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Adaptive identification of two unstable PDEs with boundary sensing and actuation

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SUMMARY

In this paper we consider a problem of on-line parameter identification of parabolic partial differential equations (PDEs). In the previous study, on the actuation side, both distributed (*SIAM J. Optim. Control* 1997; **35**:678–713; *IEEE Trans. Autom. Control* 2000; **45**:203–216) and boundary (*IEEE Trans. Autom. Control* 2000; **45**:203–216) actuations were considered in the open loop, whereas for the closed loop (unstable plants) only distributed one was addressed. On the sensing side, only distributed sensing was considered. The present study goes beyond the identification framework of (*SIAM J. Optim. Control* 1997; **35**:678–713; *IEEE Trans. Autom. Control* 2000; **45**:203–216) by considering boundary actuation for the unstable plants, resulting in the closed-loop identification, and also introducing boundary sensing. This makes the proposed technique applicable to a much broader range of practical problems. As a first step towards the identification of general reaction–advection–diffusion systems, we consider two benchmark plants: one with an uncertain parameter in the domain and the other with an uncertain parameter on the boundary. We design the adaptive identifier that consists of standard gradient/least-squares estimators and backstepping adaptive controllers. The parameter estimates are shown to converge to the true parameters when the closed-loop system is excited by an additional constant input at the boundary. The results are illustrated with simulations. Copyright © 2008 John Wiley & Sons, Ltd.

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1. INTRODUCTION

In distributed parameter systems with thermal, fluid, or chemical dynamics, which are usually modeled by parabolic partial differential equations (PDEs), it is often of interest to identify the physical parameters of the plant from the existing data. For a large class of plants, this problem

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has been considered in [1, 2] under the assumption that distributed actuation and measurements are available. Problems with boundary actuation have also been addressed in [2]; however, only open-loop identification has been considered, thus limiting the results to stable plants. In this paper we extend the previous study in two ways. First, we consider boundary-actuated plants that are open-loop unstable. Second, we assume that only boundary measurements are available.

We consider two benchmark plants,‡ one with the unknown parameter in the domain:

$$u_t(x,t) = u_{xx}(x,t) + gu(0,t), \quad 0<x<1 \tag{1}$$

$$u_x(0,t) = 0 \tag{2}$$

and the other with the unknown parameter in the boundary condition:

$$u_t(x,t) = u_{xx}(x,t), \quad 0<x<1 \tag{3}$$

$$u_x(0,t) = -qu(0,t) \tag{4}$$

Both plants are actuated from the boundary:

$$u(1,t) = U(t) \tag{5}$$

and we assume that only $u(0,t)$ is measured. Both plants are of infinite relative degree and are unstable for $g>2$ and $q>1$, respectively. Our objective is to identify the unknown parameters g and q.

The above plants capture significant features of the model of thermal instability in solid propellant rockets [4]. However, our primary motivation to study these systems comes from a recent paper [5], where it was shown that a general reaction–advection problem

$$v_t(x,t) = v_{xx}(x,t) + b(x)v_x(x,t) + \lambda(x)v(x,t) \tag{6}$$

$$v_x(0,t) = 0 \tag{7}$$

with output $v(0,t)$ and input $v(1,t)$ can be transformed into the form

$$u_t(x,t) = u_{xx}(x,t) + g(x)u(0,t) \tag{8}$$

$$u_x(0,t) = -qu(0,t) \tag{9}$$

while keeping input and output the same: $u(1,t) = v(1,t)$, $u(0,t) = v(0,t)$. The finite-dimensional analog of the form (8), (9) is the *observer canonical form*, in which parametric uncertainties multiply the measured output. The comparison of (1)–(4) with (8), (9) reveals that the benchmark problems presented here serve as a first step to the output-feedback adaptive identification of reaction–advection systems (6), (7).

The paper is organized as follows. In Section 2 we carry on identifiability analysis and design the open-loop identification scheme for the case of stable plants. The closed-loop identification scheme for the plant with parametric uncertainty in the domain is designed in Section 3. In Sections 4 and 5 we present the least-squares identification scheme for the plant with unknown parameter

‡The shortened version of the design for one of the plants, (1)–(2), was presented at the 45th Conference on Decision and Control, 2006 [3].

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in the boundary condition. In Section 6 we show numerical simulations that support theoretical results. Finally, we discuss further research directions in Section 7.

Notation: The spatial $L_2(0,1)$ norm is denoted by $\|\cdot\|$. The temporal norms are denoted by $\mathscr{L}_\infty$ and $\mathscr{L}_2$ for $t\geqslant 0$. We denote by l_1 a generic function in $\mathscr{L}_\infty \cap \mathscr{L}_1$.

2. OPEN-LOOP IDENTIFICATION

For simplicity, we start our investigation with a case of open-loop stable plants. We will only consider plant (1), (2), (5) in this section; the open-loop analysis for the other plant (3), (4), (5) is very similar.

The parameter g of plant (1), (2), (5) *is said to be identifiable* from the measured output $u(0,t)$ if there exists a control input $U(t)$ such that for initial conditions $u(x,0)\in L_2(0,1)$ and $\tilde{u}(x,0)\in L_2(0,1)$ of the plant and its model

$$\tilde{u}_t(x,t)=\tilde{u}_{xx}(x,t)+\tilde{g}\tilde{u}(0,t) \tag{10}$$

$$\tilde{u}_x(0,t)=0 \tag{11}$$

$$\tilde{u}(1,t)=U(t) \tag{12}$$

the identity $u(0,t)\equiv\tilde{u}(0,t)$ results in $g=\tilde{g}$.

2.1. Identifiability analysis

Lemma 1

Let system (1)–(2), (5) with $U(t)\equiv 0$ be asymptotically stable. Then, when forced by a constant input $U(t)=u_1\neq 0$, system (1)–(2), (5) possesses a time-invariant steady-state solution:

$$u^{\mathrm{ss}}(x)=u_1\frac{2-gx^2}{2-g} \tag{13}$$

such that $u^{\mathrm{ss}}(0)=u_0\neq 0$.

Proof

The steady-state solution has to satisfy the following ordinary differential equation (ODE):

$$\begin{aligned}\frac{\mathrm{d}^2u^{\mathrm{ss}}(x)}{\mathrm{d}x^2}&=-gu^{\mathrm{ss}}(0)\\ \frac{\mathrm{d}}{\mathrm{d}x}u^{\mathrm{ss}}(0)&=0,\quad u^{\mathrm{ss}}(1)=u_1\end{aligned} \tag{14}$$

This two-point boundary value problem has a unique solution given by (13). Introducing the error variable $\omega(x,t)=u(x,t)-u^{\mathrm{ss}}(x)$, we get the system

$$\omega_t(x,t)=\omega_{xx}(x,t)+g\omega(0,t) \tag{15}$$

$$\omega_x(0,t)=\omega(1,t)=0 \tag{16}$$

which is asymptotically stable by assumption, so that (13) is indeed a steady-state solution of (1)–(2), (5). To complete the proof it remains to note that $u^{\mathrm{ss}}(0)\neq 0$.

