assessment. The main result lies in the development of a sufficient condition from which convex semidefinite programming can be used by bounding the terms transducing the noncoincidence between the measurement and the reality. A numerical experiment supports the efficiency of the results. Further developments could be made for various types of parameter dependence, such as, for instance, the linear fractional dependence.

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Nonlinear Stabilization of Shock-Like Unstable Equilibria in the Viscous Burgers PDE

Miroslav Krstic, Lionel Magnis, and Rafael Vazquez

***Abstract*—We stabilize the unstable "shock-like" equilibrium profiles of the viscous Burgers equation using control at the boundaries. These equilibria are not stabilizable (even locally) using the standard "radiation feedback boundary conditions." Using a nonlinear spatially-scaled transformation (that employs three ingredients, of which one is the Hopf–Cole nonlinear integral transformation) and linear backstepping, we design an explicit nonlinear full-state control law that achieves exponential stability, with a region of attraction for which we give an estimate. The region of attraction is not the entire state space since the Burgers PDE is known not to be globally controllable.**

***Index Terms*—Burgers PDE.**

I. INTRODUCTION

We study nonlinear stabilization for the viscous Burgers equation. We consider a family of "shock-like" stationary profiles [9] which are unstable and not stabilizable (even locally) by simple "radiation boundary conditions." We achieve exponential stabilization (in spatial L^2 norm) of the shock profiles using two control inputs (one at each boundary) by full-state feedback.

Early effort on linear static collocated output feedback (a.k.a. "radiation" boundary conditions) proved local L^2 exponential stability in [6], with extension to L^∞ in [20]. Semi-global version of [6] and regulation to small boundary set points was proved in [7]. Global stabilization is achieved in [13], using nonlinear boundary conditions, and extended to KdV, Kuramoto–Sivashinsky, adaptive control, and other problems in [1], [2], and [15]–[18]. Reference [10] derives bounds on minimal time for null controllability with one and two inputs. In [3] stabilization is approached using nonlinear model reduction, with in-domain actuation. Control of the *inviscid* Burgers equation, a challenging problem and different than the one studied here, was studied in [4].

Our design for nonlinear (non-convective) parabolic PDEs [23], [24] is based on a feedback linearizing nonlinear Volterra series transformation. We follow a similar idea here and construct a nonlinear transformation (based on three ingredients, one of which is the Hopf–Cole [8], [11]), that transforms the system (with the help by one of the two boundary controls) into a linear reaction-diffusion PDE, which is then stabilized using linear backstepping [19], yielding a control that is nonlinear in the original state. We provide an estimate of the region of attraction, which is finite because the Burgers system unfortunately is not globally controllable [10].

II. BURGERS EQUATION AND ITS SHOCK PROFILES

Consider the viscous Burgers equation

$$u_t = u_{xx} - u_x u, \quad x \in [0, 1] \tag{1}$$

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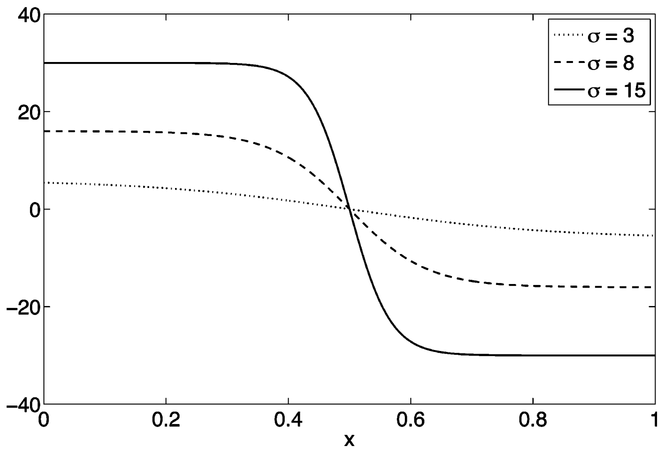

Fig. 1. Three example shock profiles for different values of σ.

with $u(x,t)$ as the state, with boundary conditions

$$u_x(0,t) = \omega_0(t), \quad u_x(1,t) = \omega_1(t), \tag{2}$$

and with $\omega_0(t)$ and $\omega_1(t)$ as the control inputs. To save space, we will drop the arguments (x,t) whenever the context allows. We are interested in the family of symmetric "shock-like" stationary solutions [5]

