Adaptive control of PDEs

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Abstract
This paper presents several recently developed techniques for adaptive control of PDE systems. Three different design methods are employed—the Lyapunov design, the passivity-based design, and the swapping design. The basic ideas for each design are introduced through benchmark plants with constant unknown coefficients. It is then shown how to extend the designs to reaction–advection–diffusion PDEs in 2D. Finally, the PDEs with unknown spatially varying coefficients and with boundary sensing are considered, making the adaptive designs applicable to PDE systems with an infinite relative degree, infinitely many unknown parameters, and open loop unstable.

Keywords: Backstepping; Distributed parameter systems; Adaptive control

1. Introduction

In systems with thermal, fluid, or chemically reacting dynamics, which are usually modelled by parabolic partial differential equations (PDEs), physical parameters are often unknown. Thus a need exists for developing adaptive controllers that are able to stabilize a potentially unstable, parametrically uncertain plant. While adaptive control of finite dimensional systems is a mature area that has produced adaptive control methods for most LTI systems of interest (Ioannou & Sun, 1996), adaptive control techniques have been developed for only a few of the classes of PDEs for which non-adaptive controllers exist. The existing results (Bentsman & Orlov, 2001; Bohm, Demetriou, Reich & Rosen, 1998; Hong & Bentsman, 1994) focus on model reference (MRAC) type schemes and the control action distributed in the PDE domain, see (Krstic & Smyshlyaev, 2008) for a more detailed literature review. One of the major obstacles in developing adaptive schemes for PDEs is the absence of parametrized families of stabilizing controllers. In a recent paper (Smyshlyaev & Krstic, 2004), the explicit formulae were introduced for boundary control of parabolic PDEs. Those formulae are not only explicit functions of the spatial coordinates, but also depend explicitly on the physical parameters of the plant. In this paper we overview three different design methods based on those explicit controllers—Lyapunov method, and certainty equivalence approaches with passive and swapping identifiers. For tutorial reasons, the presentation proceeds through a series of one-unknown-parameter benchmark examples with sketches of the proofs. The detailed proofs for the designs presented here are given in Krstic and Smyshlyaev (2008), Smyshlyaev and Krstic (2006a, 2006b); Smyshlyaev and Krstic (2007a, 2007b). We then extend the presented approaches to reaction–advection–diffusion plants in 2D and plants with spatially varying (functional) parametric uncertainties. We end the paper with the output-feedback adaptive design for reaction–advection–diffusion systems with only boundary sensing and actuation. These systems have an infinite relative degree, infinitely many unknown parameters and are open-loop unstable, representing the ultimate challenge in adaptive control for PDEs.

2. Lyapunov design

Consider the following plant:

\[ u_t(x,t) = u_{xx}(x,t) + \lambda u(x,t), \quad 0 < x < 1, \]  
\[ u(0,t) = 0, \]  
\[ u(1,t) = 0. \]
where \( \lambda \) is an unknown constant parameter. We use a Neumann boundary controller in the form (Smyshlyaev & Krstic, 2004)

\[
u_x(1) = -\frac{\hat{\lambda}}{2} u(1) - \hat{\lambda} \int_0^1 I_2 \left( \frac{\sqrt{\hat{\lambda}(1 - \xi^2)}}{1 - \xi^2} \right) u(\xi) \, d\xi,
\]

(3)

which employs the measurements of \( u(x) \) for \( x \in [0, 1] \) and an estimate \( \hat{\lambda} \) of \( \lambda \). One can show that invertible change of variables

\[
w(x) = u(x) - \int_0^x \hat{k}(x, \xi) u(\xi) \, d\xi,
\]

(4)

\[
\hat{k}(x, \xi) = -\hat{\lambda} \xi \frac{\sqrt{\hat{\lambda}(x^2 - \xi^2)}}{\sqrt{\hat{\lambda}(x^2 - \xi^2)}}.
\]

(5)

maps (1)–(3) into

\[
w_x(t) = \hat{w}_x(0) + \hat{\lambda} \int_0^t \frac{1}{2} w(\xi) \, d\xi + \hat{\lambda} w,
\]

(6)

\[
w(0) = \hat{w}_x(1) = 0,
\]

(7)

where \( \hat{\lambda} = \lambda - \hat{\lambda} \) is the parameter estimation error.

Furthermore, the update law

\[
\hat{\lambda} = \gamma \frac{\|w\|^2}{1 + \|w\|^2}, \quad 0 < \gamma < 1
\]

(8)

achieves regulation of \( u(x, t) \) to zero for all \( x \in [0, 1] \), for arbitrarily large initial data \( u(x, 0) \) and for an arbitrarily poor initial estimate \( \hat{\lambda}(0) \).

**Theorem 1.** Suppose that the system (1)–(3), (8) has a well defined classical solution for all \( t \geq 0 \). Then, for any initial condition \( v_0 \in H^1(0, 1) \) compatible with boundary conditions, and any \( \hat{\lambda}(0) \in \mathbb{R} \), the solutions \( u(x, t) \) and \( \hat{\lambda}(t) \) are uniformly bounded and \( \lim_{t \to -\infty} u(x, t) = 0 \) uniformly in \( x \in [0, 1] \).

**Proof. (Sketch).** The time derivative of the Lyapunov function

\[
V = \frac{1}{2} \log(1 + \|w\|^2) + \frac{1}{2} \gamma \hat{\lambda}^2
\]

(9)

along the solutions of (6)–(8) can be shown to be

\[
\dot{V} = -\frac{\|w_x\|^2 + \frac{\hat{\lambda}}{2} \|w(x)\|^2}{1 + \|w\|^2} + \frac{\hat{\lambda}}{2} \int_0^t w(x) (\int_0^x \hat{\lambda}(\xi) \, d\xi) \, d\xi
\]

(10)

(the calculation involves integration by parts). Note that \( V \) depends only on time and contains the logarithm of the spatial \( L_2 \) norm (Praly, 1992). This results in the normalized update law (8) and slows down the adaptation, which is necessary because the control law (3) is of certainty equivalence type. An additional measure of preventing overly fast adaptation in (8) is the restriction on the adaptation gain \( (\gamma < 1) \).

