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Brief Paper

Modular Approach to Adaptive Nonlinear Stabilization*

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Key Words—Adaptive stabilization; nonlinear control; control Lyapunov functions; input-to-state stability, backstepping.

Abstract—A modular approach to adaptive control of nonlinear systems introduced previously is presented in a general framework. This approach abandons the certainty equivalence principle ubiquitous in the estimation-based approach. The adaptive stabilization problem is studied in the setting of control Lyapunov functions, and is reduced to the problem of input-to-state stabilization with respect to the parameter estimation error considered as the input.

1. *Introduction*

The estimation-based approach, which has been successful in adaptive linear control (Egardt, 1979; Goodwin and Mayne, 1987), has been applied to nonlinear systems with only limited success. The stability phenomena in nonlinear systems can be faster than the convergence of standard parameter identifiers. The state can escape to infinity before the identifier is able to provide stabilizing values of parameter estimates. For this reason, estimation-based designs for nonlinear systems (Sastry and Isidori, 1989; Teel *et al.*, 1991) either assumed that the nonlinearities were linearly bounded or their results remained local. Only Praly *et al.* (1991) went beyond linear growth conditions and characterized relationships between *nonlinear* growth constraints and controller stabilizing properties.

These difficulties indicated that either the controllers should be stronger than those based on the certainty equivalence principle or the identifiers should be faster. In Krstić and Kokotović (1995a) we introduced a modular design that finally removed the growth restrictions in the estimation-based approach. The modular design employs stronger controllers that guarantee boundedness whenever the parameter estimates and their derivatives are bounded.

In this paper we propose a general framework for modular design, and provide further insight into the result of Krstić and Kokotović (1995a). We reduce the adaptive stabilization problem to the probem of input-to-state (Sontag, 1989a) stabilization (ISS) with respect to the parameter estimation error considered as the input.

Our study is cast in the setting of control Lyapunov functions (Artstein, 1983; Sontag, 1989b). We employ the concept of an *ISS-control Lyapunov function* (*iss-clf*), which is a particular form of a *robust control Lyapunov function* (*rclf*) introduced by Freeman and Kokotović (1996). The iss-clf formalism is suitable for modular adaptive design, because it takes advantage of the fact that the system is affine both in the control input and in the disturbance input, and yields a simple Sontag-type formula for the control law.

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We consider the problem of global feedback stabilization of systems of the form

$$\dot{x} = f(x) + F(x)\theta + g(x)u, \quad x \in \mathbb{R}^n, \quad u \in \mathbb{R}, \tag{1}$$

where θ is a constant unknown parameter vector in $\mathbb{R}^p$, the mappings $f(x)$, $F(x)$ and $g(x)$ are smooth, and $f(0) = 0$, $F(0) = 0$.

We say that the system (1) is *globally adaptively stabilizable* if there exists a function $\alpha(x, \hat{\theta})$ continuous on $(\mathbb{R}^n\backslash\{0\}) \times \mathbb{R}^p$ with $\alpha(0, \hat{\theta}) \equiv 0$, continuous functions $\tau(x, \hat{\theta}, \eta)$ and $H(x, \hat{\theta}, \eta)$, and a positive-definite symmetric $p \times p$ matrix Γ such that the dynamic controller

$$u = \alpha(x, \hat{\theta}), \tag{2}$$

$$\dot{\hat{\theta}} = \Gamma\tau(x, \hat{\theta}, \eta), \tag{3}$$

$$\dot{\eta} = H(x, \hat{\theta}, \eta) \tag{4}$$

guarantees that the solution $(x(t), \hat{\theta}(t), \eta(t))$ is globally bounded, and $x(t) \to 0$ as $t \to \infty$, for all $\theta \in \mathbb{R}^p$.

We refer to (3) and (4) as an identifier, with (3) being its update law.

In Section 2 we introduce basic concepts for modular adaptive design, and show that a system is adaptively stabilizable provided it is input-to-state stabilizable with respect to the parameter estimation error. In Section 3 we show how to use backstepping to construct input-to-state stabilizing control laws, which recovers our earlier design (Krstić and Kokotović, 1995a).

2. *ISS-control Lyapunov functions*

We start by rewriting (1) with (2) as

$$\dot{x} = f(x) + F(x)\hat{\theta} + g(x)\alpha(x, \hat{\theta}) + F(x)\tilde{\theta}, \tag{5}$$

where $\tilde{\theta}(t) = \theta - \hat{\theta}(t)$. Suppose we know how to find a control law $\alpha(x, \hat{\theta})$ that stabilizes this system with $\tilde{\theta} = 0$. In the presence of the disturbance input $\tilde{\theta}$, this control law, in general, does not preserve stability, even if $\tilde{\theta}$ is bounded and exponentially decaying. To preserve stability, we need a stronger controller. Since the standard parameter estimators guarantee that $\tilde{\theta}$ is bounded, we are interested in desiging controllers that can guarantee input-to-state stability of (5) with respect to $\tilde{\theta}$ as input. However, the time-varying character of the parameter estimate $\hat{\theta}(t)$ forces us to consider $\dot{\hat{\theta}}(t)$ as another disturbance input, even though it is not explicitly present in (5). As we shall see, the identifiers will also guarantee that $\dot{\hat{\theta}}(t)$ is bounded.