We are now in a position to state the identifiability result. □

Theorem 1
The parameter g of system (1)–(2), (5) is identifiable from the measured output $u(0,t)$ with an arbitrary constant input $U(t)=u_1\neq 0$ under the assumption $g<2$.

Proof
Let us fix a constant input $U(t)=u_1\neq 0$. Since $g<2$, system (1)–(2), (5) has asymptotically stable internal dynamics. Let us assume that the model system (10)–(12) produces the same output as the original system: $u(0,t)=\tilde{u}(0,t)$ for all $t\geqslant 0$. This implies that system (10)–(12) also has asymptotically stable internal dynamics. By Lemma 1, both systems (1)–(2), (5) and (10)–(12) have time-invariant steady-state solutions $u^{\mathrm{ss}}(x)$ and $\tilde{u}^{\mathrm{ss}}(x)$, respectively, and these solutions take non-zero values $u^{\mathrm{ss}}(0)\neq 0$ and $\tilde{u}^{\mathrm{ss}}(0)\neq 0$ at $x=0$. Setting $\delta u(x,t)=u(x,t)-\tilde{u}(x,t)$, $\delta g=g-\tilde{g}$, we obtain

$$\delta u_t(x,t)=\delta u_{xx}(x,t)+\delta g u(0,t) \tag{17}$$

$$\delta u_x(0,t)=\delta u(1,t)=0 \tag{18}$$

This system has a steady-state solution $\delta u^0(x)=u^{\mathrm{ss}}(x)-\tilde{u}^{\mathrm{ss}}(x)$, which satisfies

$$\delta u^0_{xx}+\delta g u_0=0 \tag{19}$$

$$\delta u^0_x(0)=\delta u^0(1)=0 \tag{20}$$

where $u_0=u^{\mathrm{ss}}(0)\neq 0$. It follows that

$$\delta u^0(x)=\tfrac{1}{2}\delta g u_0(1-x^2) \tag{21}$$

Since $\delta u(0,t)=0$ by assumption, we get $\delta u^0(0)=0$, and therefore from (21) we obtain $\delta g u_0=0$. Since $u_0\neq 0$ it follows that $\delta g=0$. Hence, $g=\tilde{g}$ and the parameter g is identifiable under an arbitrary non-zero constant input $U(t)=u_1\neq 0$. The proof is completed. □

2.2. Adaptive identifier design

We propose the following gradient update law with normalization for the on-line identification of the parameter g [4]:

$$\dot{\hat{g}}=\gamma\frac{\hat{e}(0,t)v(0,t)}{1+v^2(0,t)} \tag{22}$$

where $\gamma>0$ is the adaptation gain. The above identifier utilizes the prediction error

$$\hat{e}(x,t)=u(x,t)-\hat{g}(t)v(x,t)-\eta(x,t) \tag{23}$$

evaluated at $x=0$, and the filters

$$v_t(x,t)=v_{xx}(x,t)+u(0,t) \tag{24}$$

$$v_x(0,t)=v(1,t)=0 \tag{25}$$

$$\eta_t(x,t)=\eta_{xx}(x,t) \tag{26}$$

$$\eta_x(0,t)=0 \tag{27}$$

$$\eta(1,t)=u(1,t) \tag{28}$$

such that the boundary value problem

$$e_t(x,t)=e_{xx}(x,t) \tag{29}$$

$$e_x(0,t)=e(1,t)=0 \tag{30}$$

expressed in terms of the error $e(x,t)=u(x,t)-gv(x,t)-\eta(x,t)$ is asymptotically stable.

Theorem 2

Consider plant (1), (2), (5) with filters (24)–(28) and the update law (22). Let $U(t)=u_1\neq 0$. Then for any initial conditions $u(x,0), v(x,0), \eta(x,0)\in L_2(0,1)$, $\hat{g}(0)\in \mathbf{R}$ and any adaptation gain $\gamma>0$, the signals $\hat{g}(t)$, $u(x,t)$, $v(x,t)$, $\eta(x,t)$ are uniformly bounded for $t>0$ and $\lim_{t\to\infty}\hat{g}(t)=g$.

Proof

1. Following the line of reasoning used in the proof of Lemma 1, one shows that under a constant input $U(t)=u_1\neq 0$ the linear time-invariant system (1), (2), (5), (24)–(25) has a time-invariant steady-state solution:

$$u^{\text{ss}}(x)=u_1\frac{2-gx^2}{2-g},\quad v^{\text{ss}}(x)=u_1\frac{1-x^2}{2-g} \tag{31}$$

The states u, v, η of heat equations (1), (2), (5), (24)–(28) with bounded inputs are uniformly bounded for $t>0$.

2. Representing the update law (22) in terms of the deviation from the nominal value

$$\frac{\mathrm{d}}{\mathrm{d}t}\Delta g=-\gamma\frac{\hat{e}(0,t)v(0,t)}{1+v^2(0,t)} \tag{32}$$

let us introduce the positive definite functional

$$V(e,\Delta g)=\frac{1}{2}\int_0^1 e^2(x,t)\,\mathrm{d}x+\frac{1}{2\gamma}[\Delta g(t)]^2 \tag{33}$$

and compute its time derivative along the solutions of system (29), (30), (32):

$$\begin{aligned}\dot{V}(t)&=-\int_0^1 e_x^2(x,t)\,\mathrm{d}x-\frac{\Delta g(t)\hat{e}(0,t)v(0,t)}{1+v^2(0,t)}\\&=-\int_0^1 e_x^2(x,t)\,\mathrm{d}x-\frac{\hat{e}^2(0,t)}{1+v^2(0,t)}+\frac{\hat{e}(0,t)e(0,t)}{1+v^2(0,t)}\\&\leqslant-\|e_x(\cdot,t)\|^2-\frac{\hat{e}^2(0,t)}{1+v^2(0,t)}+\frac{|\hat{e}(0,t)|\|e_x(\cdot,t)\|}{\sqrt{1+v^2(0,t)}}\\&\leqslant-\frac{1}{2}\|e_x(\cdot,t)\|^2-\frac{1}{2}\frac{\hat{e}^2(0,t)}{1+v^2(0,t)}\end{aligned} \tag{34}$$

Therefore, V is bounded and thus $\Delta g(t)$ and $\hat{g}(t)$ are bounded.

