $\omega_\alpha$ are the amplitude and frequency, respectively. The solution to the auxiliary system $r(x,t)$ is found by first taking a Laplace transform in space of Eqs. (41)–(43), which reduces the PDE in space and time to the ODE in time

$$\varepsilon \ddot{R}(s,t) - ds^2\dot{R}(s,t) - s^2R(s,t) = (\Phi(s)-1)(\alpha^r(0,t) + d\dot{\alpha}^r(0,t)) \tag{48}$$

where $\Phi(s)$ denotes the Laplace transform of $\phi(x)=-b\sinh(bx)+b\int_0^x k(x,y)\sinh(by)dy$. The solution to Eq. (48), ignoring transients, is assumed to be of the form

$$R(s,t) = A_1(s)\sin(\omega_\alpha t) + A_2(s)\cos(\omega_\alpha t) \tag{49}$$

where $A_1(s)$ and $A_2(s)$ must be found.

Substituting Eqs. (47) and (49) into Eq. (48), grouping terms common in $\sin(\omega_\alpha t)$ and $\cos(\omega_\alpha t)$, then solving the resulting linear algebra problem for $A_1(s)$ and $A_2(s)$ give $A_1(s)=A_\alpha F_1(s)(1-\Phi(s))$ and $A_2(s)=A_\alpha F_2(s)(1-\Phi(s))$, where $F_1(s)=((1+\omega_\alpha^2 d^2)s^2+\varepsilon\omega_\alpha^2)/((1+\omega_\alpha^2 d^2)s^4+2\varepsilon\omega_\alpha^2 s^2+(\varepsilon\omega_\alpha^2)^2)$ and $F_2(s)=\varepsilon\omega_\alpha^3 d/((1+\omega_\alpha^2 d^2)s^4+2\varepsilon\omega_\alpha^2 s^2+(\varepsilon\omega_\alpha^2)^2)$. The inverse Laplace transforms of $A_1(s)$, $A_2(s)$, $F_1(s)$, and $F_2(s)$ are

$$a_1(x) = A_\alpha\bigg(f_1(x) - \int_0^x f_1(x-y)\phi(y)dy\bigg) \tag{50}$$

$$a_2(x) = A_\alpha\left(f_2(x) - \int_0^x f_2(x-y)\phi(y)dy\right) \quad (51)$$

$$f_1(x) = \frac{1}{2}\left(\frac{1}{\nu}\sin(\nu x) + \frac{1}{\bar{\nu}}\sin(\bar{\nu}x)\right) \quad (52)$$

$$f_2(x) = \frac{j}{2}\left(\frac{1}{\nu}\sin(\nu x) - \frac{1}{\bar{\nu}}\sin(\bar{\nu}x)\right) \quad (53)$$

with the complex valued constant $\nu = \omega_\alpha\sqrt{\varepsilon}\sqrt{(1+j\omega_\alpha d)/(1+\omega_\alpha^2 d^2)}$. The expressions in Eqs. (52) and (53) can be written as the purely real functions

$$f_1(x) = \gamma(\omega_\alpha)\sin(\beta(\omega_\alpha)x)\cosh(\hat{\beta}(\omega_\alpha)x) + \hat{\gamma}(\omega_\alpha)\cos(\beta(\omega_\alpha)x)\sinh(\hat{\beta}(\omega_\alpha)x) \quad (54)$$

$$f_2(x) = -\gamma(\omega_\alpha)\cos(\beta(\omega_\alpha)x)\sinh(\hat{\beta}(\omega_\alpha)x) + \hat{\gamma}(\omega_\alpha)\sin(\beta(\omega_\alpha)x)\cosh(\hat{\beta}(\omega_\alpha)x) \quad (55)$$

where the real valued functions $\beta(\omega_\alpha)$, $\hat{\beta}(\omega_\alpha)$, $\gamma(\omega_\alpha)$, and $\hat{\gamma}(\omega_\alpha)$ are defined in Eqs. (15), (16), (21), and (22), respectively.