Our goal is to find a control law $\alpha(x, \hat{\theta})$ continuous on $(\mathbb{R}^n\backslash\{0\}) \times \mathbb{R}^p$ with $\alpha(0, \hat{\theta}) \equiv 0$, such that the following *input-to-state stability* (ISS) property is satisfied:

$$|x(t)| \leq \beta(|x(0)|, t) + \gamma\left(\sup_{0\leq\tau\leq t}\left|\begin{bmatrix}\tilde{\theta}(\tau)\\ \dot{\hat{\theta}}(\tau)\end{bmatrix}\right|\right), \tag{6}$$

where β is a class $\mathcal{KL}$ function and γ is a class $\mathcal{K}$ function (Sontag, 1989a). Lin *et al.* (1994, Theorem 3) proved that a necessary and sufficient condition for (6) is the existence of

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an *ISS-Lyapunov function* (sufficiency was proved by Sontag, 1989a).† We say that a smooth function $V:\mathbb{R}^n \times \mathbb{R}^p \to \mathbb{R}_+$, positive definite and proper in x for each $\hat{\theta}$, is an ISS-Lyapunov function for (1) if there exists a class $\mathcal{K}_\infty$ function ρ such that the following implication holds for all $x \neq 0$, all bounded $\hat{\theta}$,‡ and all $\tilde{\theta}, \dot{\hat{\theta}} \in \mathbb{R}^p$:

$$|x| \geq \rho\left(\left|\begin{bmatrix}\tilde{\theta}\\ \dot{\hat{\theta}}\end{bmatrix}\right|\right) \quad \Downarrow \quad \frac{\partial V}{\partial x}[f(x) + F(x)\hat{\theta} + g(x)\alpha(x, \hat{\theta})] + \frac{\partial V}{\partial x}F(x)\tilde{\theta} + \frac{\partial V}{\partial \hat{\theta}}\dot{\hat{\theta}} < 0. \tag{7}$$

If there exists such a triple (α, V, ρ), we say that the system (1) is *input-to-state stabilizable with respect to* $(\tilde{\theta}, \dot{\hat{\theta}})$.

We adopt a control Lyapunov function setting for input-to-state stabilization.

Definition 2.1. A smooth function $V: \mathbb{R}^n \times \mathbb{R}^p \to \mathbb{R}_+$, positive-definite and proper in x for each $\hat{\theta}$, is called an *ISS-control Lyapunov function* (*iss-clf*) for (1) if there exists a class $\mathcal{K}_\infty$ function ρ such that the following implication holds for all $x \neq 0$, all bounded $\hat{\theta}$, and all $\tilde{\theta}, \dot{\hat{\theta}} \in \mathbb{R}^p$:

$$|x| \geq \rho\left(\left|\begin{bmatrix}\tilde{\theta}\\ \dot{\hat{\theta}}\end{bmatrix}\right|\right) \quad \Downarrow \quad \inf_{u \in \mathbb{R}}\left\{\frac{\partial V}{\partial x}[f(x) + F(x)\hat{\theta} + g(x)u] + \frac{\partial V}{\partial x}F(x)\tilde{\theta} + \frac{\partial V}{\partial \hat{\theta}}\dot{\hat{\theta}}\right\} < 0. \tag{8}$$

We now show that the existence of an iss-clf is a necessary and sufficient condition for input-to-state stabilizability. The proof of sufficiency is constructive—we design a control law starting from a given iss-clf.

Lemma 2.1. (*Input-to-state stabilization.*) The system (1) is input-to-state stabilizable with respect to $(\tilde{\theta}, \dot{\hat{\theta}})$ if and only if there exists an iss-clf.

Proof. The 'only if' part is obvious, because (7) implies that there exists a particular control law $u = \alpha(x, \hat{\theta})$ that satisfies (8). We shall now prove the 'if' part using Sontag (1989b). We show that the following control law achieves input-to-stage stabilization:

$$\alpha(x, \hat{\theta}) = \begin{cases} -\dfrac{\omega + \sqrt{\omega^2 + \left(\dfrac{\partial V}{\partial x}g\right)^4}}{\dfrac{\partial V}{\partial x}g} & \left(\dfrac{\partial V}{\partial x}g \neq 0\right), \\ 0 & \left(\dfrac{\partial V}{\partial x}g = 0\right), \end{cases} \tag{9}$$

where

$$\omega(x, \hat{\theta}) = \frac{\partial V}{\partial x}[f(x) + F(x)\hat{\theta}] + \left|\begin{bmatrix}\dfrac{\partial V}{\partial x}F & \dfrac{\partial V}{\partial \hat{\theta}}\end{bmatrix}^{\mathrm{T}}\right|\rho^{-1}(|x|). \tag{10}$$