Using Cauchy and Poincare inequalities, one gets

\[
\left\| \int_0^1 w(x) \left( \int_0^x \hat{\lambda}(\xi) \, d\xi \right) \, dx \right\| \leq \frac{2}{\sqrt{3}} \|w_x\|^2.
\]

(11)

Substituting (11) and (8) into (10) and using the fact that \( |\hat{\lambda}| < \gamma \) (see (8)), we get

\[
V \leq -\left(1 - \frac{\gamma}{\sqrt{3}}\right) \|w_x\|^2 \left(1 + \|w\|^2\right).
\]

(12)

This implies that \( V(t) \) remains bounded for all time whenever \( 0 < \gamma < \sqrt{3} \). From the definition of \( V \) it follows that \( \|w\| \) and \( \hat{\lambda} \) remain bounded for all time. To show that \( w(x, t) \) is bounded for all time and for all \( x \), we estimate (using Agmon, Young, and Poincare inequalities):

\[
\frac{1}{2} d/dt \|w_x\|^2 = -\|w_x\|^2 + \|w_x\|^2 \left[ \frac{\hat{\lambda}}{4} (w(1)^2 - \|w\|^2) \right] \leq -\left(1 - \frac{\gamma}{\sqrt{3}}\right) \|w_x\|^2 + \|w\|^2,
\]

(13)

Integrating the last inequality, we obtain

\[
\|w_x(t)\|^2 \leq \|w_x(0)\|^2 + 2 \sup_{0 \leq t \leq \tau} \|\hat{\lambda}(\tau)\| \int_0^\tau \|w_x(\tau)\|^2 \, d\tau.
\]

(14)

Using (12) and the fact that \( \|w\| \) is bounded, we get

\[
\int_0^\tau \|w_x(\tau)\|^2 \, d\tau \leq (1 + C) \int_0^\tau \|w_x(\tau)\|^2 \, d\tau < \infty,
\]

(15)

where \( C \) is the bound on \( \|w\|^2 \). From (14) and (15) we get that \( \|w_x\|^2 \) is bounded. Combining Agmon and Poincare inequalities, we get \( \max_{x \in [0, 1]} \|w(x)\|^2 \leq 4 \|w_x\|^2 \), thus \( w(x, t) \) is bounded for all \( x \) and \( t \).

Next, we prove regulation of \( w(x, t) \) to zero. Using (6) and (7), it is easy to show that

\[
\frac{1}{2} d/dt \|w_x\|^2 \leq \|w_x\|^2 + \left[ \|\hat{\lambda}\| + \frac{\gamma}{4\sqrt{3}} \right] \|w\|^2.
\]

(16)

Since \( \|w\| \) and \( \|w_x\| \) have been proven bounded, it follows that \( (d/dt)\|w_x\|^2 \) is bounded, and thus \( \|w(t)\| \) is uniformly continuous. From (15) and Poincare inequality we get that \( \|w\|^2 \) is integrable in time over the infinite time interval. By Barbalat’s lemma it follows that \( \|w\| \to 0 \) as \( t \to \infty \). The regulation in the maximum norm follows from Agmon inequality.

Having proved the boundedness and regulation of \( w \), we now set out to establish the same for \( u \). We start by noting that the inverse transformation to (4) is (Smyshlyaev & Krstic, 2004):

\[
u(x) = \hat{\lambda}(x) \int_0^x \hat{\lambda}(x, \xi) w(\xi) \, d\xi,
\]

(17)
\[
\dot{l}(x, \xi) = -\hat{\lambda} \xi \left( \sqrt{\hat{\lambda}(x^2 - \xi^2)} \right).
\]  
(18)

Since \( \hat{\lambda} \) is bounded, the function \( \dot{l}(x, \xi) \) has bounds
\[
L_1 = \max_{0 \leq \xi \leq 1} \dot{l}(x, \xi)^2, \quad L_2 = \max_{0 \leq \xi \leq 1} \dot{l}(x, \xi)^2.
\]  
(19)

It is straightforward to show that
\[
|u_e|^2 \leq 2(1 + \hat{\lambda}^2 + 4L_2)|w_e|^2.
\]  
(20)

Noting that \( u(x, t)^2 \leq 4|u_e|^2 \) for all \( (x, t) \in [0, 1] \times [0, \infty) \) and using the fact that \( |w_e| \) is bounded, we get uniform boundedness of \( u \). To prove regulation of \( u \), we estimate from (17)
\[
|u|^2 \leq 2(1 + L_1)|w|^2.
\]  
(21)

Since \( |w| \) is regulated to zero, so is \( |u| \). By Agmon’s inequality \( u(x, t)^2 \leq 2|u||w| \), where \( |u| \) is bounded. Therefore \( u(x, t) \) is regulated to zero for all \( x \in [0, 1] \).

The Lyapunov design incorporates all the states of the closed loop system into a single Lyapunov function and therefore Lyapunov adaptive controllers possess the best transient performance. However, this method is not applicable as broadly as the certainty equivalence approaches, which we consider next.

3. Certainty equivalence design with passive identifier

Consider the plant
\[
u_t = u_{xx} + \lambda u,
\]  
(22)

\[u(0) = 0,
\]  
(23)

where a constant parameter \( \lambda \) is unknown. We use a Dirichlet controller designed in (Smyshlyaev and Krstic (2004)):

\[
u(1) = -\hat{\lambda} \int_0^1 \frac{I_1}{\sqrt{\hat{\lambda}(1 - \xi^2)}} u(\xi) \, d\xi,
\]  
(24)

Following the certainty equivalence principle, first we need to design an identifier which will provide the estimate \( \hat{\lambda} \).

3.1. Identifier

Consider the following auxiliary system:

\[
\dot{u}_t = \dot{u}_{xx} + \hat{\lambda} u + \gamma^2 (u - \bar{u}) \int_0^1 u^2(x) \, dx,
\]  
(25)

\[\dot{u}(0) = 0,
\]  
(26)

\[\dot{u}(1) = u(1).
\]  
(27)

Such an auxiliary system is often called an “observer”, even though it is not used here for state estimation (the entire state \( u \) is available for measurement in our problem). The purpose of this “observer” is to help identify the unknown parameter. This identifier employs a copy of the PDE plant and an additional nonlinear term. We will refer to the system (25)–(27) as a “passive identifier”. The term “passive identifier” comes from the fact that an operator from the parameter estimation error \( \dot{\lambda} = \lambda - \hat{\lambda} \) to the inner product of \( u \) with \( u - \bar{u} \) is strictly passive. The additional term in (25) acts as nonlinear damping whose task is to slow down the adaptation.

Let us introduce the error signal \( e = u - \bar{u} \). Using (22)–(23) and (25)–(27), we obtain
\[
e_t = e_{xx} + \hat{\lambda} u - \gamma^2 e^2\|u\|^2,
\]  
(28)

\[e(0) = e(1) = 0.
\]  
(29)

Consider a Lyapunov function
\[
V = \frac{1}{2} \int_0^1 e^2(x) \, dx + \frac{\hat{\lambda}^2}{2\gamma}.
\]  
(30)

The time derivative of \( V \) along the solutions of (28)–(29) is

\[
\dot{V} = -\|e_t\|^2 - \gamma^2 \|e\|^2\|u\|^2 + \hat{\lambda} \int_0^1 e(x)u(x) \, dx - \frac{\hat{\lambda}^2}{\gamma}.
\]  
(31)

With the update law
\[
\hat{\lambda} = \gamma \int_0^1 (u(x) - \bar{u}(x))u(x) \, dx,
\]  
(32)

the last two terms in (31) cancel out and we obtain
\[
\dot{V} = -\|e_t\|^2 - \gamma^2 \|e\|^2\|u\|^2,
\]  
(33)

which implies \( V(t) \leq V(0) \). By the definition of \( V \), this means that \( \hat{\lambda} \) and \( \|e\| \) are bounded functions of time.

Integrating (33) with respect to time from zero to infinity we get that the spatial norms \( \|e_t\| \) and \( \|e\|\|u\| \) are square integrable over infinite time (belong to \( L_2 \)). From the update law (32) we get \( \|\hat{\lambda}\| \leq \gamma\|e\|\|u\| \) which shows that \( \hat{\lambda} \) is also square integrable in time.