$$U(x) = -2\sigma \tanh(\sigma(x-1/2)) \tag{3}$$

parameterized by $\sigma \geq 0$ (see Fig. 1). The result in this paper can be derived for other equilibria, but (3) is of interest as the "most unstable" case. While the Burgers PDE is often studied as $u_t = \varepsilon u_{xx} - u_x u$, to reduce notation we focus on $\varepsilon = 1$, as $U(x) = -2\varepsilon\sigma\tanh(\sigma(x-1/2))$ scales with ε, so eigenvalues scale with ε (and so does the estimate of the region of attraction). Continuing with (3), we first observe that $U'(x) = -2\sigma^2\left(1-\tanh^2(\sigma(x-1/2))\right)$, hence $U'(0) = U'(1) = -2\sigma^2\left(1-\tanh^2(\sigma/2)\right)$, so from (2) we obtain that

$$\omega_0 = \omega_1 = -2\sigma^2\left(1-\tanh^2(\sigma/2)\right) \leq 0 \tag{4}$$

produce the equilibrium (3). Let us denote the fluctuation variable around the shock profile as $\tilde{u}(x,t) = u(x,t) - U(x)$, along with $\tilde{\omega}_0(t) = \omega_0(t) - U'(0), \tilde{\omega}_1(t) = \omega_1(t) - U'(1)$, which yields

$$\tilde{u}_t = \tilde{u}_{xx} - U(x)\tilde{u}_x - U'(x)\tilde{u} - \tilde{u}_x\tilde{u} \tag{5}$$
$$\tilde{u}_x(0,t) = \tilde{\omega}_0(t), \quad \tilde{u}_x(1,t) = \tilde{\omega}_1(t). \tag{6}$$

III. Instability of Shock Profiles

To study the stability of the origin of the open-loop system (5), we linearize it, obtaining $\theta_t = \theta_{xx} + 2\sigma\left(\tanh(\sigma(x-1/2))\theta\right)_x, \theta_x(0) = \theta_x(1) = 0$, and then eliminate the advection term in this PDE using the invertible transformation $\zeta(x,t) = G(x)\theta(x,t)$, where

$$G(x) = \frac{\cosh(\sigma(x-1/2))}{\cosh(\sigma/2)} \tag{7}$$

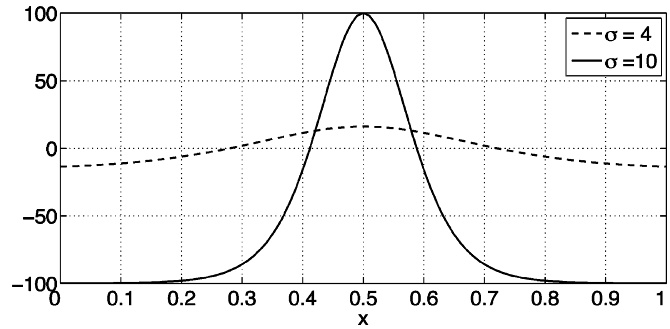

Fig. 2. Value of the reaction coefficient in (8). The coefficient is positive and large in the region around $x = 1/2$, potentially destabilizing the system.

obtaininig

$$\zeta_t = \zeta_{xx} + \sigma^2\left[\frac{2}{\cosh^2(\sigma(x-1/2))} - 1\right]\zeta \tag{8}$$
$$\zeta_x(0) = -\sigma\tanh(\sigma/2)\zeta(0) \tag{9}$$
$$\zeta_x(1) = \sigma\tanh(\sigma/2)\zeta(1). \tag{10}$$

For $\sigma = 0$ the system is neutrally stable. For $\sigma > 0$, the boundary conditions are destabilizing and so is the reaction term in (8), in the vicinity of $x = 1/2$, see Fig. 2. The larger σ, the more positive the first eigenvalue of (8)–(10). For example, for $\sigma = 15$ the first eigenvalue is $+0.6$.

IV. Instability Under "Radiation Feedback"

With "radiation boundary feedback" [6], [7]

$$\tilde{\omega}_0(t) = k\tilde{u}(0,t), \quad \tilde{\omega}_1(t) = -k\tilde{u}(1,t), \quad k > 0 \tag{11}$$

system (8)–(10) changes only in boundary conditions

$$\zeta_x(0) = (k - \sigma\tanh(\sigma/2))\zeta(0), \tag{12}$$
$$\zeta_x(1) = -(k - \sigma\tanh(\sigma/2))\zeta(1). \tag{13}$$

Stability of the system (8), (12), and (13) improves as $k \to +\infty$. However, eigenvalue computation shows that for $\sigma > \sigma^*$, where $\sigma^* \approx 10$, the system always has exactly one unstable eigenvalue, so no value of k exists that stabilizes the system, see Fig. 3. Thus, semiglobal stability in [7] holds for sufficiently small set points but not for *large set points*. So, a more sophisticated feedback (full-state or dynamic) is needed to stabilize shock-like equilibria (even locally).