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3. By taking into account the relation

$$\hat{e}(x,t)=e(x,t)+\Delta g(t)v(x,t) \tag{35}$$

resulting from (23), (30), the update law (22), rewritten in the form

$$\frac{\mathrm{d}}{\mathrm{d}t}\Delta g=-\gamma\frac{[e(0,t)+\Delta g(t)v(0,t)]v(0,t)}{1+v^2(0,t)} \tag{36}$$

and coupled to system (29), (30), turns out to be asymptotically autonomous because

$$\lim_{t\to\infty} v(0,t)=v^{\mathrm{ss}}(0)=\frac{u_1}{2-g}\neq 0 \tag{37}$$

4. By extending the invariance principle [6, Chapter VIII] to the asymptotically autonomous parabolic system (29), (30), (36) (in analogy to that of [7, Theorem 4.3.4]), there must occur a convergence of the system trajectories to the maximal invariant subset of a set of solutions of (29), (30), (36), for which $\dot{V}(t)=0$. Since (29), (30) is exponentially stable, we get

$$e(x,t)=0,\quad \hat{e}(0,t)=0,\quad \Delta g(t)v(0,t)=\hat{e}(0,t)-e(0,t)=0 \tag{38}$$

Owing to (37), we get $\lim_{t\to\infty}\Delta g(t)=0$. The theorem is proved. □

3. CLOSED-LOOP IDENTIFICATION: PLANT WITH UNKNOWN PARAMETER IN THE DOMAIN

When the internal dynamics of (1), (2), (5) are unstable, we propose the following boundary controller:

$$u(1,t)=u_1+\int_0^1 \hat{k}(1,\xi)(\hat{g}v(\xi,t)+\eta(\xi,t))\,\mathrm{d}\xi \tag{39}$$

$$\hat{k}(x,\xi)=\begin{cases}-\sqrt{\hat{g}}\sinh\sqrt{\hat{g}}(x-\xi), & \hat{g}\geqslant 0\\ \sqrt{-\hat{g}}\sin\sqrt{-\hat{g}}(x-\xi), & \hat{g}<0\end{cases} \tag{40}$$

coupled to the filters (24)–(28) and the gradient update law (22). Note that $\hat{k}(x,\xi)$ depends on time through $\hat{g}$. The above controller with $u_1=0$ was introduced in [8] (the nominal controller was designed in [9] using the backstepping method). The present controller modification with a constant component $u_1\neq 0$ makes the control signal sufficiently rich to persistently excite the system, yielding the desired parameter convergence.

Our main result is summarized as follows.

Theorem 3

Consider system (1), (2) with the boundary controller (39), (40) where $u_1\neq 0$, (24)–(28) are filters, and the update law is given by (22). Then for any $\hat{g}(0)\in\mathbf{R}$ and any initial conditions u_0, v_0, $\eta_0\in L_2(0,1)$, the signals $\hat{g}(t)$, $u(x,t)$, $v(x,t)$, $\eta(x,t)$ are uniformly bounded for $t>0$ and $\hat{g}(t)\to g$ as $t\to\infty$.

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Proof
We break the proof into several steps.

1. It is straightforward to verify that estimate (34) of the time derivative of the positive definite functional (33), computed on the solutions of (22), (29), (30), is still satisfied. This yields the following properties:

$$\frac{\hat{e}(0,t)}{\sqrt{1+v^2(0,t)}}\in\mathscr{L}_2,\quad \Delta g\in\mathscr{L}_\infty \tag{41}$$

Since

$$\frac{\hat{e}(0,t)}{\sqrt{1+v^2(0,t)}}=\frac{e(0,t)}{\sqrt{1+v^2(0,t)}}+\Delta g\frac{v(0,t)}{\sqrt{1+v^2(0,t)}} \tag{42}$$

$$\dot{\hat{g}}=\gamma\frac{\hat{e}(0,t)}{\sqrt{1+v^2(0,t)}}\frac{v(0,t)}{\sqrt{1+v^2(0,t)}} \tag{43}$$

we get

$$\frac{\hat{e}(0,t)}{\sqrt{1+v^2(0,t)}}\in\mathscr{L}_2\cap\mathscr{L}_\infty,\quad \Delta g\in\mathscr{L}_\infty,\quad \dot{\hat{g}}\in\mathscr{L}_2\cap\mathscr{L}_\infty \tag{44}$$

2. Consider the transformation

$$\hat{w}(x,t)=\hat{g}v(x,t)+\eta(x,t)-\int_0^x\hat{k}(x,\xi)(\hat{g}v(\xi,t)+\eta(\xi,t))\,\mathrm{d}\xi \tag{45}$$

with $\hat{k}(x,\xi)$ given by (40). One can show that (45) maps (1), (2), (23)–(28), (39) into the following system (see Lemma 4 in Appendix A):

$$\hat{w}_t(x,t)=\hat{w}_{xx}(x,t)+\beta(x)\hat{e}(0,t)+\dot{\hat{g}}v(x,t)+\dot{\hat{g}}\int_0^x\alpha(x-\xi)\left(\hat{g}v(\xi,t)+\hat{w}(\xi,t)\right)\mathrm{d}\xi \tag{46}$$

$$\hat{w}_x(0,t)=0 \tag{47}$$

$$\hat{w}(1,t)=u_1 \tag{48}$$

where

$$\alpha(x)=\begin{cases}\dfrac{1}{\sqrt{\hat{g}}}\sinh\sqrt{\hat{g}}x, & \hat{g}\geqslant 0,\\[2ex] \dfrac{1}{\sqrt{-\hat{g}}}\sin\sqrt{-\hat{g}}x, & \hat{g}<0,\end{cases}\qquad \beta(x)=\hat{k}_\xi(x,0)=\begin{cases}\hat{g}\cosh\sqrt{\hat{g}}x, & \hat{g}\geqslant 0\\ \hat{g}\cos\sqrt{-\hat{g}}x, & \hat{g}<0\end{cases} \tag{49}$$

Using the fact that $u(0,t)=\hat{w}(0,t)+\hat{e}(0,t)$, let us rewrite filter (24), (25) in the form

$$v_t(x,t)=v_{xx}(x,t)+\hat{w}(0,t)+\hat{e}(0,t) \tag{50}$$

$$v_x(0,t)=v(1,t)=0 \tag{51}$$

We now have two interconnected systems (46)–(48) and (50)–(51) for $\hat{w}$ and v, which are driven by three external signals: a constant u_1 and signals $\hat{e}(0,t)$, $\dot{\hat{g}}(t)$ with properties (44).

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3. Our next goal is to demonstrate that v-system and $\hat{w}$-system are asymptotically stable around the limit points

$$\hat{w}^{\mathrm{lp}}(x)=u_1, \quad v^{\mathrm{lp}}(x)=u_1\frac{1-x^2}{2} \tag{52}$$

Let us introduce the error variables $\bar{w}=\hat{w}-\hat{w}^{\mathrm{lp}}$, $\bar{v}=v-v^{\mathrm{lp}}$. The equations for $\bar{w}$ and $\bar{v}$ are

$$\bar{w}_t(x,t)=\bar{w}_{xx}(x,t)+\beta(x)\hat{e}(0,t)+\dot{\hat{g}}\bar{v}(x,t)+\dot{\hat{g}}\int_0^x \alpha(x-\xi)\left(\hat{g}\bar{v}(\xi,t)+\bar{w}(\xi,t)\right)\mathrm{d}\xi$$

$$+\dot{\hat{g}}\beta(x)\frac{u_1}{2\hat{g}} \tag{53}$$

$$\bar{w}_x(0,t)=\bar{w}(1,t)=0 \tag{54}$$

and

$$\bar{v}_t(x,t)=\bar{v}_{xx}(x,t)+\bar{w}(0,t)+\hat{e}(0,t) \tag{55}$$

$$\bar{v}_x(0,t)=\bar{v}(1,t)=0 \tag{56}$$

Consider a Lyapunov function

$$V_1=\frac{1}{2}\int_0^1 \bar{v}^2(x)\,\mathrm{d}x+\frac{1}{2}\int_0^1 \bar{v}_x^2(x)\,\mathrm{d}x \tag{57}$$