Lemma 2. The identifier (25)–(27) with update law (32) guarantees the following properties:

\[
\|e_t\|, \quad \|e\|\|u\|, \quad \|e\|, \quad \hat{\lambda} \in L_2, \quad \|e\|, \quad \hat{\lambda} \in L_{\infty}.
\]  
(34)

3.2. Main result

Theorem 3. Suppose that a closed loop system that consists of (22)–(24), identifier (25)–(27), and update law (32), has a classical solution \( (\hat{\lambda}, u, \bar{u}) \). Then for any \( \hat{\lambda}(0) \) and any initial conditions \( u_0, \bar{u}_0 \in H^1(0, 1) \), the signals \( \hat{\lambda}, u, \bar{u} \) are bounded and \( u \) is regulated to zero for all \( x \in [0, 1] \):

\[
\lim_{t \to \infty} \max_{x \in [0, 1]} |u(x, t)| = 0.
\]  
(35)
Proof. One can show that the transformation

\[
\hat{w}(x) = \hat{u}(x) - \int_0^x \hat{k}(x, y) \hat{u}(y) \, dy,
\]

with \( \hat{k} \) given by (5), maps (25)–(27) into the following “target system”

\[
\hat{w}_t = \hat{w}_{xx} + \lambda \int_0^x \frac{\xi}{2} \hat{w}(\xi) \, d\xi + (\lambda + \gamma^2 ||u||^2) e_1,
\]

\[
\hat{w}(0) = \hat{w}(1) = 0,
\]

where \( e_1 \) is the transformed estimation error

\[
e_1(x) = e(x) - \int_0^x \hat{k}(x, y)e(y) \, dy.
\]

We observe that in comparison to non-adaptive target system (plain heat equation) two additional terms appeared in (37), one is proportional to \( \lambda \) and the other to the estimation error \( e \). The identifier guarantees that both of these terms are square integrable in time.

Since \( \lambda \) is bounded, and the functions \( \hat{k}(x, y) \) and \( \hat{l}(x, y) \) are twice continuously differentiable with respect to \( x \) and \( y \), there exist constants \( M_1, M_2, M_3 \) such that

\[
||e_1|| \leq M_1 ||e||,
\]

\[
||u|| \leq ||\hat{u}|| + ||e|| \leq M_2 ||\hat{w}|| + ||e||,
\]

\[
||u_\lambda|| \leq ||\hat{u}_\lambda|| + ||e_\lambda|| \leq M_3 ||\hat{w}_\lambda|| + ||e_\lambda||.
\]

Before we proceed, we need the following lemma.

Lemma 4. (Krstic, Kanellakopoulos, & Kokotovic, 1995) Let \( v, l_1, \) and \( l_2 \) be real-valued functions of time defined on \( [0, \infty) \), and let \( c \) be a positive constant. If \( l_1 \) and \( l_2 \) are nonnegative and integrable on \( [0, \infty) \) and satisfy the differential inequality

\[
\dot{v} \leq -cv + l_1(t)v + l_2(t), \quad v(0) \geq 0,
\]

then \( v \) is bounded and integrable on \( [0, \infty) \).

Using Young, Cauchy-Schwartz, and Poincare inequalities along with the identifier properties (34) and (40)–(42) one can obtain the following estimate:

\[
\frac{1}{2} \frac{d}{dt} ||\hat{w}||^2 \leq -\frac{1}{16} ||\hat{w}||^2 + l_1 \hat{w} ||\hat{w}||^2 + l_2,
\]

where \( l_1, l_2 \) are some integrable functions of time on \( [0, \infty) \). Using Lemma 4 we get the boundedness and square integrability of \( ||\hat{w}|| \). From (41) and (34) we get boundedness and square integrability of \( ||u|| \) and \( ||\hat{u}|| \), and (32) then gives boundedness of \( \lambda \).

In order to get pointwise in \( x \) boundedness, one estimates

\[
\frac{1}{2} \frac{d}{dt} \int_0^1 \hat{w}_x^2 \, dx \leq -\frac{1}{8} ||\hat{w}_x||^2 + \frac{\lambda_0^2 ||\hat{w}||^2}{4} + (\lambda_0 + \gamma^2 ||u||^2)^2 M_1 ||e||^2,
\]

(45)

where \( \lambda_0 \) is some integrable functions of time on \( [0, \infty) \). Using Lemma 4 we get boundedness and square integrability of \( ||\hat{w}_x|| \) and \( ||\hat{w}|| \). From (34) and (40) we get boundedness and square integrability of \( ||u_x|| \) and \( ||\hat{u}_x|| \), and (32) then gives boundedness of \( \lambda \).

Since the right hand sides of (45) and (46) are integrable, using Lemma 4 we get boundedness and square integrability of \( ||\hat{w}|| \) and \( ||e|| \). Using the inverse transformation

\[
\hat{u}(x) = \hat{w}(x) + \int_0^x \hat{l}(x, y) \hat{w}(y) \, dy,
\]

with \( \hat{l} \) given by (18), we get boundedness and square integrability of \( ||u_\lambda|| \) and (42) then gives the same properties for \( ||u_x|| \). By Agmon inequality, we get the boundedness of \( \hat{u} \) and \( u \) for all \( x \in [0, 1] \).

To show the regulation of \( u \) to zero, note that

\[
\frac{1}{2} \frac{d}{dt} ||e||^2 \leq -||e||^2 + ||\hat{u}|| ||e|| ||u|| \leq ||e||^2 + ||u||^2 ||u|| \leq 2 ||e||^2 + ||u||^2 ||u|| < \infty.
\]

The boundedness of \( (d/dt)||w||^2 \) follows from (44). By Barbalat’s lemma, we get \( ||\hat{w}|| \to 0, ||\hat{w}|| \to 0 \) as \( t \to \infty \). It follows from (47) that \( ||\hat{u}|| \to 0 \) and therefore (41) gives \( ||u|| \to 0 \). Using Agmon inequality and the fact that \( ||u_x|| \) is bounded, we get \( u(x, t) \to 0 \) for all \( x \in [0, 1] \). The proof is completed.

4. Certainty equivalence design with swapping identifier

The certainty equivalence design with swapping identifier is perhaps the most common method of parameter estimation in adaptive control. Filters of the “regressor” and of the measured part of the plant are implemented to convert a dynamic parameterization of the problem (given by the plant’s dynamic model) into a static parametrization where standard gradient and least squares estimation techniques can be used.