V. Full State Feedback

We use a state transformation and feedback for $\tilde{\omega}_0$ to linearize the transformed PDE and its boundary condition at $x = 0$; then design a feedback for $\tilde{\omega}_1$ to stabilize the resulting linear system using backstepping.

A. Linearizing Transformation and the Design of $\tilde{\omega}_0$

Define $v(x,t) = \tilde{u}(x,t)e^{-\frac{1}{2}\int_0^x[\tilde{u}(y,t)+U(y)]dy}$, which is a new state variable, written using (7) as

$$v(x,t) = G(x)\tilde{u}(x,t)e^{-\frac{1}{2}\int_0^x \tilde{u}(y,t)dy}. \tag{14}$$

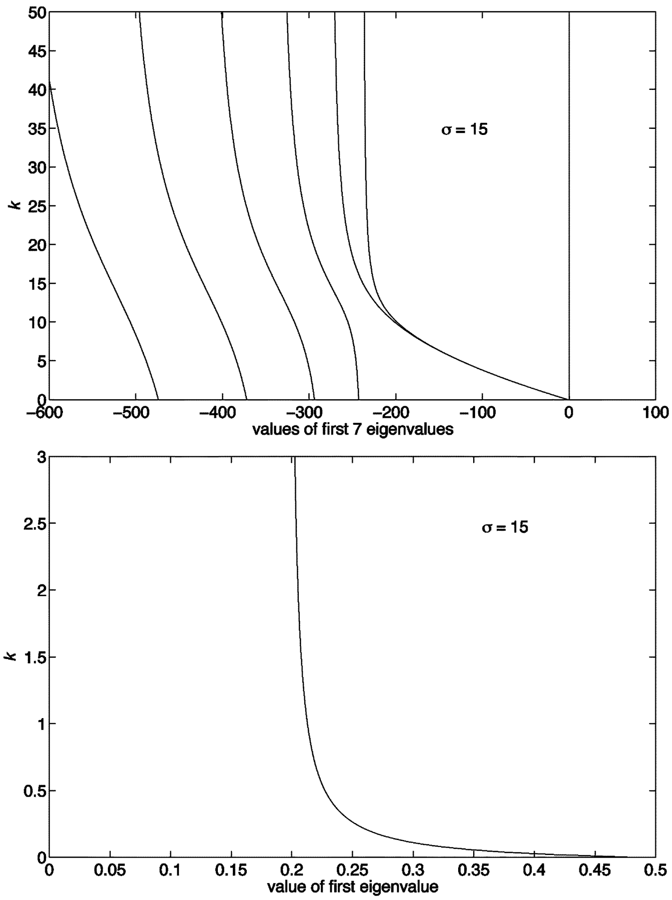

Fig. 3. Top: Seven rightmost eigenvalues of (8), (12), and (13) for growing k and fixed $\sigma = 15$. All are real and move leftward as k increases. Bottom: Detail for first eigenvalue, positive for all k.

Transformation (14) is a composition of the Hopf–Cole [8], [11] transformation on $\tilde{u}(x,t)$ and the transformation ∂_x, scaled by the "gauge" transformation $G(x)$. The transformation (14) from $\tilde{u}$ to v is invertible:

$$\tilde{u}(x,t) = \frac{v(x,t)/G(x)}{1 - \frac{1}{2}\int_0^x \frac{v(y,t)}{G(y)}dy}. \tag{15}$$

Substituting (14) into (5)–(6), we obtain

$$\begin{aligned} v_t &= v_{xx} - \left(U'(x) + \sigma^2\right) v \\ &\quad + \frac{1}{2}\left(\tilde{\omega}_0 - U(0)\tilde{u}(0) - \frac{\tilde{u}^2(0)}{2}\right) v \end{aligned} \tag{16}$$

$$v_x(0) = \tilde{\omega}_0 - \frac{1}{2}\left(\tilde{u}(0) + U(0)\right)\tilde{u}(0) \tag{17}$$

$$\begin{aligned} v_x(1) &= \left(\tilde{\omega}_1 - \frac{1}{2}\left(\tilde{u}(1) + U(1)\right)\tilde{u}(1)\right) \\ &\quad \times e^{-\frac{1}{2}\int_0^1 [\tilde{u}(y,t) + U(y)]dy}. \end{aligned} \tag{18}$$

Setting the feedback law

$$\tilde{\omega}_0 = U(0)\tilde{u}(0) + \frac{\tilde{u}^2(0)}{2} = 2\sigma \tanh(\sigma/2)\,\tilde{u}(0) + \frac{\tilde{u}^2(0)}{2} \tag{19}$$