The solution to the auxiliary system, given by the inverse Laplace transform of Eq. (49) with $a_1(x)$ and $a_2(x)$ given in Eqs. (50) and (51), is then

$$r(x,t) = A_\alpha\left(f_1(x) - \int_0^x f_1(x-y)\phi(y)dy\right)\sin(\omega_\alpha t) + A_\alpha\left(f_2(x) - \int_0^x f_2(x-y)\phi(y)dy\right)\cos(\omega_\alpha t) \quad (56)$$

THEOREM 3.3. *The shear beam model* (10), (11), (8), *and* (9) *is satisfied by the state reference trajectories* (38) *and* (45) *, where* $l(x,y)$ *satisfies Eqs.* (33)–(35), $w^r(x,t)$ *is given in Eq.* (20), *and* $r(x,t)$ *is given in Eq.* (56). *The outputs of the system satisfy the tip displacement and deflection angle reference trajectories* (12) *and* (47) *given the open-loop control inputs* (39) *and* (46).

*Proof.* The reference solutions (38) and (45) evaluated at $x=0$ satisfy the desired free-end displacement and deflection angle reference trajectories. Equations (38) and (45) substituted into Eqs. (10), (11), (8), and (9) satisfy the shear beam PDE and free-end boundary conditions. ■

## 4 Reference Tracking

Reference tracking controllers combine the open-loop motion-planning reference solutions with stabilizing feedback controllers. Their purpose is to stabilize the system and improve the rate of convergence to the reference solution when there exists a mismatch in initial conditions between the system state and reference solution.

DEFINITION 4.1. *The reference trajectory* $u^r(x,t)$ *is said to be exponentially stable if there exist positive constants M and m such that*

$$(\|u(t)-u^r(t)\|^2 + \|u_t(t)-u_t^r(t)\|^2)^{1/2} \le Me^{-mt}(\|u_0-u_0^r\|^2 + \|\dot{u}_0-\dot{u}_0^r\|^2)^{1/2} \quad (57)$$

*where* $\|\cdot\|$ *denotes the norm of* $v$, $\|v\|=(\int_0^1 v(x)^2dx)^{1/2}$, $u_0(x)=u(x,0)$, $u_0^r(x)=u^r(x,0)$, $\dot{u}_0(x)=u_t(x,0)$, *and* $\dot{u}_0^r(x)=u_t^r(x,0)$.

**4.1 String.** The tracking controller for the string is an extension of the stabilizing controller in Refs. [24,25].

THEOREM 4.1. *The state-feedback tracking controller*

$$u_x(1,t) = -c_0u(1,t) - c_1u_t(1,t) - c_0c_1\int_0^1 u_t(y,t)dy + w_x^r(1,t) + c_1w_t^r(1,t) \quad (58)$$

*exponentially stabilizes the string system* (1) *and* (2) *about the state reference trajectory* (14).

*Proof.* The expression for the boundary controller (58) is found by writing the standard boundary controller in Refs. [24,25] in terms of the reference tracking error $\tilde{u}(x,t)=u(x,t)-u^r(x,t)$, where $w_x^r(1,t)+c_1w_t^r(1,t)=u_x^r(1,t)+c_0u^r(1,t)+c_1u_t^r(1,t)+c_0c_1\int_0^1 u_t^r(y,t)dy$.

The string reference solution (14) satisfies string model (1), (2), and (58), and therefore has the same dynamics. Therefore, the tracking error dynamics can be written as

$$\varepsilon\tilde{u}_{tt} = (1+d\partial_t)\tilde{u}_{xx} \quad (59)$$

$$\tilde{u}_x(0,t) = 0 \quad (60)$$

$$\tilde{u}_x(1,t) = -c_0\tilde{u}(1,t) - c_1\tilde{u}_t(1,t) - c_0c_1\int_0^1 \tilde{u}_t(y,t)dy \quad (61)$$

which resemble the closed-loop string dynamics. The direct and inverse backstepping transformations $\tilde{w}(x,t)=\tilde{u}(x,t)+c_0\int_0^x\tilde{u}(y,t)dy$ and $\tilde{u}(x,t)=\tilde{w}(x,t)-c_0\int_0^x e^{-c_0(x-y)}\tilde{w}(y,t)dy$ relate the tracking error dynamics (59)–(61) and the exponentially stable tracking error target system