Consider the plant

\[
u_t = u_{xx} + \lambda u, \quad 0 < x < 1,
\]

(49)

\[
u(0) = 0,
\]

(50)

with unknown constant parameter \( \lambda \). We start by employing two filters: the state filter

\[
\nu_t = \nu_{xx} + u,
\]

(51)

\[
\nu(0) = \nu(1) = 0,
\]

(52)

and the input filter

\[
\eta_t = \eta_{xx},
\]

(53)

\[
\eta(0) = 0,
\]

(54)

\[
\eta(1) = u(1).
\]

(55)

The “estimation” error

\[
e = u - \lambda \nu - \eta,
\]

(56)

is then exponentially stable:

\[
e_t = e_{xx},
\]

(57)

\[
e(0) = e(1) = 0.
\]

(58)
Using the static relationship (56) as a parametric model, we implement a “prediction error” as
\[\hat{e} = u - \hat{\lambda}v - \eta, \quad \hat{e} = e + \hat{\lambda}v.\] 
(59)

We choose the gradient update law with normalization
\[\dot{\hat{\lambda}} = \gamma \frac{\int_0^1 \hat{e}(x)v(x) \, dx}{1 + \|v\|^2}.
\] 
(60)

With a Lyapunov function
\[V = \frac{1}{2} \int_0^1 e^2 \, dx + \frac{1}{8\gamma} \lambda^2,
\] 
(61)
we get
\[\dot{V} \leq -\int_0^1 \hat{e}_x^2 \, dx - \frac{\int_0^1 \hat{e}^2(x) \, dx}{4(1 + \|v\|^2)} + \frac{\int_0^1 \hat{e}(x)e(x) \, dx}{4(1 + \|v\|^2)}
\leq -\frac{\|e_x\|^2}{4(1 + \|v\|^2)} + \frac{\|e\|\|e_x\|}{2\sqrt{1 + \|v\|^2}}
\leq -\frac{1}{2}\|e_x\|^2 - \frac{1}{8}\|\hat{e}\|^2/\sqrt{1 + \|v\|^2}.
\] 
(62)

This gives the following properties:
\[\|\hat{e}\|/\sqrt{1 + \|v\|^2} \in L_2 \cap L_\infty,\] 
(63)
\[\hat{\lambda} \in L_\infty, \quad \hat{\lambda} \in L_2 \cap L_\infty.\] 
(64)

In contrast with the passive identifier, the normalization in the swapping identifier is employed in the update law. This makes \(\hat{\lambda}\) not only square integrable but also bounded.

We use the controller (24) with the state \(u\) replaced by its estimate \(\hat{\lambda}v + \eta\):
\[u(1) = -\hat{\lambda} \int_0^1 I_1 \left( \frac{\sqrt{\hat{\lambda}(1 - \xi^2)}}{\sqrt{\hat{\lambda}(1 - \xi^2)}} \right) \left( \hat{\lambda}v(\xi) + \eta(\xi) \right) \, d\xi.
\] 
(65)

**Theorem 5.** Suppose that a closed loop system that consists of the plant (49)–(50), the controller (65), the filters (51)–(55), and the update law (60), has a classical solution (\(\hat{\lambda}, u, v, \eta\)). Then for any \(\hat{\lambda}(0)\) and any initial conditions \(u_0, v_0, \eta_0 \in H^1(0, 1)\), the signals \(\hat{\lambda}, u, v, \eta\) are bounded and \(u\) is regulated to zero for all \(x \in [0, 1]\):  
\[\lim_{t \to \infty} \max_{x \in [0, 1]} |u(x, t)| = 0.
\] 
(66)

**Proof.** (Sketch) Consider the transformation
\[\hat{\nu}(x) = \hat{\lambda}v(x) + \eta(x) - \int_0^x \hat{\nu}(x, \xi) \dot{\hat{\lambda}}v(\xi) + \eta(\xi) \, d\xi,
\] 
(67)

with the same \(\hat{\nu}(x)\) as in (36). Using (51)–(55) and the inverse transformation
\[\hat{\nu}(x) + \eta(x) = \hat{\nu}(x) + \int_0^x \hat{\nu}(x, \xi) \dot{\hat{\lambda}}v(\xi) \, d\xi,
\] 
(68)

one can get the following PDE for \(\hat{\nu}(x)\):
\[\hat{\nu}(x, \xi) = -\hat{\nu}(x, \xi) - \int_0^\xi \hat{\nu}(x, \xi) \dot{\hat{\nu}}(\xi) \, d\xi
\] 
(69)
\[\hat{\nu}(0) = \hat{\nu}(1) = 0.
\] 
(70)

In order to prove boundedness of all signals we rewrite the filter (51)–(52) as follows:
\[v_t = v_{xx} + \hat{\nu}v(x) + \hat{\nu}_x v(x) + \hat{\nu}_{xx} v(x) + \hat{\nu}_{xx} v(x)
\] 
(71)
\[v(0) = v(1) = 0.
\] 
(72)

We have now two interconnected systems for \(v\) and \(\hat{\nu}\), (70)–(73), which are driven by the signals \(\hat{\lambda}, \hat{\lambda}, \hat{\nu}\) with properties (64). Note that the situation here is more complicated than in the passive design where we had to analyze only the \(\hat{\nu}\)-system (37) and (38). While the signal \(v\) in (70) and (71) is multiplied by a square integrable signal \(\hat{\lambda}\), the signal \(\hat{\nu}\) in the \(v\)-system (72) and (73) is multiplied by a bounded but possibly large gain \(\hat{I}\). To prove the boundedness of \(||\hat{\nu}||\) and \(||v||\) we use a weighted Lyapunov function
\[W = A||\hat{\nu}||^2 + ||v||^2,
\] 
(73)
where \(A\) is a large enough constant. One can then show that
\[\dot{W} \leq -\frac{1}{4A} W + I_1 W,
\] 
(74)
and with the help of Lemma 4 we get the boundedness of \(||\hat{\nu}||\) and \(||v||\). Using this result it can be shown that
\[\frac{d}{dt} \left( ||\hat{\nu}||^2 + ||v_x||^2 \right) \leq -||\hat{\nu}||^2 - ||v_x||^2 + I_1,
\] 
(75)
which proves that \(||\hat{\nu}||\) and \(||v_x||\) are bounded. From Agmon’s inequality we get that \(\hat{\nu}\) and \(v\) are bounded pointwise in \(x\). By Barbalat’s lemma we get \(||\hat{\nu}|| \to 0\) and \(||v_x|| \to 0\) as \(t \to \infty\). From (68) and (56) we get the pointwise boundedness of \(\eta\) and \(u\) and \(||u|| \to 0\). Finally, the pointwise regulation of \(u\) to zero follows from Agmon’s inequality. \(\square\)

The swapping method uses the highest order of dynamics of all identifier approaches. Lyapunov design has the lowest dynamic order as it only incorporates the dynamics of the parameter update, and the passivity-based method is better than the swapping method because it uses only one filter, as opposed
5. Extension to reaction–advection–diffusion systems in higher dimensions

All the approaches presented in Sections 2–4 can be readily extended to reaction–advection–diffusion plants and higher dimensions (2D and 3D). As an illustration, consider a 2D plant with four unknown parameters $\epsilon$, $b_1$, $b_2$, and $\lambda$:

$$u_t = \epsilon(u_{xx} + u_{yy}) + b_1 u_x + b_2 u_y + \lambda u, \quad (76)$$

on the rectangle $0 \leq x \leq 1$, $0 \leq y \leq L$ with actuation applied on the side with $x = 1$ and Dirichlet boundary conditions on the other three sides. We choose to design the scheme with passive identifier. We introduce the following “observer”

$$\hat{u}_t = \hat{\epsilon}(u_{xx} + u_{yy}) + \hat{b}_1 \hat{u}_x + \hat{b}_2 \hat{u}_y + \hat{\lambda} u + \gamma^2 \langle u - \hat{u} \rangle ||\nabla u||^2, \quad (77)$$

$$\hat{\epsilon} = 0, \quad (x, y) \in \{(0, 1) \times [0, 1]\} \setminus \{x = 1\}, \quad (78)$$

$$\hat{u} = u, \quad x = 1, 0 \leq y \leq 1. \quad (79)$$

There are two main differences compared to 1D case with one parameter considered in Section 3. First, since the unknown diffusion coefficient $\epsilon$ is positive, we must use projection to ensure $\hat{\epsilon} > \epsilon > 0$:

$$\text{Proj}_\epsilon\{\tau\} = \begin{cases} 0, & \hat{\epsilon} = \epsilon \quad \text{and} \quad \tau < 0 \\ \tau, & \text{else}. \text{ (80)} \end{cases}$$

Although this operator is discontinuous, it can be easily modified to avoid dealing with Filippov solutions and noise due to frequent switching of the update law, see (Krstic and Smyshlyaev 2008) for more details. However, we use (80) here for notational clarity. Note that $\hat{\epsilon}$ does not require the projection from above and all other parameters do not require projection at all.