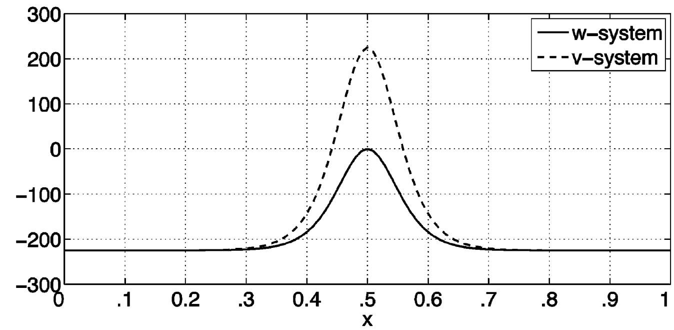

Fig. 4. Target system reaction coefficient in (24), compared with the coefficient of the v plant in (20), for $\sigma = 15$. Notice that the coefficient only has been changed close to $x = 1/2$, where it was destabilizing. The new coefficient is negative everywhere.

we obtain the following *linear* reaction-diffusion system akin to (8):

$$v_t = v_{xx} + \sigma^2 \left[\frac{2}{\cosh^2(\sigma(x - 1/2))} - 1\right] v \tag{20}$$

$$v_x(0) = \sigma \tanh(\sigma/2)\, v(0) \tag{21}$$

$$\begin{aligned} v_x(1) &= \sigma \tanh(\sigma/2)\, v(1) + \left(\tilde{\omega}_1 - \frac{\tilde{u}(1)^2}{2}\right) \\ &\quad \times \left(1 - \frac{1}{2}\int_0^1 \frac{v(y)}{G(y)}dy\right). \end{aligned} \tag{22}$$

B. *Design of $\tilde{\omega}_1$ Using the Backstepping Method*

To find the feedback $\tilde{\omega}_1$ to stabilize (20)–(22), we use backstepping parabolic PDEs [19]. We define a new state

$$w(x,t) = v(x,t) - \int_0^x k(x,y) v(y,t) dy \tag{23}$$

which we require to satisfy the *target* PDE

$$\begin{aligned} w_t &= w_{xx} + \sigma^2 \left[\frac{1}{\cosh^2(\sigma(x - 1/2))} - 1\right] w \\ &\quad - cw \end{aligned} \tag{24}$$

$$w_x(0) = \sigma \tanh(\sigma/2)\, w(0) \tag{25}$$

$$w_x(1) = -\sigma \tanh(\sigma/2)\, w(1) \tag{26}$$

where $c \geq 0$ is a control parameter. In (24) we do not eliminate the reaction term from the original system (20) but only lower it to eliminate its positive part, without wasting control to change its negative part, see Fig. 4. The coefficient c can be set to zero whenever $\sigma > 0$. When $\sigma = 0$, we need $c > 0$ because the system $w_t = w_{xx}$, $w_x(0,t) = w_x(1,t) = 0$ is only neutrally stable.

Following [19], we find that the kernel k has to verify

$$\begin{aligned} k_{xx} &= k_{yy} + \sigma^2 \left[1 - 2\tanh^2(\sigma(y - 1/2)) \right. \\ &\quad \left. + \tanh^2(\sigma(x - 1/2))\right] k + ck \end{aligned} \tag{27}$$

$$k(x,x) = -\frac{\sigma}{2}\left[\tanh(\sigma(x - 1/2)) + \tanh(\sigma/2)\right] - \frac{cx}{2} \tag{28}$$

$$k_y(x,0) = \sigma \tanh(\sigma/2)\, k(x,0) \tag{29}$$

which is a linear hyperbolic PDE in the domain $\mathcal{T} = \{(x, y) : 0 \leq y \leq x \leq 1\}$. In [19] it is shown that (27)–(29) is well-posed and that $k \in \mathcal{C}^2(\mathcal{T})$. The kernel k can be computed from (27)–(29) numerically or symbolically using procedures outlined in [19].

From (26) and (23) we obtain the condition $v_x(1) = \int_0^1 (k_x(1, y) + \sigma \tanh(\sigma/2) k(1, y)) v(y) dy + (k(1, 1) - \sigma \tanh(\sigma/2)) v(1)$. Substituting (22) we find the control law

$$\begin{aligned}\tilde{\omega}_1(t) = &\frac{\tilde{u}(1,t)^2}{2} + (k(1,1) - 2\sigma \tanh(\sigma/2))\,\tilde{u}(1,t) \\ &+ \int_0^1 (k_x(1,y) + \sigma \tanh(\sigma/2)\, k(1,y)) \\ &\times G(y) e^{\frac{1}{2}\int_y^1 \tilde{u}(\xi,t)d\xi} \tilde{u}(y,t) dy \end{aligned} \tag{30}$$

where we have used (15) to express v in terms of $\tilde{u}$.