Using Young's, Poincare's, and Agmon's inequalities[§] we have[¶]

$$\begin{aligned}
\dot{V}_1 &= -\int_0^1 \bar{v}_x^2\,\mathrm{d}x+(\bar{w}(0)+\hat{e}(0))\int_0^1 \bar{v}\,\mathrm{d}x-\int_0^1 \bar{v}_{xx}^2\,\mathrm{d}x-(\bar{w}(0)+\hat{e}(0))\int_0^1 \bar{v}_{xx}\,\mathrm{d}x\\
&\leqslant -\|\bar{v}_x\|^2+\frac{1}{8}\|\bar{v}\|^2+4\frac{\hat{e}^2(0)}{1+v^2(0)}(1+v^2(0))\\
&\quad +4\|\bar{w}_x\|^2-\|\bar{v}_{xx}\|^2+\frac{1}{2}\|\bar{v}_{xx}\|^2+\|\bar{w}_x\|^2+\frac{\hat{e}^2(0)}{1+v^2(0)}(1+v^2(0))\\
&\leqslant -\frac{1}{2}\|\bar{v}_x\|^2-\frac{1}{2}\|\bar{v}_{xx}\|^2+5\|\bar{w}_x\|^2+5\frac{\hat{e}^2(0)}{1+v^2(0)}\left(1+2\|\bar{v}_x\|^2+\frac{u_1^2}{2}\right)\\
&\leqslant -\frac{1}{2}\|\bar{v}_x\|^2-\frac{1}{2}\|\bar{v}_{xx}\|^2+5\|\bar{w}_x\|^2+l_1\|\bar{v}_x\|^2+l_1
\end{aligned} \tag{58}$$

where by l_1 we denote a generic function of time in $\mathscr{L}_1\cap\mathscr{L}_\infty$.

§We use the following versions of these inequalities: $\|w(x)\|\leqslant 2\|w_x\|$, $|w(0)|\leqslant\|w_x\|$, $w^2(x)\leqslant 4\|w\|\|w_x\|$, where w is any function satisfying $w(1)=0$.

¶We drop the dependence on time in the proofs to simplify the notation.

Using the following Lyapunov function for the $\bar{w}$–system

$$V_2 = \frac{1}{2}\int_0^1 \bar{w}^2\,\mathrm{d}x \tag{59}$$

we get

$$\begin{aligned}\dot{V}_2 = & -\int_0^1 \bar{w}_x^2\,\mathrm{d}x + \hat{e}(0)\int_0^1 \beta\bar{w}\,\mathrm{d}x + \dot{\hat{g}}\int_0^1 \bar{w}\bar{v}\,\mathrm{d}x + \dot{\hat{g}}\int_0^1 \bar{w}(x)v^{\mathrm{lp}}(x)\,\mathrm{d}x \\ & + \dot{\hat{g}}\int_0^1 \bar{w}(x)\int_0^x \alpha(x-y)(\hat{g}v^{\mathrm{lp}}(y)+u_1+\hat{g}\bar{v}(y)+\bar{w}(y))\,\mathrm{d}y\,\mathrm{d}x\end{aligned} \tag{60}$$

Before we proceed, we note that Δg is bounded and therefore $\hat{g}$ is also bounded; let us denote this bound by g_0. The functions α and β are also bounded; let us denote these bounds by α_0 and β_0. With the help of Young's, Poincare's, and Agmon's inequalities, we get the following estimate:

$$\begin{aligned}\dot{V}_2 \leqslant & -\|\bar{w}_x\|^2 + \frac{c_1}{2}\|\bar{w}\|^2 + \frac{|\dot{\hat{g}}|^2}{8c_1}u_1^2\cosh^2(\sqrt{g_0}) + c_1\|\bar{w}\|^2 + \frac{|\dot{\hat{g}}|^2(1+\alpha_0 g_0)^2}{2c_1}\|\bar{v}\|^2 \\ & + \frac{\beta_0^2}{2c_1}\frac{\hat{e}^2(0)}{1+v^2(0)}\left(1+\|\bar{v}_x\|^2+u_1^2\right) + c_1\|\bar{w}\|^2 + \frac{|\dot{\hat{g}}|^2\alpha_0^2}{2c_1}\|\bar{w}\|^2 \\ \leqslant & -(1-10c_1)\|\bar{w}_x\|^2 + l_1\|\bar{w}\|^2 + l_1\|\bar{v}_x\|^2 + l_1\end{aligned} \tag{61}$$

Choosing $c_1 = \frac{1}{40}$ and using the Lyapunov function $V = V_2 + \frac{1}{20}V_1$, we get

$$\begin{aligned}\dot{V} & \leqslant -\frac{1}{2}\|\bar{w}_x\|^2 - \frac{1}{40}\|\bar{v}_x\|^2 - \frac{1}{40}\|\bar{v}_{xx}\|^2 + l_1\|\bar{w}\|^2 + l_1\|\bar{v}_x\|^2 + l_1 \\ & \leqslant -\frac{1}{4}V + l_1 V + l_1\end{aligned} \tag{62}$$

and by Lemma 3 we obtain boundedness and square integrability of $\|\bar{w}\|$, $\|\bar{v}\|$, and $\|\bar{v}_x\|$. Using these properties we can compute

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\bar{w}_x\|^2 & \leqslant -\|\bar{w}_{xx}\|^2 + \beta_0|\hat{e}(0)|\|\bar{w}_{xx}\| + |\dot{\hat{g}}|\|\bar{w}_{xx}\|((1+\alpha_0 g_0)\|\bar{v}\| + \alpha_0\|\bar{w}\|) \\ & \quad + |\dot{\hat{g}}|\|\bar{w}_{xx}\||u_1|\frac{\cosh^2(\sqrt{g_0})}{2} \\ & \leqslant -\frac{1}{8}\|\bar{w}_x\|^2 + l_1\end{aligned} \tag{63}$$

so that by Lemma 3 $\|\bar{w}_x\| \in \mathscr{L}_2 \cap \mathscr{L}_\infty$. Using the fact that $\|\bar{v}_x\|$, $\|\bar{w}_x\|$ are bounded, it is easy to see that

$$\left|\frac{\mathrm{d}}{\mathrm{d}t}(\|\bar{v}\|^2 + \|\bar{w}\|^2)\right| < \infty \tag{64}$$

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By Barbalat's lemma we get $\|\bar{w}\| \to 0$, $\|\bar{v}\| \to 0$. By Agmon's inequality, we have

$$\max_{x\in[0,1]} |\bar{v}| \leqslant 2\|\bar{v}\|\|\bar{v}_x\| \tag{65}$$

and therefore $\bar{v}(x,t)$ is uniformly bounded and is regulated to zero as $t\to\infty$. By the same argument we get the boundedness and regulation of $\bar{w}(x,t)$. Thus, we proved that v-system and $\hat{w}$-system are globally asymptotically stable around the limit points $v^{\mathrm{lp}}(x)$ and $w^{\mathrm{lp}}(x)$, respectively.