Second, we can see in (77) that while the diffusion and advection coefficients multiply the operators of $\hat{u}$, the reaction coefficient multiplies $u$ in the observer. This is necessary in order to eliminate any $\lambda$-dependence in the error system so that it is stable.

The update laws are

$$\hat{\epsilon} = -\gamma_0 \text{Proj}_\epsilon\{0\} \int_0^1 \int_0^1 u_x (u_x - \hat{u}_x) + u_y (u_y - \hat{u}_y) \, dx \, dy, \text{ (81)}$$

$$\hat{b}_1 = \gamma_1 \int_0^1 \int_0^1 (u - \hat{u}) u_x \, dx \, dy, \quad (82)$$

$$\hat{b}_2 = \gamma_1 \int_0^1 \int_0^1 (u - \hat{u}) u_y \, dx \, dy, \quad (83)$$

$$\dot{\lambda} = \gamma_2 \int_0^1 \int_0^1 (u - \hat{u}) u \, dx \, dy, \quad (84)$$

and the controller is

$$u(1, y) = -\int_0^1 \frac{\hat{b}_1 (1 - \xi)}{2\hat{\epsilon}} \left( \frac{\hat{b}_1 (1 - \xi)}{2\hat{\epsilon}} \right) \, dx. \quad (85)$$

The results of the simulation of the above scheme are presented in Figs. 1 and 2. The true parameters are set to $\epsilon = 1$, $b_1 = 1$, $b_2 = 2$, $\lambda = 22$, $L = 2$. With this choice the open-loop plant has two unstable eigenvalues at 8.4 and 1. All estimates come close to the true values at approximately $t = 0.5$ and after that the controller stabilizes the system.

6. Plants with spatially varying uncertainties

The designs presented in Sections 2–4 can be extended to plants with spatially varying unknown parameters. For example, for the plant

$$u_t = u_{xx} + \lambda(x) u, \quad (86)$$

$$u_x(0) = 0, \quad (87)$$

the Lyapunov adaptive controller would be

$$u(1) = \tilde{\lambda}(1, 1) u(1) + \int_0^1 \tilde{\lambda}(1, \xi) u(\xi) \, d\xi, \quad (88)$$

with

$$\dot{\tilde{\lambda}}(t, x) = \gamma \frac{u(t, x)(w(t, x) - \int_0^1 \tilde{\lambda}(\xi, x) w(t, \xi) \, d\xi)}{1 + ||w(t)||^2},$$

where $\tilde{\lambda}(t, x)$ is the online functional estimate of $\lambda(x)$, $w(t, x) = u(t, x) - \int_0^1 \tilde{\lambda}(\xi, x) u(\xi) \, d\xi$, and the kernel $\tilde{\lambda}(x, \xi) = \tilde{\lambda}_\alpha(x, \xi)$ is obtained recursively from

$$\tilde{\lambda}_0 = -\frac{1}{2} \int_{(x-\xi/2)}^{(x+\xi/2)} \tilde{\lambda}(\xi) \, d\xi; \quad (89)$$

$$\tilde{\lambda}_{i+1} = \tilde{\lambda}_i + \int_{(x-\xi/2)}^{(x+\xi/2)} \int_{(x-\xi/2)}^{(x+\xi/2)} \tilde{\lambda}(\xi - \sigma) \tilde{\lambda}(\sigma, \xi - \sigma) \times d\sigma \, d\xi, \quad i = 0, 1, \ldots, n$$

for each new update of $\tilde{\lambda}(t, x)$. Stability is guaranteed for sufficiently small $\gamma$ and sufficiently high $n$. The recursion (89) was proved convergent in (Smyshlyaev and Krstic 2004). The certainty equivalence designs with passive and swapping identifiers can also be extended to the case of functional unknown parameters using the same recursive procedure. For further details, the reader is referred to (Smyshlyaev and Krstic 2006a).
7. Output-feedback design

Consider the plant

\[ u_t = u_{xx} + \lambda(x)u, \quad 0 < x < 1, \quad (90) \]

\[ u_x(0, t) = 0, \quad (91) \]

\[ u(1, t) = U(t). \quad (92) \]

where \( \lambda(x) \) is an unknown continuous function and only the boundary value \( u(0, t) \) is measured.

The key step in our design is the transformation of the original plant (90)–(92) into a system in which unknown parameters multiply the measured output.
7.1. Transformation to observer canonical form

One can show that the transformation

\[ v(x) = u(x) - \int_0^x p(x,y)u(y) \, dy, \tag{93} \]

where \( p(x,y) \) is a solution of the PDE

\[ p_{xx}(x,y) - p_{yy}(x,y) = \lambda(y)p(x,y), \tag{94} \]

\[ p(1,y) = 0, \tag{95} \]

\[ p(x,x) = \frac{1}{2} \int_x^1 \lambda(s) \, ds, \tag{96} \]

maps the system (90)–(92) into

\[ v_t = v_{xx} + \theta(x)v(0), \tag{97} \]

\[ v_x(0) = \theta_1 v(0), \tag{98} \]

\[ v(1) = u(1), \tag{99} \]

where

\[ \theta(x) = -p_x(x,0), \quad \theta_1 = -p(0,0), \tag{100} \]

are the new unknown functional parameters.

The system (97)–(99) is the PDE analog of observer canonical form. Note from (93) that \( v(0) = u(0) \) and therefore \( v(0) \) is measured. The transformation (93) is invertible so that stability of \( v \) implies stability of \( u \). Therefore it is enough to design the stabilizing controller for the \( v \)-system and then use the condition \( u(1) = v(1) \) (which follows from (95)) to obtain the controller for the original system. We are going to directly estimate the new unknown parameters \( \theta(x) \) and \( \theta_1 \) instead of estimating \( \lambda(x) \). Thus, we do not need to solve the PDE (94)–(96) for the control scheme implementation.