As in [19], the transformation (23) can be inverted to obtain an expression for v in terms of w as follows:

$$v(x,t) = w(x,t) + \int_0^x l(x,y) w(y,t) dy. \tag{31}$$

The inverse kernel $l(x, y)$ satisfies a well-posed hyperbolic linear partial differential equation like (27)–(29).

VI. Stability Under Full-State Feedback

Given a function $f(x)$, we define the norm $\|f\|_{L^2(0,1)} = \left(\int_0^1 f(x)^2 dx\right)^{1/2}$ and the space $L^2(0, 1) = \{f : \|f\|_{L^2(0,1)} < \infty\}$. Similarly we define the norms $\|f\|_{H^1} = \|f'\|_{L^2} + \|f\|_{L^2}$, $\|f\|_{H^2} = \|f''\|_{L^2} + \|f'\|_{L^2} + \|f\|_{L^2}$, and the function spaces $H^1 = \{f : \|f\|_{H^1} < \infty\}$ and $H^2 = \{f : \|f\|_{H^2} < \infty\}$, where $H^2 \subset H^1 \subset L^2$. For a function $f(x, t)$, its time-varying spatial norms $\|f(t)\|$ are defined as above. Norms in time and space are given by $\|f\|_{H_T^{2,0}} = \left(\int_0^T \|f(t)\|_{H^2}^2 dt\right)^{1/2}$, $\|f\|_{H_T^{2,1}} = \|f_t\|_{H_T^{2,0}} + \|f\|_{H_T^{2,0}}$, and we denote $H^{2,0} = H_\infty^{2,0}$ and $H^{2,1} = H_\infty^{2,1}$ We denote the initial condition as $\tilde{u}_0(x) = \tilde{u}(x, 0)$, and define a class $\mathcal{K}_\infty$ [12] function $g(r) = \frac{r}{2} e^{\frac{r}{2}}$.

Theorem 1: Assume that $\tilde{u}_0 \in H^2$ is such that

$$\|\tilde{u}_0\|_{L^2} < g^{-1}\left(\frac{1}{m}\sqrt{\frac{\sigma}{\sinh \sigma}}\right) \tag{32}$$

and that it verifies compatibility with feedback boundary conditions (6), (19), (30). Then the equilibrium $\tilde{u} \equiv 0$ of system (5)–(6) with feedback laws (19) and (30) is exponentially stable in the L^2 norm, i.e., for all $t > 0$

$$\|\tilde{u}(t)\| \leq \sqrt{2\sigma \coth(\sigma/2)} \frac{g(\|\tilde{u}_0\|_{L^2})}{\frac{1}{m}\sqrt{\frac{\sigma}{\sinh \sigma}} - g(\|\tilde{u}_0\|_{L^2})} e^{-\alpha t} \tag{33}$$

where $m = \left(1 + \max_{(x,y)\in\mathcal{T}} |k(x, y)|\right)\left(1 + \max_{(x,y)\in\mathcal{T}} |l(x, y)|\right)$, and $\alpha = c + \min\{1/4, \sigma \tanh(\sigma/2)\}$. Moreover, the solution $\tilde{u}$ belongs to $H^{2,1}$.

VII. Proof of the Main Result

Lemma 1: Given $\tilde{u} \in L^2$ and v defined in terms of $\tilde{u}$ as given in (14), then $v \in L^2$ and $\|v\|_{L^2} \leq 2g(\|\tilde{u}\|_{L^2})$, where $g(r) = r/2e^{r/2}$. Moreover, if $\tilde{u} \in H^1$ (resp. H^2) then $v \in H^1$ (resp. H^2).

Proof: From the transformation (14) we obtain

$$\|v\|_{L^2}^2 \leq \max_{x\in[0,1]} |G(x)| \|\tilde{u}\|_{L^2}^2 e^{\int_0^1 |\tilde{u}(y,t)|dy} \leq \|\tilde{u}\|_{L^2}^2 e^{\|\tilde{u}\|_{L^2}} \tag{34}$$

since $\max_{x\in[0,1]} |G(x)| = 1$. The last part of the lemma is proved analogously, by taking derivatives of (14). ■

Lemma 2: Given $v \in L^2$ such that $\|v\|_{L^2} < 2\sqrt{\sigma/\sinh \sigma}$ and $\tilde{u}$ defined in terms of v as given in (15), then $\tilde{u} \in L^2$ and

$$\|\tilde{u}\|_{L^2} \leq \frac{\cosh(\sigma/2)\|v\|_{L^2}}{1 - \frac{1}{2}\sqrt{\frac{\sinh\sigma}{\sigma}}\|v\|_{L^2}}. \tag{35}$$

If $v \in H^1$ (resp. H^2) then $\tilde{u} \in H^1$ (resp. H^2).