We first show that (9) is continuous on $(\mathbb{R}^n\backslash\{0\}) \times \mathbb{R}^p$. Sontag (1989b) proved that the function (9) is smooth provided its arguments ω and $(\partial V/\partial x)g$ are such that

$$\frac{\partial V}{\partial x}g = 0 \Rightarrow \omega < 0. \tag{11}$$

† To be precise, the proof of Sontag (1989a) can be easily extended to the time-varying case (note that $\hat{\theta}(t)$ appears in (5)).

‡ The boundedness of $\hat{\theta}$ will be independently guaranteed by the parameter identifier (cf. Lemma 2.3). In fact, $|\hat{\theta}(t)| \leq \tilde{\theta}(0)|$. The uniform boundedness of $\hat{\theta}$ is sufficient to guarantee the existence of class $\mathcal{K}_\infty$ functions α_1 and α_2 and a class $\mathcal{K}$ function α_3 such that $\alpha_1(|x|) \leq V(x, \hat{\theta}) \leq \alpha_2(|x|)$ and $\dot{V} \leq -\alpha_3(|x|)$ in (7).

We show that V being an iss-clf implies that (11) is satisfied. By Definition 2.1, if $x \neq 0$ and $(\partial V/\partial x)g = 0$ then

$$|x| \geq \rho\left(\left|\begin{bmatrix}\tilde{\theta}\\ \dot{\hat{\theta}}\end{bmatrix}\right|\right) \quad \Downarrow \quad \frac{\partial V}{\partial x}[f(x) + F(x)\hat{\theta}] + \frac{\partial V}{\partial x}F(x)\tilde{\theta} + \frac{\partial V}{\partial \hat{\theta}}\dot{\hat{\theta}} < 0. \tag{12}$$

Let us consider the particular input

$$\begin{bmatrix}\tilde{\theta}\\ \dot{\hat{\theta}}\end{bmatrix} = \frac{\begin{bmatrix}\dfrac{\partial V}{\partial x}F & \dfrac{\partial V}{\partial \hat{\theta}}\end{bmatrix}^{\mathrm{T}}}{\left|\begin{bmatrix}\dfrac{\partial V}{\partial x}F & \dfrac{\partial V}{\partial \hat{\theta}}\end{bmatrix}^{\mathrm{T}}\right|}\rho^{-1}(|x|). \tag{13}$$

This input satisfies the upper part of the implication (12):

$$\rho\left(\left|\begin{bmatrix}\tilde{\theta}\\ \dot{\hat{\theta}}\end{bmatrix}\right|\right) = |x|. \tag{14}$$

Therefore, substituting (13) into the lower part of (12), we conclude that, if $x \neq 0$ and $(\partial V/\partial x)g = 0$ then

$$\frac{\partial V}{\partial x}[f(x) + F(x)\hat{\theta}] + \left|\begin{bmatrix}\dfrac{\partial V}{\partial x}F & \dfrac{\partial V}{\partial \hat{\theta}}\end{bmatrix}^{\mathrm{T}}\right|\rho^{-1}(|x|) < 0; \tag{15}$$

that is, (11) is satisfied for $x \neq 0$. Therefore (9) is a smooth function of ω and $(\partial V/\partial x)g$ whenever $x \neq 0$. Since $\omega(x, \hat{\theta})$ is continuous and $(\partial V/\partial x)g(x, \hat{\theta})$ is smooth, the control law (9) is continuous for $x \neq 0$. We note that the control law $\alpha(x, \hat{\theta})$ given by (9) is also continuous at $x = 0$ if and only if the iss-clf V satisfies the following *small control property* (Sontag, 1989b): for each $\hat{\theta} \in \mathbb{R}^p$ and for any $\varepsilon > 0$, there is a $\delta > 0$ such that, if $x \neq 0$ satisfies $\rho\left(\left|\begin{bmatrix}\tilde{\theta}\\ \dot{\hat{\theta}}\end{bmatrix}\right|\right) \leq |x| \leq \delta$ then there is some u with $|u| \leq \varepsilon$ such that

$$\frac{\partial V}{\partial x}[f(x) + F(x)\hat{\theta} + g(x)u] + \frac{\partial V}{\partial x}F(x)\tilde{\theta} + \frac{\partial V}{\partial \hat{\theta}}\dot{\hat{\theta}} < 0. \tag{16}$$