7.2. Estimator

The unknown parameters \( \theta \) and \( \theta(x) \) enter the boundary condition and the domain of the \( v \)-system. Therefore we will need the following output filters:

\[ \phi_t = \phi_{xx}, \tag{101} \]

\[ \phi_x(0) = u(0), \tag{102} \]

\[ \phi(1) = 0, \tag{103} \]

and

\[ \phi_1 = \phi_{xx} + \delta(x - \xi)u(0), \tag{104} \]

\[ \phi_x(0) = \phi(1) = 0. \tag{105} \]

Here the filter \( \Phi = \Phi(x, \xi) \) is parametrized by \( \xi \in [0,1] \) and \( \delta(x - \xi) \) is a delta function. The reason for this parametrization is the presence of the functional parameter \( \theta(x) \) in the domain. Therefore, loosely speaking, we need an infinite “array” of filters, one for each \( x \in [0,1] \) (since the swapping design normally requires one filter per unknown parameter). We also introduce the input filter

\[ \psi_t = \psi_{xx}, \tag{106} \]

\[ \psi_x(0) = 0, \tag{107} \]

\[ \psi(1) = u(1). \tag{108} \]

It is straightforward to show now that the error

\[ \tilde{e}(x) = v(x) - \psi(x) - \theta_1 \phi(x) - \int_0^x \theta(\xi) \Phi(x, \xi) \, d\xi, \tag{109} \]

satisfies the exponentially stable PDE

\[ \tilde{e}_t = \tilde{e}_{xx}, \tag{110} \]

\[ \tilde{e}_x(0) = \tilde{e}(1) = 0. \tag{111} \]

Typically the swapping method requires one filter per unknown parameter and since we have functional parameters, infinitely many filters are needed. However, we reduce their number down to only two by representing the state \( \Phi(x, \xi) \) algebraically through \( \phi(x) \) at each instant \( t \).

**Lemma 6.** The signal

\[ e(x) = v(x) - \psi(x) - \theta_1 \phi(x) - \int_0^x \theta(\xi)F(x, \xi) \, d\xi, \tag{112} \]

where \( F(x, \xi) \) is given by

\[ F_{xx}(x, \xi) = F_{\xi\xi}(x, \xi), \tag{113} \]

\[ F(0, \xi) = -\phi(\xi), \tag{114} \]

\[ F_1(0, \xi) = F_{\xi}(x, 0) = F(x, 1) = 0, \tag{115} \]

is governed by the exponentially stable heat equation:

\[ e_t = e_{xx}, \tag{116} \]

\[ e_x(0) = e(1) = 0. \tag{117} \]

**Proof.** The initial conditions for the filters \( \phi \) and \( \Phi \) are the design choice so let us assume that they are continuous functions in \( x \) and \( \xi \). We now write down the explicit solutions to the filters:

\[ \phi(x,t) = 2 \sum_{n=0}^{\infty} \cos(\sigma_n x)e^{-\sigma_n^2 t} \int_0^1 \phi_0(s) \cos(\sigma_n s) \, ds \]

\[ -2 \sum_{n=0}^{\infty} \cos(\sigma_n x) \int_0^t u(0, \tau)e^{-\sigma_n^2(1-\tau)} \, d\tau, \tag{118} \]

and

\[ \Phi(x, \xi, t) = 2 \sum_{n=0}^{\infty} \cos(\sigma_n x) \cos(\sigma_n \xi) \int_0^t \Phi_0(s, \xi) e^{-\sigma_n^2(1-t)} \, ds \]

\[ +2 \sum_{n=0}^{\infty} \cos(\sigma_n x) e^{-\sigma_n^2 t} \int_0^1 \Phi_0(s, \xi) \cos(\sigma_n s) \, ds, \tag{119} \]
where $\sigma_n = \pi(n + 1/2)$. Multiplying (118) with $\cos(\sigma_n x)$ and using the orthogonality of these functions on $[0, 1]$ we can rewrite the $\Phi$-filter in the form

$$
\Phi(x, \xi, t) = -2\sum_{n=0}^{\infty} \cos(\sigma_n x) \cos(\sigma_n \xi) \times \int_0^1 \cos(\sigma_n s) \phi(s, t) \, ds \times 2\sum_{n=0}^{\infty} \cos(\sigma_n x) e^{-\sigma_n^2 t} \int_0^1 \cos(\sigma_n s) \times \phi_0(s) \cos(\sigma_n \xi) + \Phi_0(s, \xi)) \, ds.
$$

(120)

Here the first term represents the explicit solution of the system (113)–(115) and the second term is the effect of filters’ initial conditions. Therefore we can represent $\Phi$ as

$$
\Phi(x, \xi, t) = F(x, \xi, t) + \Delta F(x, \xi, t),
$$

(121)

where $\Delta F$ satisfies

$$
\Delta F_x = \Delta F_{xx},
$$

(122)

$$
\Delta F_x(0, \xi, t) = \Delta F(1, \xi, t) = 0,
$$

(123)

and using (110) and (111) we get (116) and (117).

Lemma 6 allows us to avoid the need to solve an infinite “array” of parabolic Eqs. (104) and (105) by computing the solution of the standard wave Eqs. (113)–(115) at each time step. Therefore we only have two dynamic equations to solve. □

7.3. Update laws

We take the following equation as a parametric model:

$$
e(0) = v(0) - \psi(0) - \theta_1 \phi(0) + \int_0^1 \theta(\xi) \phi(\xi) \, d\xi.
$$

(124)

The estimation error is

$$
\hat{e}(0) = v(0) - \psi(0) - \hat{\theta}_1 \phi(0) + \int_0^1 \hat{\theta}(\xi) \phi(\xi) \, d\xi.
$$

(125)

We employ the gradient update laws with normalization

$$
\hat{\theta}(x, t) = -\gamma \frac{\hat{e}(0) \phi(x)}{1 + ||\phi||^2 + \phi^2(0)}
$$

(126)

$$
\hat{\gamma}_1 = \gamma_1 \frac{\hat{e}(0) \phi(0)}{1 + ||\phi||^2 + \phi^2(0)},
$$

(127)

where $\gamma(x)$ and $\gamma_1$ are positive adaptation gains.

Lemma 7. The adaptive laws (126) and (127) guarantee the following properties:

$$
\hat{e}(0) \in \mathcal{L}_2 \cap \mathcal{L}_\infty,
$$

(128)

$$
||\hat{\theta}||, \quad \hat{\theta}_1 \in \mathcal{L}_\infty, \quad ||\hat{\theta}_1||, \quad \hat{\gamma}_1 \in \mathcal{L}_2 \cap \mathcal{L}_\infty.
$$

(129)

Proof. Using a Lyapunov function

$$
\dot{V} = \frac{1}{2} ||e||^2 + \frac{1}{2\gamma_1} \dot{\hat{\theta}}_1^2 + \int_0^1 \frac{\dot{\theta}^2(x)}{2\gamma(x)} \, dx,
$$

(130)

we get

$$
\dot{V} = -\int_0^1 e^2 \, dx + \int_0^1 \frac{\hat{\theta}(x) \phi(x) \, dx - \hat{\theta}_1 \phi(0) \dot{\theta}(0)}{1 + ||\phi||^2 + \phi^2(0)} \dot{\theta}(0)
\leq -||e||^2 + \frac{e^2(0) - \hat{e}^2(0)}{1 + ||\phi||^2 + \phi^2(0)} - \frac{\dot{\theta}^2(0)}{1 + ||\phi||^2 + \phi^2(0)}
\leq -\frac{1}{2} ||e||^2 - \frac{1}{2} \dot{\theta}^2(0),
$$