$$\varepsilon\tilde{w}_{tt} = (1+d\partial_t)\tilde{w}_{xx} \quad (62)$$

$$\tilde{w}_x(0,t) = c_0\tilde{w}(0,t) \quad (63)$$

$$\tilde{w}_x(1,t) = -c_1\tilde{w}_t(1,t) \quad (64)$$

The state of the tracking error system $\tilde{u}(x,t)$ can be bounded by the state of the tracking error target system $\tilde{w}(x,t)$ by $\|\tilde{u}(t)\|\le(1+c_0)\|\tilde{w}(t)\|$, and the same is true for the time derivatives, so the closed-loop system (1), (2), and (58) is exponentially stable around the reference solution (14). ■

The string boundary controller (58) requires slope/force actuation at the base, but can also be written in a form that requires displacement actuation. When combined with full-state observers [24,25], the output-feedback tracking controller requires sensing of the free-end displacement and velocity.

**4.2 Shear Beam.** The tracking controllers for the shear beam are extensions of the stabilizing controllers in Refs. [17–20].

THEOREM 4.2. *The state-feedback tracking controllers*

$$u_x(1,t) = k(1,1)u(1,t) + \int_0^1 k_x(1,y)u(y,t)dy - c_1u_t(1,t) + c_1\int_0^1 k(1,y)u_t(y,t)dy + w_x^r(1,t) + c_1w_t^r(1,t) - r_x(1,t) - c_1r_t(1,t) \quad (65)$$

$$\alpha(1,t) = \cosh(b)\alpha^r(0,t) + b\sinh(b)u(0,t) - b^2\int_0^1 \cosh(b(1-y))u(y,t)dy \quad (66)$$

*exponentially stabilize the shear beam* (10), (11), (8), *and* (9) *about the state reference trajectories* (38) *and* (45). *The tip displacement and deflection angle track Eqs.* (12) *and* (47), *respectively.*

*Proof.* The expression for the boundary controller (65) is found by expressing the target system boundary condition (5) in terms of the tracking error $[w_x(1,t)-w_x^r(1,t)]=-c_1[w_t(1,t)-w_t^r(1,t)]$, and