4. In order to show the boundedness of $\eta(x,t)$ and $u(x,t)$, we express η in terms of v and $\hat{w}$ with the inverse transformation to (45):

$$\hat{g}v(x,t)+\eta(x,t) = \hat{w}(x,t) - \hat{g}\int_0^x (x-\xi)\hat{w}(\xi,t)\,\mathrm{d}\xi \tag{66}$$

Since $v(x,t)$ and $\hat{w}(x,t)$ are bounded, we see from (66) that $\eta(x,t)$ is also bounded. Finally, the boundedness of $u(x,t)$ is obtained from the relationship $u = e + gv + \eta$.

5. We showed that

$$\lim_{t\to\infty} v(0,t) = v^{\mathrm{lp}}(0) = \frac{u_1}{2} \neq 0 \tag{67}$$

Thus, the update law (22), being rewritten in the form (36) and coupled to the parabolic system (29), (30), turns out to be asymptotically autonomous. To complete the proof it remains to apply the invariance principle to the asymptotically autonomous system (29), (30), (36) and note that due to (67), the convergence of the solution to the maximal invariant set $\Delta g(t)v(0,t) = 0$ results in $\lim_{t\to\infty} \Delta g(t) = 0$. Theorem 3 is thus proved.

In the proof we presented, we assumed the existence and uniqueness of the solution of the nonlinear closed-loop system (1), (2), (22)–(28), (39), which is defined in a mild sense as the solution of a corresponding integral equation expressed by means of a strongly continuous semigroup, generated by an appropriate infinitesimal operator. □

4. CLOSED-LOOP IDENTIFICATION: PLANT WITH UNKNOWN PARAMETER IN THE BOUNDARY CONDITION

Consider the following plant:

$$u_t(x,t) = u_{xx}(x,t) \tag{68}$$

$$u_x(0,t) = -qu(0,t) \tag{69}$$

$$u(1,t) = U(t) \tag{70}$$

where $U(t)$ is the control signal. With $U(t)\equiv 0$ this PDE is unstable for $q>1$. One can establish identifiability of (68)–(70) for $q<1$ with constant non-zero input following the approach in Section 2. We are going to skip this calculation and start directly with the unstable case.

For the case of known q the transformation

$$w(x,t) = u(x,t) + \int_0^x q e^{q(x-\xi)} u(\xi,t)\,\mathrm{d}\xi \tag{71}$$

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along with the feedback

$$U(t) = -\int_0^1 q e^{q(1-\xi)} u(\xi, t)\,\mathrm{d}\xi \tag{72}$$

was used in [8] to map (68)–(70) into the target system

$$w_t(x,t) = w_{xx}(x,t) \tag{73}$$

$$w_x(0,t) = w(1,t) = 0 \tag{74}$$

The first step in our identification algorithm is to design the least-squares adaptive identifier.

4.1. Least-squares adaptive identifier

We first introduce input and output filters:

$$v_t(x,t) = v_{xx}(x,t) \tag{75}$$

$$v_x(0,t) = -u(0,t) \tag{76}$$

$$v(1,t) = 0 \tag{77}$$

$$\eta_t(x,t) = \eta_{xx}(x,t) \tag{78}$$

$$\eta_x(0,t) = 0 \tag{79}$$

$$\eta(1,t) = u(1,t) \tag{80}$$

The error $e = u - qv - \eta$ satisfies the exponentially stable heat equation (29)–(30). Let us define the prediction error as

$$\hat{e}(x,t) = u(x,t) - \hat{q}v(x,t) - \eta(x,t) \tag{81}$$

The least-squares update law is

$$\dot{\hat{q}}(t) = \gamma(t)\frac{\hat{e}(0,t)v(0,t)}{1+\gamma(t)v^2(0,t)} \tag{82}$$

$$\dot{\gamma}(t) = -\frac{\gamma^2(t)v^2(0,t)}{1+\gamma(t)v^2(0,t)}, \quad \gamma(0) > 0 \tag{83}$$

Lemma 2
The identifier (75)–(83) guarantees the following:

1. $0 < \gamma(t) < \infty$, $|\dot{\gamma}(t)| < \infty$ for all $t \geqslant 0$.
2. $\Delta q(t)$ is bounded.
3. $\hat{e}(0)/\sqrt{1+v^2(0,t)} \in \mathscr{L}_2 \cap \mathscr{L}_\infty$ and $\dot{\hat{q}} \in \mathscr{L}_2 \cap \mathscr{L}_\infty$.
4. There exist γ_∞, q_∞ such that $\lim_{t\to\infty} \gamma(t) = \gamma_\infty$, $\lim_{t\to\infty} q(t) = q_\infty$.

Proof
1. Rewriting (83) as

$$\frac{\mathrm{d}}{\mathrm{d}t}(\gamma(t)^{-1}) = \frac{\gamma(t)^{-1}v^2(0,t)}{\gamma(t)^{-1}+v^2(0,t)} \tag{84}$$

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we can see that $\gamma(t)^{-1}\geqslant\gamma(0)^{-1}>0$. Therefore, $\gamma(t)$ is bounded and positive. From (83) we get $|\dot{\gamma}(t)|\leqslant|\gamma(t)|<\infty$ for all $t\geqslant 0$.

2. Consider the Lyapunov function

$$V=\frac{1}{2}\|e\|^2+\frac{1}{2\gamma(t)}\Delta q^2 \tag{85}$$

where $\Delta q=q-\hat{q}$. We get

$$\begin{aligned}\dot{V}&=-\|e_x\|^2+\frac{1}{2}\frac{\Delta q^2 v^2(0)}{1+\gamma v^2(0)}-\frac{\Delta q v(0)\hat{e}(0)}{1+\gamma v^2(0)}=-\|e_x\|^2+\frac{1}{2}\frac{\Delta q v(0)}{1+\gamma v^2(0)}(\Delta q v(0)-2\hat{e}(0))\\&=-\|e_x\|^2+\frac{1}{2}\frac{(\hat{e}(0)-e(0))(-e(0)-\hat{e}(0))}{1+\gamma v^2(0)}=-\|e_x\|^2+\frac{1}{2}\frac{e^2(0)-\hat{e}^2(0)}{1+\gamma(t)v^2(0)}\\&\leqslant-\frac{1}{2}\|e_x\|^2-\frac{1}{2}\frac{\hat{e}^2(0)}{1+\gamma(t)v^2(0)}\end{aligned} \tag{86}$$

Therefore, V is bounded, which in turn implies that Δq is bounded.