(131)

This gives

$$
\frac{\dot{\theta}(0)}{\sqrt{1 + ||\phi||^2 + \phi^2(0)}} \in \mathcal{L}_2, \quad ||\hat{\theta}||, \quad \hat{\theta}_1 \in \mathcal{L}_\infty.
$$

(132)

The rest of the properties (128) and (129) follows from the relation $\dot{e}(0) = e(0) + \dot{\theta}_1 \phi(0) - \int_0^1 \hat{\theta}(x) \phi(x) \, dx$ and the update laws. □

7.4. Main result

Theorem 8. Consider the system (90)–(92) with the controller

$$
u(1) = \int_0^1 \left( \psi(y) + \hat{\theta}_1 \phi(y) + \int_0^1 F(y, \xi) \hat{\theta}(\xi) \, d\xi \right) \times \hat{k}(1, y) \, dy,
$$

(133)

where $\hat{k}(x, y) = \hat{k}(x - y)$ with $\hat{k}(x)$ determined from the equation

$$
\kappa'(x) = -\hat{\theta}_1 \kappa(x) - \hat{\theta}(x) + \int_0^x \kappa(x - y) \hat{\theta}(y) \, dy,
$$

(134)

$$
\kappa(0) = \hat{\theta}_1,
$$

(135)

the filters $\phi$ and $\psi$ are given by (101)–(103), (106)–(108) and the update laws for $\hat{\theta}(x)$ and $\hat{\theta}_1$ are given by (126) and (127). If the closed loop system has a solution $(u, \phi, \psi, \hat{\theta}, \hat{\theta}_1)$ with $u, \phi, \psi \in H_1(0, 1)$ then for any $\hat{\theta}(x, 0), \hat{\theta}_1(0)$ and any initial conditions $u_0, \phi_0, \psi_0 \in H_1(0, 1)$ the signals $||\hat{\theta}||, ||\hat{\theta}_1||, ||u||, ||\phi||, ||\psi||$ are bounded and $||u||$ is regulated to zero:

$$
\lim_{t \to \infty} ||u|| = 0.
$$

(136)
Proof. (Sketch).

Denote

\[ h(x) = \psi(x) + \dot{\varphi}_1 \varphi(x) + \int_{0}^{1} F(x, \xi) \dot{\varphi}(\xi) \, d\xi, \]  

(137)

and use the following backstepping transformation

\[ w(x) = h(x) - \int_{0}^{x} \hat{k}(x, y) h(y) \, dy := T[h](x). \]  

(138)

One can show that the inverse transformation to (138) is

\[ h(x) = w(x) + \int_{0}^{x} \hat{l}(x, y) w(y) \, dy, \]  

(139)

where

\[ \hat{l}(x, y) = \hat{\varphi}_1 - \int_{y}^{x} \hat{\varphi}(\xi) \, d\xi. \]  

(140)

Using Lemma 6, the equations for the plant, filters \( \varphi \) and \( \psi \), and the Volterra relationship between \( \hat{l} \) and \( \hat{k} \)

\[ \hat{l}(x, y) = \hat{k}(x, y) + \int_{y}^{x} \hat{l}(x, \xi) \hat{k}(\xi, y) \, d\xi, \]  

(141)

one can derive the following target system

\[ w_t = w_{xx} + \hat{\varphi}(0) \hat{k}(x, 0) - \int_{0}^{x} w(y) \left( \hat{l}(x, y) - \int_{y}^{x} \hat{k}(x, \xi) \hat{l}(\xi, y) \, d\xi \right) \, dy \]  

+ \hat{\varphi}_1 T[\varphi] + T \left[ \int_{0}^{1} F(x, \xi) \dot{\varphi}_i(\xi) \, d\xi \right]. \]  

(142)

\[ w_x(0) = \hat{\varphi}_1 \hat{\varphi}(0), \]  

(143)

\[ w(1) = 0. \]  

(144)

Let us rewrite \( \varphi \) filter as

\[ \varphi_t = \varphi_{xx}, \]  

(145)

\[ \varphi(0) = w(0) + \hat{\varphi}(0), \]  

(146)

\[ \varphi(1) = 0. \]  

(147)

We now have interconnection of two systems \( \varphi \) and \( w \) with forcing terms that have properties (128) and (129).

Let us establish bounds on the gains \( \hat{k}(x, y) \) and \( \hat{l}(x, y) \). The boundedness of the parameter estimates \( \hat{\varphi}_1 \) and \( ||\hat{\varphi}|| \) has been shown in Lemma 7. From (140) we get

\[ ||\hat{l}(x, y)|| \leq \hat{\varphi}_1 + \hat{\varphi}, \]  

(148)

where we denote \( \hat{\varphi}_1 = \max_{\tau \geq 0} ||\hat{\varphi}_1|| \) and \( \hat{\varphi} = \max_{\tau \geq 0} ||\hat{\varphi}|| \).

Using (141) and Gronwall inequality it is easy to get the following bound:

\[ ||\hat{k}(x, y)|| \leq (\hat{\varphi}_1 + \hat{\varphi}) e^{\hat{\varphi}_1 + \hat{\varphi}} := K_1. \]  

(149)

If we look at the right-hand side of the \( w \)-system, we can see that the estimates for \( \hat{k}(x, 0) \) and \( \hat{l}(x, y) \) are also needed. They are readily obtained from (134) and (140):

\[ \hat{k}(x, 0) \leq (\hat{\varphi}_1 + \hat{\varphi}) K_1 + \hat{\varphi} := K_2, \]  

(150)

\[ ||\hat{l}(x, y)|| \leq ||\hat{\varphi}_1|| + ||\hat{\varphi}||. \]  

(151)

We are now ready to start with stability analysis of (142)–(147). Consider a Lyapunov function

\[ V_1 = \frac{1}{2} \int_{0}^{1} \phi^2 \, dx. \]  

(152)

Computing its derivative along the solutions of the \( \varphi \)-system and using Young, Poincare, and Agmon inequalities, we get

\[ V_1 = \frac{1}{2} \int_{0}^{1} \phi^2 \, dx \leq \frac{1}{2} \phi^2(0) + \frac{1}{2} \hat{\varphi}^2(0) - ||\varphi_x||^2 
\]  

+ \frac{1 + ||\varphi||^2 + \phi^2(0)}{2} \left[ 1 + ||\phi||^2 + 2||\phi|| ||\varphi|| \right] 
\]  

\[ \leq - \frac{1}{2} ||\varphi_x||^2 + \frac{1}{2} ||\varphi_x||^2 + c_1 ||\varphi_x||^2 
\]  

+ \frac{1}{4c_1} + ||\varphi||^2 + \phi^2(0) 
\]  

\[ \leq - \frac{3}{2} c_1 ||\varphi_x||^2 + \frac{1}{2} ||\varphi_x||^2 + l_1 ||\varphi||^2 + l_1. \]  

(153)

Here \( c_1 \) is a positive constant that will be chosen later and \( l_1 \) denotes a generic bounded and square integrable function of time.