We now show that the control law (9) achieves input-to-state stabilization. Along the solutions of (5) and (9), the derivative of V is

$$\begin{aligned}\dot{V} &= -\left|\begin{bmatrix}\dfrac{\partial V}{\partial x}F & \dfrac{\partial V}{\partial \hat{\theta}}\end{bmatrix}^{\mathrm{T}}\right|\rho^{-1}(|x|) + \frac{\partial V}{\partial x}F(x)\tilde{\theta} + \frac{\partial V}{\partial \hat{\theta}}\dot{\hat{\theta}} \\ &\quad - \sqrt{\left(\frac{\partial V}{\partial x}[f(x) + F(x)\hat{\theta}] + \left|\begin{bmatrix}\dfrac{\partial V}{\partial x}F & \dfrac{\partial V}{\partial \hat{\theta}}\end{bmatrix}^{\mathrm{T}}\right|\rho^{-1}(|x|)\right)^2 + \left(\frac{\partial V}{\partial x}g\right)^4} \\ &\leq -\left|\begin{bmatrix}\dfrac{\partial V}{\partial x}F & \dfrac{\partial V}{\partial \hat{\theta}}\end{bmatrix}^{\mathrm{T}}\right|\left(\rho^{-1}(|x|) - \left|\begin{bmatrix}\tilde{\theta}\\ \dot{\hat{\theta}}\end{bmatrix}\right|\right) \\ &\quad - \sqrt{\left(\frac{\partial V}{\partial x}[f(x) + F(x)\hat{\theta}] + \left|\begin{bmatrix}\dfrac{\partial V}{\partial x}F & \dfrac{\partial V}{\partial \hat{\theta}}\end{bmatrix}^{\mathrm{T}}\right|\rho^{-1}(|x|)\right)^2 + \left(\frac{\partial V}{\partial x}g\right)^4}.\end{aligned} \tag{17}$$

In view of (15), this proves that $\dot{V} < 0, \forall x \neq 0$, whenever $|x| \geq \rho\left(\left|\begin{bmatrix}\tilde{\theta}\\ \dot{\hat{\theta}}\end{bmatrix}\right|\right)$, that is, V is an ISS-Lyapunov function, which, by Sontag (1989a, Claim on p. 441), establishes that (1) is input-to-state stable with respect to $(\tilde{\theta}, \dot{\hat{\theta}})$. □

Global asymptotic stabilizability for each θ is a necessary condition for the existence of an iss-clf. This becomes obvious on setting $\hat{\theta}(t) \equiv \theta$, which implies $\tilde{\theta}(t) = \dot{\hat{\theta}}(t) \equiv 0$, into (6).

Next we give sufficient conditions under which $x(t) \to 0$ as $t \to \infty$. Owing to the ISS property (6), one sufficient condition is that both $\tilde{\theta}$ and $\dot{\hat{\theta}}$ tend to zero. However, in general, identifiers cannot guarantee that $\tilde{\theta}$ goes to zero, so the next lemma gives a less demanding condition.

Lemma 2.2. (*Regulation.*) Suppose the control law $u = \alpha(x, \hat{\theta})$ guarantees that the system (1) is ISS with respect to $(\tilde{\theta}, \dot{\hat{\theta}})$. If $\tilde{\theta}(t)$ and $\dot{\hat{\theta}}(t)$ are bounded and continuous, and both $F(x(t))\tilde{\theta}(t)$ and $\dot{\hat{\theta}}(t)$ converge to zero as $t \to \infty$, then $\lim_{t\to\infty} x(t) = 0$.

Proof. Since the system

$$\dot{x} = f(x) + F(x)\theta + g(x)\alpha(x, \hat{\theta}) \tag{18}$$

is ISS with respect to $(\tilde{\theta}, \dot{\hat{\theta}})$, the same system with $\tilde{\theta} = \dot{\hat{\theta}} = 0$, namely the system

$$\dot{x} = f(x) + F(x)\hat{\theta} + g(x)\alpha(x, \hat{\theta}), \tag{19}$$

is globally asymptotically stable. Therefore, by Sontag (1990, Theorem 2), there exist $\beta \in \mathcal{KL}$, $\gamma \in \mathcal{K}$, and a continuous function $\sigma: \mathbb{R}_+ \to \mathbb{R}_+$, $\sigma(s) > 0$ for $s > 0$, such that, for each continuous and bounded input $w(t) \triangleq \begin{bmatrix} F(x(t))\tilde{\theta}(t) \\ \dot{\hat{\theta}}(t) \end{bmatrix}$, for each $x(t_0) \in \mathbb{R}^n$ and for all $t \geq t_0 \geq 0$, the following implication holds:

$$\begin{gathered} \sup_{t_0 \leq \tau \leq t} |w(\tau)| \leq \sigma(|x(t_0)|) \\ \Downarrow \\ |x(t)| \leq \beta(|x(t_0)|, t - t_0) + \gamma\Big(\sup_{t_0 \leq \tau \leq t} |w(\tau)|\Big). \end{gathered} \tag{20}$$