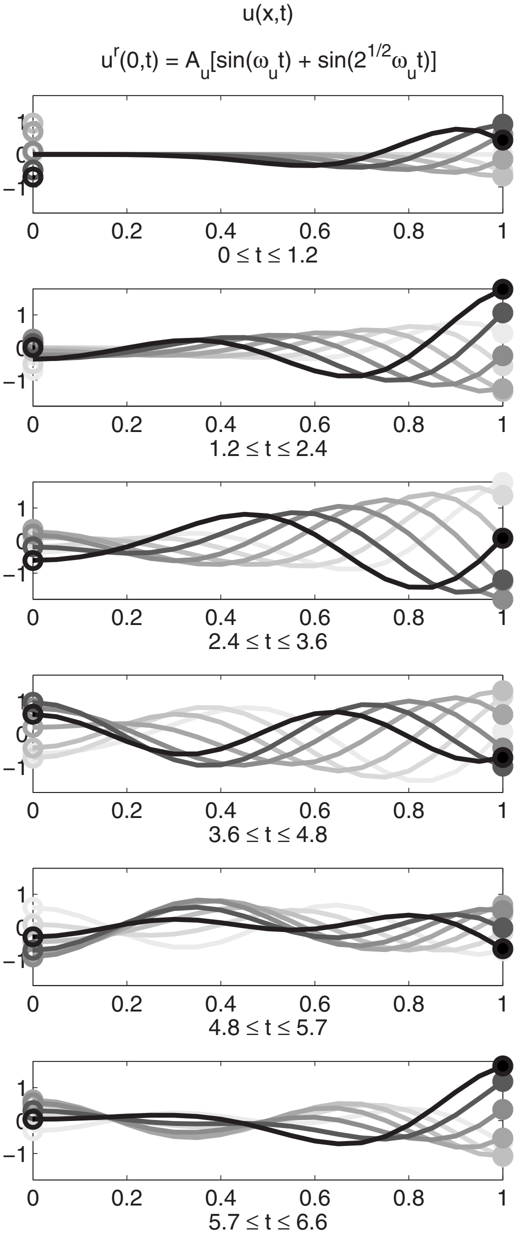


**Fig. 6 String simulation showing the state as snapshots in time**

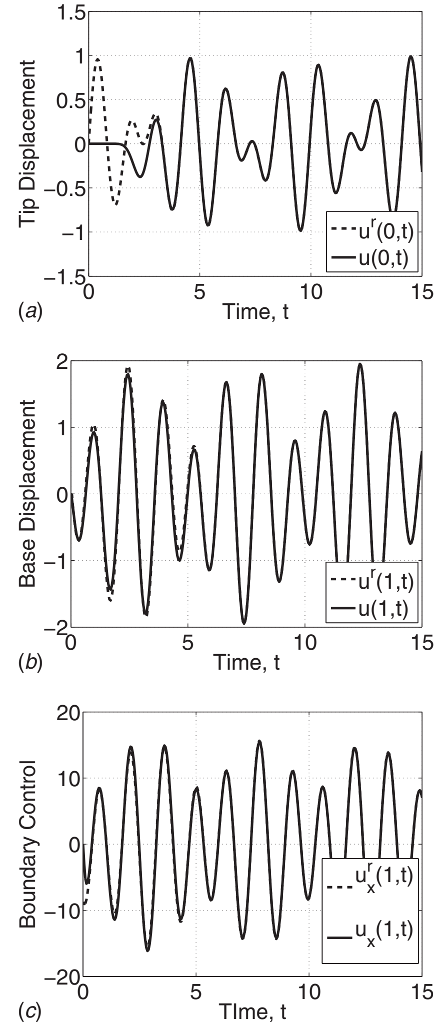


**Fig. 7 String simulation results comparing the (*a*) tip displacement $u(0,t)$ and reference trajectory $u^r(0,t)$, (*b*) base displacement $u(1,t)$ and reference displacement $u^r(1,t)$, and (*c*) boundary control input $u_x(1,t)$ and reference input $u_x^r(1,t)$**

using the transformation (40) to substitute for $w_x(1,t)$ and $w_t(1,t)$.

Application of the boundary controller (66) makes the shear beam spatially causal. The shear beam reference solution (45) satisfies the strict-feedback shear beam model for motion planning (36), (37), and (65), and therefore has the same dynamics. The tracking error dynamics can therefore be written as

$$\varepsilon\tilde{u}_{tt} = (1 + d\partial_t)\left(\tilde{u}_{xx} + b^2\tilde{u} - b^2\cosh(bx)\tilde{u}(0,t) + b^3\int_0^x \sinh(b(x-y))\tilde{u}(y,t)dy\right) \tag{67}$$

$$\tilde{u}_x(0,t) = 0 \tag{68}$$

$$\tilde{u}_x(1,t) = k(1,1)\tilde{u}(1,t) + \int_0^1 k_x(1,y)\tilde{u}(y,t)dy - c_1\tilde{u}_t(1,t) + c_1\int_0^1 k(1,y)\tilde{u}_t(y,t)dy \tag{69}$$

which resemble the closed-loop strict-feedback shear beam model dynamics. The direct and inverse backstepping transformations $\tilde{w}(x,t)=\tilde{u}(x,t)-\int_0^x k(x,y)\tilde{u}(y,t)dy$ and $\tilde{u}(x,t)=\tilde{w}(x,t)+\int_0^x l(x,y)\tilde{w}(y,t)dy$ relate the tracking error dynamics (67)–(69) to the exponentially stable tracking error target system (62)–(64). The state of the tracking error system $\tilde{u}(x,t)$ can be upper bounded by the state of the tracking error target system $\tilde{w}(x,t)$ by $\|\tilde{u}(t)\| \leq (1+\|l(1,y)\|_\infty)\|\tilde{w}(t)\|$, and the closed-loop system (10), (11), (8), (9), (65), and (66) is exponentially stable around the reference solutions (38) and (45). ■

The shear beam boundary controllers (65) and (66) require actuation of the slope (or displacement) and bending moment at the base. When combined with full-state observers [17–20], the output-feedback tracking controllers require sensing of the free-end displacement and velocity.