With a Lyapunov function

\[ V_2 = \frac{1}{2} \int_{0}^{1} w^2 \, dx, \]  

we get

\[ V_2 = - \int_{0}^{1} \int_{0}^{1} w_x^2 \, dx + \hat{\varphi}(0) \int_{0}^{1} \hat{k}(x, 0) w(x) \, dx 
\]  

+ \int_{0}^{1} w(x) T \left[ \int_{0}^{1} F(x, \xi) \dot{\varphi}_i(\xi) \, d\xi \right] \, dx 
\]  

\[ + \hat{\varphi}_1 w(0) \hat{\varphi}(0) + \hat{\varphi}_1 \int_{0}^{1} w(x) T[\varphi](x) \, dx 
\]  

\[ + \int_{0}^{1} w(x) \int_{0}^{1} w(y) \right) \times (\hat{l}(x, y) - \int_{0}^{1} \hat{l}(x, \xi) \hat{l}(\xi, y) \, d\xi) \, dy \] 

\[ \leq c_1 ||\varphi_x||^2 - (1 - c_2) ||\varphi_x||^2 
\]  

+ l_1 ||w||^2 + l_1 ||\varphi||^2 + l_1. \]  

(154)

For \( V = V_1 + V_2 \) we get

\[ V \leq - \left( \frac{1}{2} - c_2 \right) ||\varphi_x||^2 - \left( \frac{1}{2} - 3c_1 - c_3 \right) ||\varphi_x||^2 
\]  

+ l_1 ||w||^2 + l_1 ||\varphi||^2 + l_1. \]  

(155)
Choosing $c_2 = 1/4$, $3c_1 = 1/16$, $c_3 = 3/16$, we get
\[
\dot{V} \leq -\frac{1}{8} V + l_1 V + l_1, \tag{156}
\]
and by Lemma 4 we get $\|w\|, \|\phi\| \in \mathcal{L}_2 \cap \mathcal{L}_\infty$. From the transformation (139) we get $\|\psi\| \in \mathcal{L}_2 \cap \mathcal{L}_\infty$ and therefore $\|\psi\| \in \mathcal{L}_2 \cap \mathcal{L}_\infty$ follows from (137). From (112) and (93) we get $\|v\|, \|u\| \in \mathcal{L}_2 \cap \mathcal{L}_\infty$. It is easy to see from (156) that $V$ is bounded from above. By using an alternative to Barbalat’s lemma (Liu and Krstic, 2001, Lemma 3.1) we get $V \to 0$, that is $\|\psi\| \to 0$, $\|\phi\| \to 0$. From the transformation (139) we get $\|h\| \to 0$ and from (137) $\|\psi\| \to 0$ follows. From (112) and (93) we get $\|v\| \to 0$ and $\|u\| \to 0$. □

7.5. Reaction–advection–diffusion systems

The approach presented in the paper can also be applied to general reaction–advection–diffusion system
\[
u_t = \varepsilon(x)u_{xx} + b(x)u_t + \lambda(x)u + g(x)u(0) + \int_0^x f(x, y)u(y) \, dy, \tag{157}
\]
\[u_x(0) = -qu(0), \tag{158}
\]
where $\varepsilon(x), b(x), \lambda(x), g(x), f(x, y), q$ are unknown parameters. The parameters $g(x), f(x, y)$, and $q$ can be easily handled because the observer canonical form (97)–(99) is not changed in this case, only the PDE (94)–(96) and expressions (100) for the new unknown parameters are modified. Since we are not concerned with identification, the adaptive scheme stays exactly the same.

With unknown parameters $b(x)$ and $\varepsilon(x)$, however, additional difficulties arise. The transformed plant is changed to
\[
u_t = \theta_0 \nu_{xx} + \theta_1 \nu(0), \tag{159}
\]
\[\nu_x(0) = \theta_1 \nu(0), \tag{160}
\]
\[\nu(1) = \theta_2 u(1), \tag{161}
\]
where the new constant parameters $\theta_0$ and $\theta_2$ appear due to $\varepsilon(x)$ and $b(x)$ respectively. We can see that one of the issues is the need of projection to keep the estimates of $\theta_0$ and $\theta_2$ positive since the filters should be stable and the controller is given as $u(1) = \hat{\theta}_2^{-1} \nu(1)$. This issue, although making the closed loop stability proof more challenging, does not pose a conceptual problem. The real difficulty comes from the fact that the parameter $\theta_0$, which comes from the unknown $\varepsilon(x)$, multiplies the second derivative of the state which is not measured. Therefore, while an unknown $b(x)$ is allowed, $\varepsilon(x)$ should be known.

7.6. Simulations

We now present the results of numerical simulations of the designed adaptive scheme. The parameters of the plant (157) and (158) are taken to be $b(x) = 3 - 2x^2$ and $\lambda(x) = 16 + 3\sin(2\pi x)$, $\varepsilon \equiv 1$, $g(x) = q = 0$, so that the plant is unstable. The evolution of the closed loop state is shown in Fig. 3. We can see that the regulation is achieved. The parameter estimates, shown in Figs. 4 and 5, converge to some stabilizing values.
8. Conclusion

We presented several approaches to adaptive control of parabolic PDEs. In future work, the extension to hyperbolic PDEs (strings, beams, and plates) will be considered. Another important problem is the identification of systems with spatially varying (functional) unknown parameters using only boundary sensing and actuation. Since no boundary input is capable of persistently exciting such a system, one possible approach would be to approximate the spatially varying parameter by a polynomial of a sufficiently high degree and identify the coefficients of this polynomial with any prescribed accuracy using sufficiently rich input.

The developments reported on in this paper represent the results obtained in an attempt to reproduce the adaptive backstepping control designs (Krstic et al., 1995) in an infinite-dimensional setting. Actually the result achieved for PDEs only partly mirrors the results for ODEs in (Krstic et al., 1995). We have been successful in developing the schemes in all of the three categories introduced in (Krstic et al., 1995): the Lyapunov schemes (Krstic et al., 1995, Chap. 4), the modular schemes with passive identifiers (Krstic et al., 1995, Chap. 5), and the modular schemes with swapping identifiers (Krstic et al., 1995, Chap. 5). We have also been successful in reproducing the output-feedback schemes with a Kreisselmeier-type observer and a swapping identifier as in (Krstic et al., 1995, Chap. 10, Section 10.6.3).

However, the nonlinear results of (Krstic et al., 1995) remain elusive. By this we don’t only mean the possible extension to nonlinear PDEs—this is a formidable problem. We mean also an extension of the nonlinear tools (for nonlinear and linear plants) such as the tuning function design in (Krstic et al., 1995, Chap. 4, Section 10.2) and nonlinear damping in (Krstic et al. 1995, Chaps. 5 and 6). Tuning functions and nonlinear damping are hard to extend to the infinite dimensional case because they cannot be iterated infinitely many times to a bounded limit. In the case of tuning functions the polynomial growth of nonlinearities dependent on the state produces feedback laws with infinite polynomial powers in the limit. In the case of nonlinear damping the situation is more delicate. For linear plants, nonlinear damping results in a polynomial growth of the feedback law in the parameter estimate. For infinite-dimensional plant this process results in an unbounded growth of the feedback law in the parameter estimate, which is a state of the compensator.

References


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