Since $\tilde{\theta}$ and $\dot{\hat{\theta}}$ are bounded, so is $x(t)$. Let M be such that $|x(t)| \leq M$ for all $t \geq 0$. Let $\varepsilon = \min\{\sigma(r) \mid r \leq M\}$, and let $T \geq 0$ be such that $|w(t)| \leq \varepsilon$ for all $t \geq T$. Then, from (20), we obtain

$$|x(t)| \leq \beta(|x(t_0)|, t - t_0) + \gamma\Big(\sup_{t_0 \leq \tau \leq t} |w(\tau)|\Big) \tag{21}$$

for all $t \geq t_0 \geq T$. To complete the proof, we have to show that the ISS property (21) implies that $x(t) \to 0$ as $t \to \infty$. Our computations follow those in the proof of Sontag (1989a, Proposition 7.2). First, we note that there exists a monotonically decreasing to zero function η continuous on $[T, \infty)$ such that

$$|w(t)| \leq \eta(t - t_0) \quad \forall t \geq t_0 \geq T. \tag{22}$$

Then we have

$$\begin{aligned} |x(t)| &\leq \beta\Big(\Big|x\Big(\frac{t+T}{2}\Big)\Big|, \frac{t-T}{2}\Big) + \gamma\Big(\sup_{(t+T)/2 \leq \tau \leq t} |w(\tau)|\Big) \\ &\leq \beta\Big(\beta\Big(|x(T)|, \frac{t-T}{2}\Big) + \gamma\Big(\sup_{T \leq \tau \leq (t+t)/2} |w(\tau)|\Big), \frac{t-T}{2}\Big) \\ &\quad + \gamma\Big(\sup_{(t+T)/2 \leq \tau \leq t} \eta(\tau - T)\Big). \end{aligned} \tag{23}$$

Noting that for any class $\mathcal{K}$ function δ, $\delta(a+b) \leq \delta(2a) + \delta(2b)$ for any nonnegative a and b, we proceed from (23) with

$$\begin{aligned} |x(t)| &\leq \beta\Big(2\beta\Big(|x(T)|, \frac{t-T}{2}\Big), \frac{t-T}{2}\Big) \\ &\quad + \beta\Big(2\gamma\Big(\sup_{T \leq \tau \leq (t+T)/2} |w(\tau)|\Big), \frac{t-T}{2}\Big) + \gamma\Big(\eta\Big(\frac{t-T}{2}\Big)\Big) \\ &\leq \beta\Big(2\beta\Big(\Big|x(T)\Big|, \frac{t-T}{2}\Big), \frac{t-T}{2}\Big) \\ &\quad + \beta\Big(2\beta\Big(\sup_{T \leq \tau \leq (t+T)/2} \eta(\tau - T)\Big), \frac{t-T}{2}\Big) + \gamma\Big(\eta\Big(\frac{t-T}{2}\Big)\Big) \\ &\leq \beta\Big(2\beta\Big(M, \frac{t-T}{2}\Big), \frac{t-T}{2}\Big) + \beta\Big(2\gamma(\eta(0)), \frac{t-T}{2}\Big) \\ &\quad + \gamma\Big(\eta\Big(\frac{t-T}{2}\Big)\Big), \end{aligned} \tag{24}$$

which converges to zero as $t \to \infty$. □

Returning to the modular controller–identifier design, Lemma 2.2 serves as a list of conditions that an identifier has to satisfy: generate a bounded estimate $\hat{\theta}$ with a bounded derivative $\dot{\hat{\theta}}$, as well as ensure that $f(x)\tilde{\theta}$ and $\dot{\hat{\theta}}$ tend to zero.

2.1. *Identifier.* We give just one form among several possible identfiers with these properties. It employs the filters

$$\dot{\Omega}^{\mathrm{T}} = [A_0 - \lambda F(x)F(x)^{\mathrm{T}}P]\Omega^{\mathrm{T}} + F(x), \tag{25}$$

$$\dot{\Omega}_0 = [A_0 - \lambda F(x)F(x)^{\mathrm{T}}P](\Omega_0 - x) + f(x) + g(x)u, \tag{26}$$

where $\lambda \geq 0$, and A_0 is an arbitrary constant matrix such that $PA_0 + A_0^{\mathrm{T}}P = -I$, $P = P^{\mathrm{T}} > 0$. The *estimation error* is given by

$$\epsilon = x - \Omega_0 - \Omega^{\mathrm{T}}\hat{\theta}. \tag{27}$$

The update law for $\hat{\theta}$ is either the gradient,

$$\dot{\hat{\theta}} = \Gamma\frac{\Omega\epsilon}{1 + \nu|\Omega|_{\mathcal{F}}^2}, \quad \Gamma = \Gamma^{\mathrm{T}} > 0, \quad \nu \geq 0, \tag{28}$$