3. Integrating (86) in time, we get $\hat{e}(0)/\sqrt{1+\gamma(t)v^2(0,t)}\in\mathscr{L}_2$. We also get

$$\frac{\hat{e}(0,t)}{\sqrt{1+v^2(0,t)}}=\frac{\hat{e}(0,t)}{\sqrt{1+\gamma(t)v^2(0,t)}}\frac{\sqrt{1+\gamma(t)v^2(0,t)}}{\sqrt{1+v^2(0,t)}}\leqslant\frac{\hat{e}(0,t)\sqrt{1+\gamma(0)}}{\sqrt{1+\gamma(t)v^2(0,t)}}\in\mathscr{L}_2 \tag{87}$$

and

$$\frac{\hat{e}(0,t)}{\sqrt{1+v^2(0,t)}}=\frac{e(0,t)}{\sqrt{1+v^2(0,t)}}+\frac{\Delta q v(0,t)}{\sqrt{1+v^2(0,t)}}<\infty \tag{88}$$

From (82) we have

$$\dot{\hat{q}}=\frac{\hat{e}(0)}{\sqrt{1+\gamma(t)v^2(0,t)}}\frac{\gamma(t)v(0,t)}{\sqrt{1+\gamma(t)v^2(0,t)}}\leqslant\frac{\sqrt{\gamma(t)}\hat{e}(0)}{\sqrt{1+\gamma(t)v^2(0,t)}}\in\mathscr{L}_2\cap\mathscr{L}_\infty \tag{89}$$

4. Since $\gamma(t)$ is monotonically decreasing and is bounded from below, it has a limit: $\lim_{t\to\infty}\gamma(t)=\gamma_\infty$. We rewrite (82) as

$$\frac{\mathrm{d}}{\mathrm{d}t}\Delta q(t)=-\gamma\frac{v(0,t)(e(0,t)+\Delta q(t)v(0,t))}{1+\gamma(t)v^2(0,t)}=-\gamma\frac{v(0,t)e(0,t)}{1+\gamma(t)v^2(0,t)}+\frac{\Delta q(t)}{\gamma(t)}\dot{\gamma}(t) \tag{90}$$

The solution to this ODE is

$$\Delta q(t)=\frac{\Delta q(0)}{\gamma(0)}\gamma(t)-\gamma(t)\int_0^t\frac{v(0,\tau)e(0,\tau)}{1+\gamma(\tau)v^2(0,\tau)}\,\mathrm{d}\tau \tag{91}$$

Therefore,

$$\lim_{t\to\infty}\hat{q}(t)=q-\frac{\Delta q(0)}{\gamma(0)}\gamma_\infty+\gamma_\infty\int_0^\infty\frac{v(0,\tau)e(0,\tau)}{1+\gamma(\tau)v^2(0,\tau)}\,\mathrm{d}\tau=q_\infty \tag{92}$$

since the integral in (92) is obviously bounded. □

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4.2. Main result

Our main result is stated in the following theorem.

Theorem 4
Consider system (68)–(69) with the controller

$$u(1,t)=u_1-\int_0^1 \hat{q}e^{\hat{q}(1-\xi)}(\hat{q}v(\xi,t)+\eta(\xi,t))\,\mathrm{d}\xi \tag{93}$$

update law (82)–(83), and filters (75)–(80). Then for any $\hat{q}(0)$ and any initial conditions u_0, v_0, $\eta_0\in L_2(0,1)$, the signals $\hat{q}(t)$, $u(x,t)$, $v(x,t)$, $\eta(x,t)$ are uniformly bounded and $\hat{q}(t)\to q$ as $t\to\infty$.

5. PROOF OF THEOREM 4

We break the proof in several steps.

1. Following (71), we introduce the transformation

$$\hat{w}(x,t)=\hat{q}v(x,t)+\eta(x,t)+\int_0^x \hat{q}e^{\hat{q}(x-\xi)}(\hat{q}v(\xi,t)+\eta(\xi,t))\,\mathrm{d}\xi \tag{94}$$

where $\hat{q}v(x,t)+\eta(x,t)$ is the estimate of the state u. One can show that this transformation maps (68)–(69), (93) into the following system (see Lemma 5 in Appendix A):

$$\hat{w}_t(x,t)=\hat{w}_{xx}(x,t)+\hat{q}^2e^{\hat{q}x}\hat{e}(0,t)+\dot{\hat{q}}v+\dot{\hat{q}}\int_0^x e^{\hat{q}(x-\xi)}(\hat{q}v(\xi,t)+\hat{w}(\xi,t))\,\mathrm{d}\xi \tag{95}$$

$$\hat{w}_x(0,t)=-\hat{q}\hat{e}(0,t) \tag{96}$$

$$\hat{w}(1,t)=u_1 \tag{97}$$

Noting that $u(0,t)=\hat{w}(0,t)+\hat{e}(0,t)$, we rewrite v-filter as

$$v_t(x,t)=v_{xx}(x,t) \tag{98}$$

$$v_x(0,t)=-\hat{w}(0,t)-\hat{e}(0,t) \tag{99}$$

$$v(1,t)=0 \tag{100}$$

We now have two interconnected systems for $\hat{w}$ and v excited by the constant boundary input u_1 and by the signal $\hat{e}(0,t)$ with properties established in Lemma 2.

2. Our next goal is to demonstrate that these systems are asymptotically stable around the limit points

$$\hat{w}^{\mathrm{lp}}(x)=u_1,\quad v^{\mathrm{lp}}(x)=u_1(1-x) \tag{101}$$

Introducing the error variables $\bar{w}=\hat{w}-\hat{w}^{\mathrm{lp}}$, $\bar{v}=v-v^{\mathrm{lp}}$, we get

$$\bar{w}_t(x,t)=\bar{w}_{xx}(x,t)+\hat{q}^2e^{\hat{q}x}\hat{e}(0,t)+\dot{\hat{q}}\left[\bar{v}+u_1e^{\hat{q}x}+\int_0^x e^{\hat{q}(x-\xi)}(\hat{q}\bar{v}(\xi,t)+\bar{w}(\xi,t))\,\mathrm{d}\xi\right] \tag{102}$$

$$\bar{w}_x(0,t)=-\hat{q}\hat{e}(0,t) \tag{103}$$

$$\bar{w}(1,t)=0 \tag{104}$$

$$\bar{v}_t(x,t)=\bar{v}_{xx}(x,t) \tag{105}$$

$$\bar{v}_x(0,t)=-\bar{w}(0,t)-\hat{e}(0,t) \tag{106}$$

$$\bar{v}(1,t)=0 \tag{107}$$

Consider the Lyapunov function

$$V=\frac{1}{2}\int_0^1 \bar{w}^2(x)\,\mathrm{d}x+\frac{1}{2}\int_0^1 \bar{v}^2(x)\,\mathrm{d}x \tag{108}$$

We get

$$\begin{aligned}
\dot{V} &= \hat{q}\bar{w}(0)\hat{e}(0)-\int_0^1 \bar{w}_x^2\,\mathrm{d}x+\dot{\hat{q}}\int_0^1 \bar{w}(x)\bar{v}(x)\,\mathrm{d}x+\hat{e}(0)\int_0^1 \hat{q}^2 e^{\hat{q}x}\bar{w}(x)\,\mathrm{d}x-\int_0^1 \bar{v}_x^2\,\mathrm{d}x\\
&\quad+\dot{\hat{q}}\int_0^1 \bar{w}(x)\int_0^x e^{\hat{q}(x-\xi)}(\hat{q}\bar{v}(\xi)+\bar{w}(\xi))\,\mathrm{d}\xi\,\mathrm{d}x+\bar{v}(0)(\bar{w}(0)+\hat{e}(0))+\dot{\hat{q}}u_1\int_0^1 e^{\hat{q}x}\bar{w}(x)\,\mathrm{d}x\\
&\leqslant -\|\bar{w}_x\|^2+|\hat{e}(0)|(q_0|\bar{w}(0)|+q_0^2e^{q_0}\|\bar{w}\|)+\frac{(1+q_0e^{q_0})^2|\dot{\hat{q}}|^2}{c_1}\|\bar{v}\|^2+\frac{e^{2q_0}|\dot{\hat{q}}|^2}{c_1}\|\bar{w}\|^2\\
&\quad+\frac{c_1}{2}\|\bar{w}\|^2-\|\bar{v}_x\|^2+\frac{1}{2}\|\bar{v}_x\|^2+\frac{1}{2}\|\bar{w}_x\|^2+|\bar{v}(0)||\hat{e}(0)|+\frac{c_1}{2}\|\bar{w}\|^2+\frac{|\dot{\hat{q}}|^2u_1^2e^{2q_0}}{2c_1}\\
&\leqslant -\left(\frac{1}{2}-4c_1\right)\|\bar{w}_x\|^2-\frac{1}{2}\|\bar{v}_x\|^2+l_1\|\bar{w}\|^2+l_1\|\bar{v}\|^2+l_1\\
&\quad+q_0|\hat{e}(0)||\bar{w}(0)|+q_0^2e^{q_0}|\hat{e}(0)|\|\bar{w}\|+|\bar{v}(0)||\hat{e}(0)|
\end{aligned} \tag{109}$$