Proof: We first show that, from the assumption $\|v\|_{L^2} \leq 2\sqrt{\sigma/\sinh \sigma}$, it follows that $\frac{1}{2}\int_0^x \frac{v(y)}{G(y)} dy < 1$, which implies that the denominator of (15) is nonzero, so $\tilde{u}$ is well-defined. Using Cauchy–Schwartz

$$\begin{aligned}\frac{1}{2}\int_0^x \frac{v(y)}{G(y)} dy &\leq \frac{1}{2}\|v\|_{L^2}\sqrt{\int_0^1 \frac{1}{G(y)^2} dy} \\ &= \frac{\cosh(\sigma/2)}{2}\|v\|_{L^2}\sqrt{\int_0^1 \frac{1}{\cosh^2(\sigma(y-1/2))} dy} \\ &= \frac{\cosh(\sigma/2)}{2}\|v\|_{L^2}\sqrt{\frac{2\tanh(\sigma/2))}{\sigma}} \\ &= \frac{1}{2}\sqrt{\frac{\sinh\sigma}{\sigma}}\|v\|_{L^2} < 1. \end{aligned} \tag{36}$$

Hence $\tilde{u}$ is well defined in (15) and

$$\begin{aligned}\|\tilde{u}(x)\|_{L^2}^2 &= \int_0^1 \left(\frac{v(x)/G(x)}{1 - \frac{1}{2}\int_0^x \frac{v(y)}{G(y)} dy}\right)^2 dx \\ &\leq \frac{\max_{x\in[0,1]} \frac{1}{G^2(x)} \|v\|_{L^2}^2}{\left(1 - \frac{1}{2}\|v\|_{L^2}\sqrt{\int_0^1 \frac{1}{G(y)^2} dy}\right)^2} \\ &= \frac{\cosh^2(\sigma/2)\|v\|_{L^2}^2}{\left(1 - \frac{1}{2}\sqrt{\frac{\sinh\sigma}{\sigma}}\|v\|_{L^2}\right)^2} \end{aligned} \tag{37}$$

since the maximum of $1/G(x)$ is $\cosh \sigma/2$. The lemma's last part is proved analogously, by taking derivatives of (15), and noting that the denominator is always the same, hence the condition $\|v\|_{L^2} < 2\sqrt{\sigma/\sinh \sigma}$ is enough to guarantee that the derivatives of $\tilde{u}$ are well-defined.

We now establish Theorem 1. Consider the system (24)–(26) with initial condition

$$\begin{aligned} w_0(x) = G(x)\tilde{u}_0(x) e^{-\frac{1}{2}\int_0^x \tilde{u}_0(y)dy} - \int_0^x k(x,y) \\ \times G(y)\tilde{u}_0(y) e^{-\frac{1}{2}\int_0^y \tilde{u}_0(\xi)d\xi} dy. \end{aligned} \tag{38}$$

Since $\tilde{u}_0 \in H^2$ and $k \in \mathcal{C}^2(\mathcal{T})$, by taking twice the derivative with respect to x in (38), $\tilde{w}_0 \in H^2$. Since w_0 verifies the boundary conditions (25)–(26), from [14],[1] we obtain that there is a unique solution $w \in H_T^{2,1}$ [i.e., the solution is defined at least for some interval $t \in (0,T)$]. Consider now the following Lyapunov functional $L(t) = \frac{1}{2}\int_0^1 w(x,t)^2 dx = \frac{1}{2}\|w(t)\|_{L^2}^2$. We have that

$$\begin{aligned}\dot{L} = &- \int_0^1 w^2\sigma^2 \tanh^2\left(\sigma\left(x-\frac{1}{2}\right)\right)dx \\ &+ \langle w, w_{xx}\rangle - c\|w\|^2 \\ \leq &- q\left(\|w_x\|^2 + \frac{1}{4}(w(1)^2+w(0)^2)\right) - c\|w\|^2 \end{aligned} \tag{39}$$

where $q = \min\{1, 4\sigma\tanh(\sigma/2)\}$. Since $w(t) \in H^1$ for some T, we can use Poincare's inequality [13], obtaining $\frac{d}{dt}L(t) \leq -\left(c+\frac{q}{4}\right)\int_0^1 w^2 dx = -\left(c+\frac{q}{4}\right)2L$. Hence, for $\alpha = c + q/4 = c + \min\{1/4, \sigma\tanh(\sigma/2)\}$, we obtain that

$$\|w(t)\|_{L^2} \leq \|w_0\|_{L^2}e^{-\alpha t}. \tag{40}$$

Similar estimates can be derived in the H^1 and H^2 norms (see [21] for examples on how to obtain those estimates), which can be used to deduce that $w \in H^{2,1}$, i.e., that the solution is defined for all $t \in (0,\infty)$.