or the least-squares,

$$\begin{aligned} \dot{\hat{\theta}} &= \Gamma\frac{\Omega\epsilon}{1 + \nu|\Omega|_{\mathcal{F}}^2}, \\ \dot{\Gamma} &= -\Gamma\frac{\Omega\Omega^{\mathrm{T}}}{1 + \nu|\Omega|_{\mathcal{F}}^2}\Gamma, \quad \Gamma(0) = \Gamma(0)^{\mathrm{T}} > 0, \quad \nu \geq 0, \end{aligned} \tag{29}$$

where $|\cdot|_{\mathcal{F}}$ denotes the Frobenius norm. The terms with coefficients $\lambda \geq 0$ in (25) and (26), and $\nu \geq 0$ in (28) and (29), are two alternative forms of slowing down the identifier so that $\dot{\hat{\theta}}$ is kept bounded. We require that either λ or ν be nonzero.

Lemma 2.3. The identifier (25)–(27) with either (28) or (29) guarantees that

- $\tilde{\theta}(t)$ and $\dot{\hat{\theta}}(t)$ are bounded;
- if $x(t)$ is bounded then $F(x(t))\tilde{\theta}(t)$ and $\dot{\hat{\theta}}(t)$ converge to zero.

Proof. First, we note that (27) can also be expressed as

$$\epsilon = \Omega^{\mathrm{T}}\tilde{\theta} + \tilde{\epsilon}, \tag{30}$$

where $\dot{\tilde{\epsilon}} = [A_0 - \lambda F(x)F(x)^{\mathrm{T}}P]\tilde{\epsilon}$. With (30), the proof of boundedness of $\tilde{\theta}$ is standard (Sastry and Bodson, 1989). So is the proof of boundedness and convergence of $\dot{\hat{\theta}}(t)$ when $\nu > 0$. For the case $\lambda > 0$, see Krstić and Kokotović (1995a).

Here we only prove that $F(x(t))\tilde{\theta}(t)$ converges to zero. The boundedness of x implies the boundedness of all other signals and their derivatives. It is straightforward to show that ϵ is also in $\mathcal{L}_2$ (Sastry and Bodson, 1989). Hence, by Barbalat's lemma, $\epsilon(t)$ converges to zero. Therefore

$$\lim_{t\to\infty} \int_0^t \dot{\epsilon}(\tau)\, d\tau = \lim_{t\to\infty} \epsilon(t) - \epsilon(0) = -\epsilon(0) < \infty. \tag{31}$$

Combining (25)–(27), we get

$$\dot{\epsilon} = [A_0 - \lambda F(x)F(x)^{\mathrm{T}}P]\epsilon + F(x)\tilde{\theta} - \Omega^{\mathrm{T}}\dot{\hat{\theta}}. \tag{32}$$

Owing to the smoothness of F, $\ddot{\epsilon}$ is bounded, and therefore $\dot{\epsilon}$ is uniformly continuous. Then, by Barbalat's lemma, $\dot{\epsilon}(t) \to 0$. Since $\dot{\epsilon}(t)$, $\epsilon(t)$ and $\dot{\hat{\theta}}(t)$ converge to zero, from (32) we conclude that $F(x(t))\tilde{\theta}(t) \to 0$. Note that while $\dot{\hat{\theta}}$ is in $\mathcal{L}_2$ (Sastry and Bodson, 1989), $F(x)\tilde{\theta}$ may not be. □

Consider now the adaptive controller consisting of the control law (9), (10) and the idenitfier (25)–(29). By combining Lemmas 2.1–2.3 the following conclusion is immediate.†

Theorem 2.1. If there exists an *iss-clf* for the system (1) then the system is globally adaptively stabilizable.

3. *ISS backstepping*

With Theorem 2.1, the problem of adaptive stabilization is reduced to the problem of finding an iss-clf. Using backstepping, an iss-clf for a higher-order system is recursively constructed starting with an iss-clf for a lower-order system.

Lemma 3.1. If the system

$$\dot{x} = f(x) + F(x)\theta + g(x)u \tag{33}$$

is input-to-state stabilizable with respect to $(\tilde{\theta}, \dot{\hat{\theta}})$ using $\alpha \in C^1$, and $\hat{\theta}$ is bounded, then the augmented system

$$\begin{aligned} \dot{x} &= f(x) + F(x)\theta + g(x)\xi, \\ \dot{\xi} &= u \end{aligned} \tag{34}$$

is also input-to-state stabilizable with respect to $(\tilde{\theta}, \dot{\hat{\theta}})$.

† We have avoided the issue of existence of solutions, which was delat with in detail in Krstić and Kokotović (1995a).