The Timoshenko beam control design in Refs. [19,20] is done using a singular perturbation approach to reduce the Timoshenko

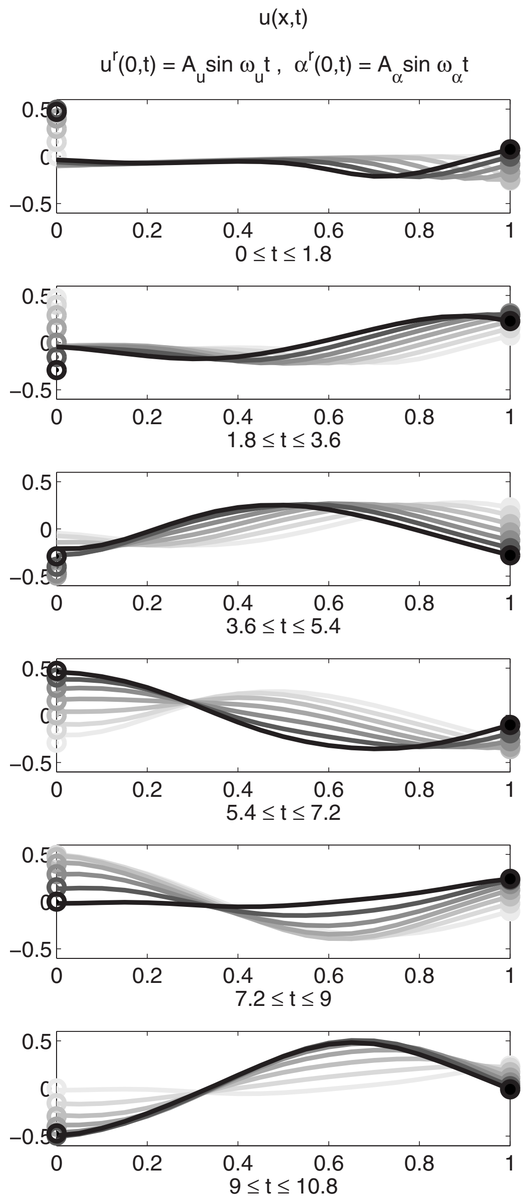


**Fig. 8 Timoshenko beam simulation results showing snapshots of the beam state $u(x,t)$**

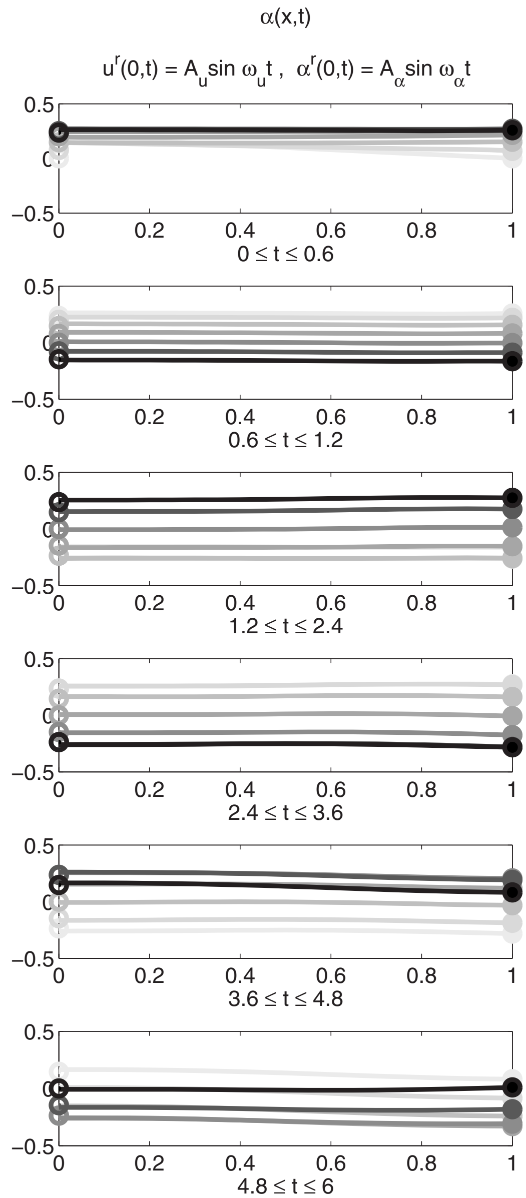


**Fig. 9 Timoshenko beam simulation results showing snapshots of the beam state $\alpha(x,t)$**

beam to the shear beam model. The design is analogous to the shear beam design [17,18], and all results for the shear beam apply *approximately* to the Timoshenko beam. Therefore the reference tracking results presented in Theorem 4.2 also apply approximately to the Timoshenko beam modulo an $O(\mu)$ residual in the tracking error.

## 5 Simulation Results

Simulations employ finite-differences to resolve partial derivatives in space, and the Crank–Nicolson method to march the equations forward in time.

**5.1 String.** This section presents simulation results for the string (1) and (2) in closed-loop with the boundary controller (58). The spatial and temporal step sizes used in the simulations are $\Delta x=\frac{1}{100}$ and $\Delta t=\frac{1}{100}$, respectively. The string parameters are $d=0.08$ and $\varepsilon=5$, and the controller parameters are $c_0=100$ and $c_1=0.99\sqrt{5}$. The reference trajectory parameters are $A_u=\frac{1}{2}$ and $\omega_u=\pi$. The simulation was initialized with zero initial displacement and velocity.

Figures 6 and 7 present simulation results for the reference trajectory $u^r(0,t)=A_u[\sin(\omega_u t)+\sin(\sqrt{2}\omega_u t)]$. Generation and tracking of two sinusoids are achieved by implementing the boundary controller as a function of the linear combination of the target system reference solutions for each sinusoid. Figure 6 shows the evolution of the string state $u(x,t)$ as a sequence of snapshots in time, with increasing darkness corresponding to increasing time in each sequence. The reference trajectory at the corresponding time is represented by a circle at $x=0$ of the same shade. Figure 7($a$) compares the tip displacement $u(0,t)$ to the tip reference trajectory $u^r(0,t)$. Figure 7($b$) compares the base displacement $u(1,t)$ to the reference displacement $u^r(1,t)$. Figure 7($c$) compares the boundary input $u_x(1,t)$ to the reference boundary input $u_x^r(1,t)$.

**5.2 Timoshenko Beam.** This section presents simulation results for the Timoshenko beam model (6)–(9) in closed-loop with the state-feedback controllers (65) and (66). The spatial and tem-