From the w estimate of (40) we can derive estimates for v as follows. The transformation (23) and its inverse (31) determine that

$$\|w(t)\|_{L^2} \leq \left(1 + \max_{(x,y)\in\mathcal{T}}|k(x,y)|\right)\|v(t)\|_{L^2} \tag{41}$$

$$\|v(t)\|_{L^2} \leq \left(1 + \max_{(x,y)\in\mathcal{T}}|l(x,y)|\right)\|w(t)\|_{L^2} \tag{42}$$

hence $\|v(t)\|_{L^2} \leq \left(1+\max_{(x,y)\in\mathcal{T}}|l(x,y)|\right)\|w_0\|_{L^2}e^{-\alpha t} \leq m\|v_0\|_{L^2}e^{-\alpha t}$. Note that since $k, l \in \mathcal{C}^2(\mathcal{T})$, estimates in the H^1 and H^2 norms are also derived [21], and it follows that $v \in H^{2,1}$.

From Lemma VII, we get that $\|v(t)\|_{L^2} \leq 2mg(\|\tilde{u}_0\|_{L^2})e^{-\alpha t}$. Using assumption (32) in the Theorem, we obtain that $\|v(t)\|_{L^2} < 2\sqrt{\sigma/\sinh\sigma}$, thus the conditions of Lemma VII are verified and $\tilde{u}$ belongs to L^2 (and to H^1 and H^2, hence $\tilde{u} \in H^{2,1}$), and satisfies the estimate

$$\begin{aligned}\|\tilde{u}(t)\|_{L^2} &\leq \frac{\cosh(\sigma/2)\|v(t)\|_{L^2}}{1-\frac{1}{2}\sqrt{\frac{\sinh\sigma}{\sigma}}\|v(t)\|_{L^2}} \\ &\leq \frac{\cosh(\sigma/2)\,2mg(\|\tilde{u}_0\|_{L^2})e^{-\alpha t}}{1-\sqrt{\frac{\sinh\sigma}{\sigma}}mg(\|\tilde{u}_0\|_{L^2})e^{-\alpha t}}\end{aligned} \tag{43}$$

and from (43) the estimate (33) follows.

VIII. Simulations

The shock-like equilibrium of the open-loop system (1)–(2), (4) is unstable for any $\sigma > 0$. In Fig. 5 one can see it undergoing a finite time blow-up for $\sigma = 3$.

Before showing the results with our nonlinear design, we show the results with a linearized full-state backstepping controller

$$\tilde{\omega}_0(t) = 2\sigma\tanh(\sigma/2)\,\tilde{u}(0,t) \tag{44}$$

[1] We use Theorem 9.2 on page 343, which is proved for Dirichlet boundary conditions, but also stated to be valid under Robin boundary conditions (which we have here) in [14, pp. 341 and 351].

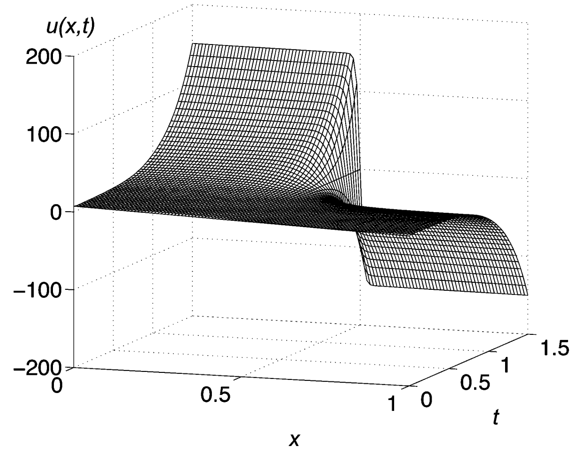

Fig. 5. Finite time blow-up of open-loop system (with constant inputs (4)) for $\sigma = 3$ and $u_0(x) = U(0) + 2 + (U(1) - U(0) - 4)x$.