Proof. Since (33) is input-to-state stabilizable with respect to $(\tilde{\theta}, \dot{\hat{\theta}})$, there exists a triple (α, V, ρ) and a class $\mathcal{K}$ function μ such that

$$|x| \geq \rho\left(\left|\begin{bmatrix}\tilde{\theta}\\ \dot{\hat{\theta}}\end{bmatrix}\right|\right)$$
$$\Downarrow \tag{35}$$
$$\frac{\partial V}{\partial x}[f(x) + F(x)\hat{\theta} + g(x)\alpha(x, \hat{\theta})] + \frac{\partial V}{\partial x}F(x)\tilde{\theta} + \frac{\partial V}{\partial \hat{\theta}}\dot{\hat{\theta}} \leq -\mu(|x|).$$

In fact, without loss of generality, we assume that μ is class $\mathcal{K}_\infty$. It was shown by Sontag (1989a) that if μ is only in class $\mathcal{K}$ then the given Lyapunov function V can be modified so that the new μ be in class $\mathcal{K}_\infty$. For $\mu \in \mathcal{K}_\infty$, it was shown by Sontag and Wang (1994) that (35) is equivalent to the following 'dissipation' type of characterization:

$$\frac{\partial V}{\partial x}[f(x) + F(x)\hat{\theta} + g(x)\alpha(x, \hat{\theta})] + \frac{\partial V}{\partial x}F(x)\tilde{\theta} + \frac{\partial V}{\partial \hat{\theta}}\dot{\hat{\theta}} \leq -\mu(|x|) + \pi\left(\left|\begin{bmatrix}\tilde{\theta}\\ \dot{\hat{\theta}}\end{bmatrix}\right|\right), \tag{36}$$

where π is a class $\mathcal{K}$ function. Since the proof of the affine case considered here is simple, we give it for completeness. It is clear that (36) implies (35). To see the converse, one only needs to consider the case $|x| \leq \rho\left(\left|\begin{bmatrix}\tilde{\theta}\\ \dot{\hat{\theta}}\end{bmatrix}\right|\right)$. Since $\hat{\theta}$ is bounded, with Young's inequality one obtains

$$\begin{aligned}
&\frac{\partial V}{\partial x}[f(x) + F(x)\hat{\theta} + g(x)\alpha(x, \hat{\theta})] + \frac{\partial V}{\partial x}F(x)\tilde{\theta} + \frac{\partial V}{\partial \hat{\theta}}\dot{\hat{\theta}} + \mu(|x|)\\
&\leq \frac{\partial V}{\partial x}[f(x) + F(x)\hat{\theta} + g(x)\alpha(x, \hat{\theta})] + \mu(|x|)\\
&\quad + \frac{1}{4}\left|\begin{bmatrix}\frac{\partial V}{\partial x}F(x) & \frac{\partial V}{\partial \hat{\theta}}\end{bmatrix}^{\mathrm{T}}\right|^2 + \left|\begin{bmatrix}\tilde{\theta}\\ \dot{\hat{\theta}}\end{bmatrix}\right|^2\\
&\leq \bar{\mu}(|x|) + \left|\begin{bmatrix}\tilde{\theta}\\ \dot{\hat{\theta}}\end{bmatrix}\right|^2 \leq \bar{\mu} \circ \rho\left(\left|\begin{bmatrix}\tilde{\theta}\\ \dot{\hat{\theta}}\end{bmatrix}\right|\right) + \left|\begin{bmatrix}\tilde{\theta}\\ \dot{\hat{\theta}}\end{bmatrix}\right|^2\\
&\triangleq \pi\left(\left|\begin{bmatrix}\tilde{\theta}\\ \dot{\hat{\theta}}\end{bmatrix}\right|\right),
\end{aligned} \tag{37}$$

where $\bar{\mu}$ is a class $\mathcal{K}_\infty$ function. This completes the proof of (36). We shall now use (36) to show that

$$V_1(x, \xi, \hat{\theta}) = V(x, \hat{\theta}) + \tfrac{1}{2}[\xi - \alpha(x, \hat{\theta})]^2 \tag{38}$$

is an iss-clf for (34). We do this by showing that the control law

$$\begin{aligned}
u = \alpha_1(x, \xi, \hat{\theta}) = &-\frac{\partial V}{\partial x}g - (\xi - \alpha) + \frac{\partial \alpha}{\partial x}(f + F\hat{\theta} + g\xi)\\
&- \left|\begin{bmatrix}\frac{\partial \alpha}{\partial x}F(x) & \frac{\partial \alpha}{\partial \hat{\theta}}\end{bmatrix}^{\mathrm{T}}\right|^2(\xi - \alpha)
\end{aligned} \tag{39}$$