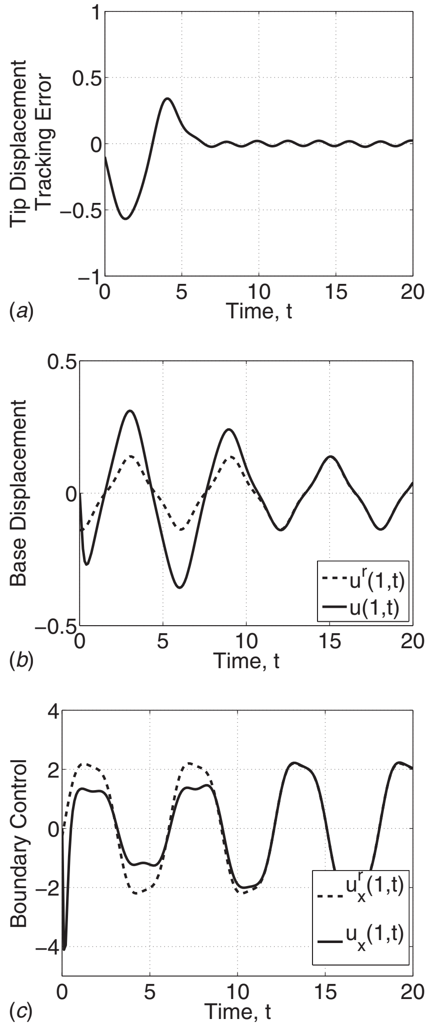


**Fig. 10 Timoshenko beam simulation results showing (*a*) the tip displacement tracking error $u(0,t)-u^r(0,t)$, (*b*) the base displacement $u(1,t)$ and the reference displacement $u^r(1,t)$, and (*c*) the boundary control $u_x(1,t)$ and the reference control $u_x^r(1,t)$**

poral step sizes used in simulation are $\Delta x=\frac{1}{100}$ and $\Delta t=\frac{1}{50}$, respectively. The beam parameters are $a=5$, $d=0.1$, $\varepsilon=20$, and $\mu=0.02$. The controller parameters are $c_0=100$ and $c_1=0.99\sqrt{20}$. The reference trajectory parameters are $A_u=\frac{1}{2}$, $\omega_u=\frac{\pi}{3}$ and $A_\alpha=\frac{1}{4}$, $\omega_\alpha=\pi$. The beam is initialized with an initial displacement $u(x,0)=-\frac{1}{10}(1-x)^2$, an initial deflection angle of $\alpha(x,0)=\frac{1}{5}(1-x)$, and zero initial velocity.

Figures 8–11 present results for the simultaneous tracking of the sinusoidal tip reference trajectories $u^r(0,t)=A_u\sin(\omega_u t)$ and $\alpha^r(0,t)=A_\alpha\sin(\omega_\alpha t)$. Figures 8 and 9 show the evolution of the beam states $u(x,t)$ and $\alpha(x,t)$. Figures 10(*a*)–10(*c*) show the tip displacement tracking error $u(0,t)-u^r(0,t)$, the base displacement $u(1,t)$ and reference displacement $u^r(1,t)$, and the boundary control $u_x(1,t)$ and reference control $u_x^r(1,t)$, respectively. Figures 11(*a*) and 11(*b*) show the tip deflection angle tracking error $\alpha(0,t)-\alpha^r(0,t)$, and the boundary control $\alpha(1,t)$ and reference control $\alpha^r(1,t)$. Both tracking error plots show a periodic steady state error on the order of $\mu=0.02$, with frequency $\omega_\alpha$.

Simultaneous tracking simulations have also been done for reference trajectories where either $u^r(0,t)$ or $\alpha^r(0,t)$ are zero. Simulations with $u^r(0,t)=0$ and $\alpha^r(0,t)=A_\alpha\sin(\omega_\alpha t)$ show how the approximate nature of the shear beam results applied to the Timoshenko beam appear as a periodic disturbance to the $u$-system. The $u_x(1,t)$ controller is not able to fully suppress the disturbance, and $u(x,t)$ exhibits $O(\mu)$ oscillations of frequency $\omega_\alpha$. Simulations with $u^r(0,t)=A_u\sin(\omega_u t)$ and $\alpha^r(0,t)=0$ do not exhibit the $O(\mu)$ tracking error, and the $\alpha(1,t)$ boundary controller stabilizes $\alpha(0,t)$ to zero.

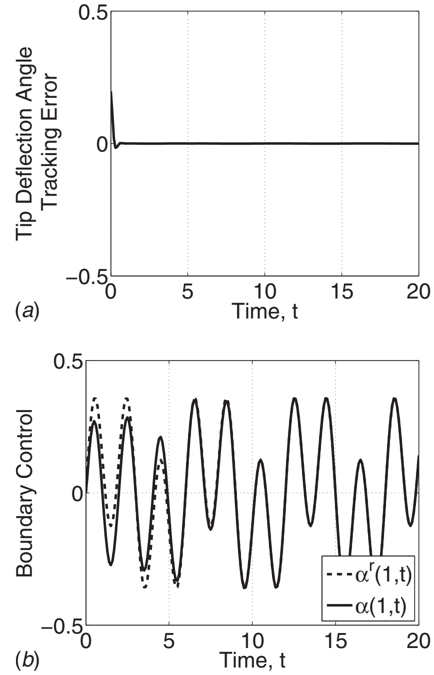


**Fig. 11 Timoshenko beam simulation results showing (*a*) the tip deflection angle tracking error $\alpha(0,t)-\alpha^r(0,t)$, and (*b*) the boundary control $\alpha(1,t)$ and reference control $\alpha^r(1,t)$**