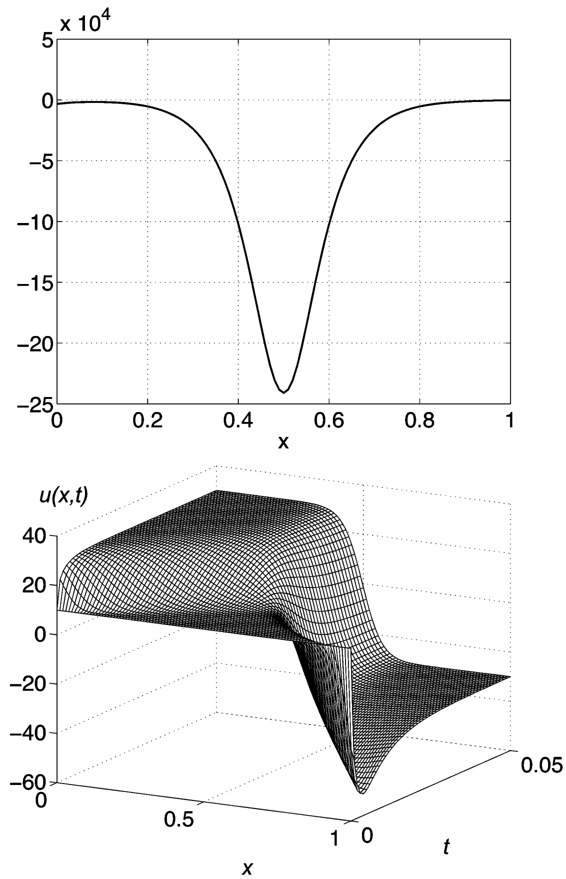

Fig. 6. The backstepping kernel $-\rho(x)$ (left) and the closed-loop solution under the linear backstepping feedback (right) for $\sigma = 15$.

$$\begin{aligned}\tilde{\omega}_1(t) = &-\left(3\sigma\tanh(\sigma/2)+\frac{c}{2}\right)\tilde{u}(1,t) \\ &- \int_0^1 \rho(y)G(y)\tilde{u}(y,t)dy.\end{aligned} \tag{45}$$

The kernel $-\rho(x)$ and a closed-loop solution with the controller (44) and (45) are shown in Fig. 6. In Fig. 7 we show solutions from various initial conditions under *nonlinear* controller (19), (30).

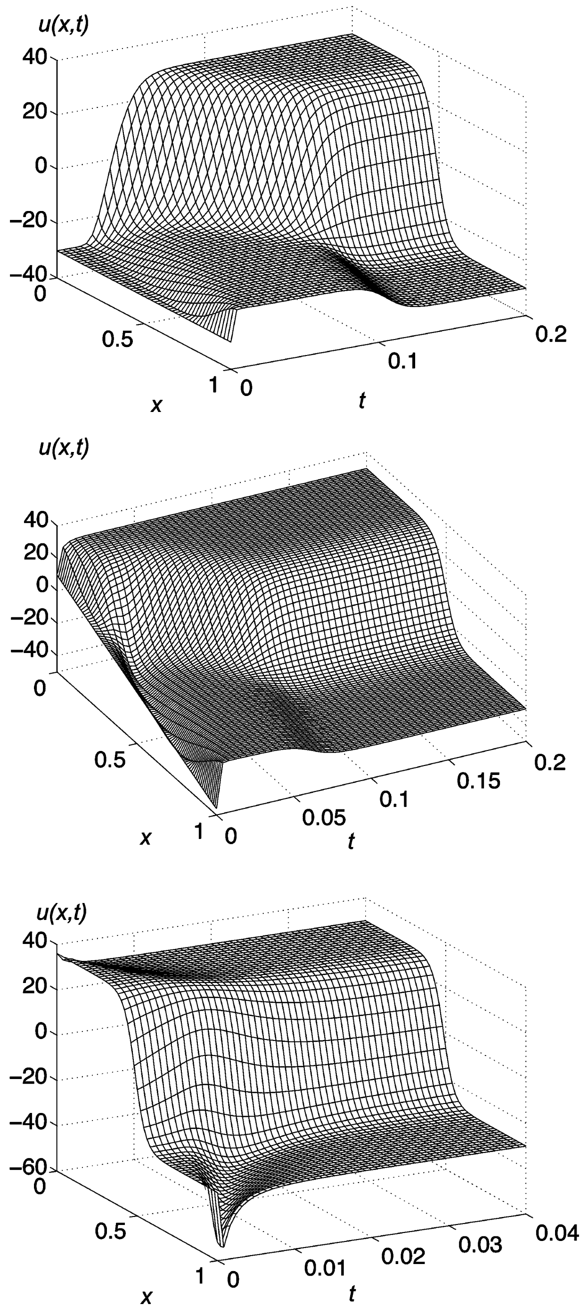

Fig. 7. Convergence of closed-loop system under the nonlinear full-state feedback for various initial conditions and $\sigma = 15$.

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