achieves input-to-state stabilization of (34). Towards this end, consider

$$\begin{aligned}
\dot{V}_1 = &\frac{\partial V}{\partial x}[f(x) + F(x)\hat{\theta} + g(x)\alpha(x, \hat{\theta})]\\
&+ \frac{\partial V}{\partial x}F(x)\tilde{\theta} + \frac{\partial V}{\partial \hat{\theta}}\dot{\hat{\theta}} + \frac{\partial V}{\partial x}g(\xi - \alpha)\\
&+ (\xi - \alpha)\left[u - \frac{\partial \alpha}{\partial x}(f + F\theta + g\xi) - \frac{\partial \alpha}{\partial \hat{\theta}}\dot{\hat{\theta}}\right]\\
\leq &-\mu(|x|) + \pi\left(\left|\begin{bmatrix}\tilde{\theta}\\ \dot{\hat{\theta}}\end{bmatrix}\right|\right)\\
&+ (\xi - \alpha)\left[u + \frac{\partial V}{\partial x}g - \frac{\partial \alpha}{\partial x}(f + F\hat{\theta} + g\xi)\right]\\
&- (\xi - \alpha)\frac{\partial \alpha}{\partial x}F\tilde{\theta} - (\xi - \alpha)\frac{\partial \alpha}{\partial \hat{\theta}}\dot{\hat{\theta}}\\
\leq &-\mu(|x|) + \pi\left(\left|\begin{bmatrix}\tilde{\theta}\\ \dot{\hat{\theta}}\end{bmatrix}\right|\right) - (\xi - \alpha)^2\\
&- \left|\begin{bmatrix}\frac{\partial \alpha}{\partial x}F & \frac{\partial \alpha}{\partial \hat{\theta}}\end{bmatrix}^{\mathrm{T}}\right|^2(\xi - \alpha)^2\\
&- (\xi - \alpha)\begin{bmatrix}\frac{\partial \alpha}{\partial x}F & \frac{\partial \alpha}{\partial \hat{\theta}}\end{bmatrix}^{\mathrm{T}}\begin{bmatrix}\tilde{\theta}\\ \dot{\hat{\theta}}\end{bmatrix}\\
\leq &-\mu(|x|) + \pi\left(\left|\begin{bmatrix}\tilde{\theta}\\ \dot{\hat{\theta}}\end{bmatrix}\right|\right) - (\xi - \alpha)^2 + \frac{1}{4}\left|\begin{bmatrix}\tilde{\theta}\\ \dot{\hat{\theta}}\end{bmatrix}\right|^2.
\end{aligned} \tag{40}$$

Denoting $\pi_1(r) = \pi(r) + \frac{1}{4}r^2$ and picking a class $\mathcal{K}_\infty$ function $\mu_1(r) \leq \min\{\mu(r), r^2\}$, because of the boundedness of $\hat{\theta}$, we get

$$\begin{aligned}
\dot{V}_1 &\leq -\mu_1\left(\left|\begin{bmatrix}x\\ \xi - \alpha(x, \xi, \hat{\theta})\end{bmatrix}\right|\right) + \pi_1\left(\left|\begin{bmatrix}\tilde{\theta}\\ \dot{\hat{\theta}}\end{bmatrix}\right|\right)\\
&\leq -\mu_2\left(\left|\begin{bmatrix}x\\ \xi\end{bmatrix}\right|\right) + \pi_1\left(\left|\begin{bmatrix}\tilde{\theta}\\ \dot{\hat{\theta}}\end{bmatrix}\right|\right),
\end{aligned} \tag{41}$$

where μ_2 is a class $\mathcal{K}_\infty$ function. Thus V_1 is an ISS-Lyapunov function. By Sontag, 1989a, Claim on p. 441), the system (34) with control law (39) is ISS with respect to $(\tilde{\theta}, \dot{\hat{\theta}})$. □

The control law $\alpha_1(x, \xi, \theta)$ in (39) is only one out of many possible control laws. Once we have shown that V_1 given by (38) is an iss-clf for (34) (with $\rho = \mu_2^{-1} \circ 2\pi_1$), we can use, for example, the C^0 control law α_1 given by (9).

Repeated application of Lemma 3.1, and then Theorem 2.1, recovers our earlier result (Krstić and Kokotović, 1995a):

Corollary 3.1. The following system is globally adaptively stabilizable:

$$\begin{aligned}
\dot{x}_i &= x_{i+1} + \varphi_i(x_1, \ldots, x_i)^{\mathrm{T}}\theta, \quad i = 1, \ldots, n-1,\\
\dot{x}_n &= u + \varphi_n(x_1, \ldots, x_n)^{\mathrm{T}}\theta.
\end{aligned} \tag{42}$$

4. *Conclusions*

The clf framework is convenient, because it eliminates adaptation from consideration, and reduces the problem of adaptive stabilization to the probem of (nonadaptive) input-to-state stabilization with respect to particular disturbance inputs.

The clf framework for modular design parallels the clf framework for Lyapunov design that we introduced in Krstić and Kokotović (1995b). The problem of adaptive stabilization was approached there as a probem of (nonadaptive) stabilization of a modified system, and the tuning functions design of Krstić *et al.* (1992) was recovered.

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