Here by q_0 and γ_0 we denoted the bounds on $\hat{q}$ and γ, respectively, and l_1 is a generic function of time in $\mathscr{L}_1\cap\mathscr{L}_\infty$. We now separately estimate the last three terms of (109) using Poincare's, Agmon's, and Young's inequalities:

$$\begin{aligned}
q_0|\hat{e}(0)||\bar{w}(0)| &\leqslant q_0|\bar{w}(0)|\frac{\hat{e}(0)}{\sqrt{1+v^2(0)}}(1+|u_1|+|\bar{v}(0)|)\\
&\leqslant c_2\|\bar{w}_x\|^2+\frac{q_0^2(1+|u_1|)^2}{4c_2}\frac{\hat{e}^2(0)}{1+v^2(0)}+2q_0\sqrt{\|\bar{w}\|\|\bar{w}_x\|\|\bar{v}\|\|\bar{v}_x\|}\frac{|\hat{e}(0)|}{\sqrt{1+v^2(0)}}\\
&\leqslant c_2\|\bar{w}_x\|^2+l_1+\frac{q_0|\hat{e}(0)|}{\sqrt{1+v^2(0)}}(\|\bar{w}\|\|\bar{w}_x\|+\|\bar{v}\|\|\bar{v}_x\|)\\
&\leqslant c_2\|\bar{w}_x\|^2+l_1+c_3\|\bar{w}_x\|^2+c_4\|\bar{v}_x\|^2+q_0^2\frac{\hat{e}^2(0)}{1+v^2(0)}\left(\frac{\|\bar{v}\|^2}{4c_3}+\frac{\|\bar{w}\|^2}{4c_4}\right)\\
&\leqslant c_2\|\bar{w}_x\|^2+c_3\|\bar{w}_x\|^2+c_4\|\bar{v}_x\|^2+l_1\|\bar{v}\|^2+l_1\|\bar{w}\|^2+l_1
\end{aligned} \tag{110}$$

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$$q_0^2 e^{q_0}|\hat{e}(0)|\|\bar{w}\| \leqslant q_0^2 e^{q_0}\|\bar{w}\|\frac{\hat{e}(0)}{\sqrt{1+v^2(0)}}(1+|u_1|+|v(0)|)$$

$$\leqslant c_5\|\bar{w}\|^2+\frac{q_0^4 e^{2q_0}(1+|u_1|)^2}{4}\frac{\hat{e}^2(0)}{1+v^2(0)}\left(\frac{1}{c_5}+\frac{1}{c_6}\|\bar{w}\|^2\right)+c_6\|\bar{v}_x\|^2$$

$$\leqslant c_5\|\bar{w}\|^2+c_6\|\bar{v}_x\|^2+l_1\|\bar{w}\|^2+l_1 \tag{111}$$

$$|\bar{v}(0)||\hat{e}(0)| \leqslant \frac{|\bar{v}(0)||\hat{e}(0)|}{1+v^2(0)}(1+2u_1^2+4\|\bar{v}\|\|\bar{v}_x\|)$$

$$\leqslant \frac{c_7}{2}\|\bar{v}_x\|^2+\frac{(1+2u_1^2)^2}{2c_7}\frac{\hat{e}^2(0)}{1+v^2(0)}+\frac{c_7}{2}\|\bar{v}_x\|^2+\frac{8}{c_7}\left(\frac{|\bar{v}(0)||\hat{e}(0)|}{1+v^2(0)}\right)^2\|\bar{v}\|^2$$

$$\leqslant c_7\|\bar{v}_x\|^2+l_1\|\bar{v}\|^2+l_1 \tag{112}$$

Substituting (110), (111), and (112) into (109), we get

$$\dot{V} \leqslant -(\tfrac{1}{2}-4c_1-c_2-c_3-4c_5)\|\bar{w}_x\|^2+l_1\|\bar{w}\|^2$$

$$-(\tfrac{1}{2}-c_4-c_6-c_7)\|\bar{v}_x\|^2+l_1\|\bar{v}\|^2+l_1 \tag{113}$$

Choosing $4c_1=c_2=c_3=4c_5=\frac{1}{16}$, $c_4=c_6=c_7=\frac{1}{12}$, we get

$$\dot{V}\leqslant -\tfrac{1}{8}V+l_1 V+l_1 \tag{114}$$

and by Lemma 3 we obtain $\|\bar{w}\|,\|\bar{v}\|\in\mathscr{L}_2\cap\mathscr{L}_\infty$. By integrating (113) we also get $\|\bar{w}_x\|$, $\|\bar{v}_x\|\in\mathscr{L}_2$.

3. To show the parameter convergence, we note that

$$|\hat{e}(0,t)|\leqslant\frac{\hat{e}(0,t)}{\sqrt{1+v(0,t)^2}}(1+\|\bar{v}_x\|+|u_1|)\in\mathscr{L}_2 \tag{115}$$

because $\|\bar{v}_x\|\in\mathscr{L}_2$ and $\hat{e}(0,t)/\sqrt{1+v(0,t)^2}\in\mathscr{L}_2\cap\mathscr{L}_\infty$ by Lemma 2. Using the definition of $\hat{e}$, we express

$$\hat{e}(0,t)=e(0,t)+\Delta q(t)v(0,t)=e(0,t)+\Delta q(t)\bar{v}(0,t)+\Delta q(t)u_1 \tag{116}$$

from which we have for $u_1\neq 0$:

$$|\Delta q(t)|\leqslant\frac{|\hat{e}(0,t)|+|e(0,t)|+|\Delta q(t)|\|\bar{v}_x\|}{|u_1|} \tag{117}$$

Since $\Delta q(t)$ is bounded and $\hat{e}(0,t)$, $e(0,t)$, and $\|\bar{v}_x\|$ are all square integrable, from (117) we conclude that $\Delta q(t)$ is square integrable. By Lemma 2 $\Delta q(t)$ and $\mathrm{d}/\mathrm{d}t\Delta q$ are bounded. By Barbalat's lemma, we have $\lim_{t\to\infty}\Delta q(t)=0$.