Figures 12(*a*) and 12(*b*) show the control gains $k(1,y)$ and

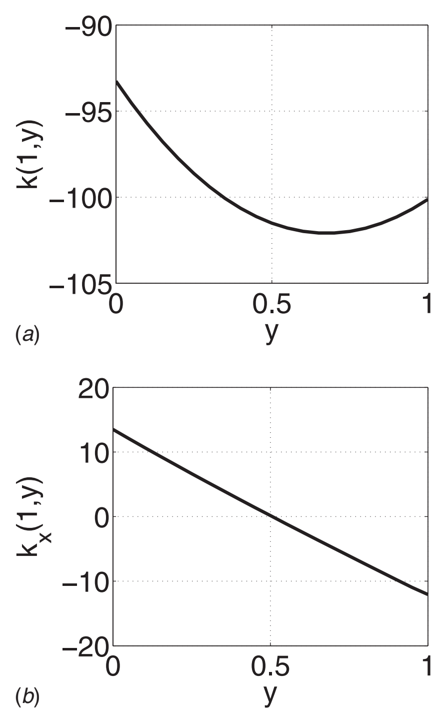


**Fig. 12 Timoshenko beam gains (*a*) $k(1,y)$ and (*b*) $k_x(1,y)$**

$k_x(1,y)$ on the interval $0 \le y \le 1$. The curves are relatively simple and can be approximated by quadratic and linear functions, respectively.

## 6 Conclusion

This paper has presented explicit reference solutions to the motion-planning problem for the wave equation (string and target system) and shear beam models with Kelvin–Voigt damping. The displacement reference solution was first found for the string, which is the simplest model being considered, then PDE backstepping transformations were used to find the displacement solutions for the target system and shear beam as a function of the string solution. PDE backstepping techniques were also used to find the deflection angle reference solution for the shear beam. Combining PDE boundary backstepping methods with classical trajectory generation methods simplified the problem of solving the motion-planning problem for the shear beam, described by coupled wave equations, to finding the reference solution for the much simpler target system.

While this paper has focused on motion planning for periodic trajectories, this approach extends to a far broader class of temporal waveforms that includes polynomials, exponentials, sinusoids, and products thereof as special cases. With a slight modification one can obtain solutions to motion planning for all output reference trajectories that can be written in the form $u^r(0,t) = CX(t)$ where $X(t)$ is a solution of the autonomous linear "exosystem" $\dot{X} = AX$ for a given initial condition $X(0)$. For example, if the reference output is $u^r(0,t) = te^{-t}\sin t$, the parameters of the exosystem would be chosen as $C = [1 \quad 0 \quad 0 \quad 0]$,

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -4 & -8 & -8 & -4 \end{bmatrix},$$

$X(0) = [2 \quad 2 \quad 0 \quad 0]^T$, and finding the motion-planning solution would proceed using the matrix exponentials of $A$.


## Acknowledgment

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## Appendix: Key Terms

**collocated control:** control architecture with actuation and sensing at the same location

**Kelvin–Voigt damping:** internal/material damping

**motion planning:** solving for an open-loop input that generates a desired output

**noncollocated control:** control architecture with actuation and sensing at different locations

**PDE backstepping transformation:** infinite dimensional state transformation relating plant and target system states

**reference solution:** solution to a motion-planning problem

**reference trajectory:** desired trajectory to be generated and tracked

**target system:** exponentially stable reference model used in PDE backstepping control design

**trajectory generation:** see Sec. 3

**trajectory tracking:** combining reference solution with feedback control to stabilize system to desired reference trajectory

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