4. We already showed boundedness in L_2-norm. Unlike in the identification proof for the g-plant, here it is considerably difficult to show spatially uniform boundedness. The main difficulty is the presence of non-homogeneous terms in the boundary conditions (103) and (106), which does

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not allow us to use the H_1-norm as the standard Lyapunov function. To remove this difficulty, we consider the following transformation from $(\bar{w},\bar{v})$ into $(\breve{w},\breve{v})$:

$$\breve{w}(x)=\bar{w}(x)+\hat{q}(1-x)\int_0^x \hat{e}(y)\,\mathrm{d}y \tag{118}$$

$$\breve{v}(x)=\bar{v}(x)+(1-x)\int_0^x (\hat{e}(y)+\bar{w}(y))\,\mathrm{d}y \tag{119}$$

The purpose of this transformation is to make boundary conditions (103) and (106) homogeneous. One can easily check that in the new variables we have $\breve{w}_x(0,t)=\breve{w}(1,t)=\breve{v}_x(0,t)=\breve{v}(1,t)=0$. The right-hand side of the resulting PDEs for $\breve{w}$ and $\breve{v}$ is quite complicated, but has a simple structure with all the terms proportional either to $\dot{\hat{q}}$, $\hat{e}(0)$, or Δq, all of which are square integrable and bounded as we have shown. After this crucial step, the rest of the proof closely follows the proof for the g-case. One first shows the boundedness of $\|\breve{w}_x\|$ and $\|\breve{v}_x\|$ with the Lyapunov function:

$$V=\tfrac{1}{2}(\|\breve{w}\|^2+\|\breve{v}\|^2+\|\breve{w}_x\|^2+\|\breve{v}_x\|^2) \tag{120}$$

By Agmon's inequality this implies boundedness of $\breve{w}(x,t)$ and $\breve{v}(x,t)$. Then from transformation (118), (119) it follows that $\bar{w}(x,t)$ and $\bar{v}(x,t)$ are bounded (after the substitution $\hat{e}=e+\Delta q\bar{v}+\Delta q u_1(1-x)$). From the invertible transformation (94) it then follows that $\eta(x,t)$ is bounded. Finally, since $u=e+qv+\eta$, we get the boundedness of $u(x,t)$.

6. SIMULATION RESULTS

To illustrate the proposed identification scheme, we consider plant (1), (2) with unknown parameter $g=4$ (the open-loop system is unstable for this value of the parameter). The initial estimate of the parameter is $\hat{g}(0)=2$. A constant boundary input $u_1=2$ is added to the feedback. The results of closed-loop simulations are presented in Figures 1 and 2. In Figure 1 the evolution of the parameter estimate is shown in comparison with the case $u_1=0$. We can see that the unknown parameter is successfully identified. The additional constant input is turned off at $t=3$ to achieve regulation to zero. In Figure 2 the closed-loop state is shown.

7. CONCLUSION

We developed on-line identification schemes for two unstable parabolic systems with boundary sensing and actuation. Further research will be focused on extending the results of the paper to reaction–advection–diffusion PDEs with functional (spatially varying) parametric uncertainties. One possible approach to this challenging problem is to approximate the unknown functional parameter in (8) as

$$g(x)\approx\sum_{n=0}^{N} g_n x^n \tag{121}$$

and consider a problem of simultaneous identification of the parameters $g_0, g_1, \ldots, g_N$. There are no fundamental obstacles in using our methodology for that problem. However, one would need

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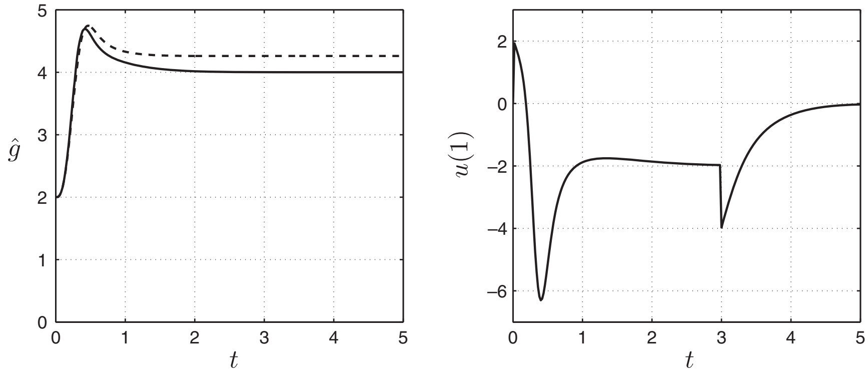

Figure 1. Left: convergence of the parameter estimate $\hat{g}$ to the true value $g=4$ with (solid line) and without (dashed line) additional constant boundary input. Right: the control effort; the additional constant input is turned off at $t=3$.

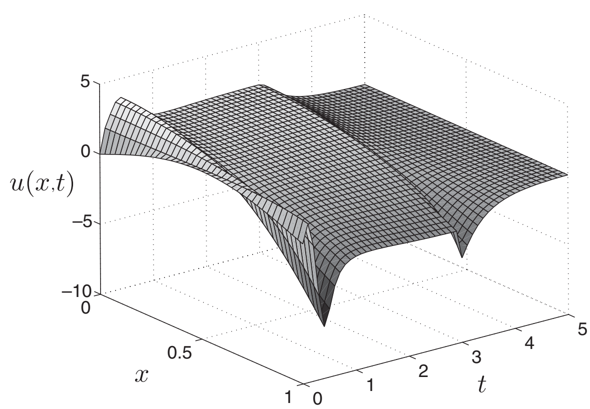

Figure 2. The closed-loop state $u(x,t)$.

to establish the persistency of excitation condition on the boundary input, since a constant input is not sufficient for the identification of more than one parameter. This condition amounts to proving that the states of the corresponding filters evaluated at $x=0$ $[v_0(0,t),\ v_1(0,t),\ldots,\ v_N(0,t)]$ are linearly independent functions of time, provided the input is sufficiently rich of order $(N/2)+1$ (based on the finite-dimensional intuition). Then one can choose N sufficiently large to identify $g(x)$ with any prescribed accuracy.

APPENDIX A

Lemma 3 (*Lemma B.6 in* [*10*])
Let v, l_1, and l_2 be real-valued functions defined on $\mathbf{R}_+$, and let c be a positive constant. If l_1 and l_2 are non-negative integrable functions of time and $\dot{v}\leqslant -cv+l_1(t)v+l_2(t)$, $v(0)\geqslant 0$, then $v\in\mathscr{L}_1\cap\mathscr{L}_\infty$.

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Lemma 4
The transformation (45) maps (1)–(2), (39) into system (46)–(48).

Proof
First, we verify boundary conditions (47), (48):

$$\hat{w}_x(0,t) = -\hat{g}v_x(0,t) - \hat{k}(0,0)\hat{g}v(0,t) - \hat{k}(0,0)\eta(0,t) = 0 \tag{A1}$$

$$\hat{w}(1,t) = u(1,t) - \int_0^1 \hat{k}(1,\xi)(\hat{g}v(\xi,t)+\eta(\xi,t))\,\mathrm{d}\xi = u_1 \tag{A2}$$

Next, we compute the derivatives of $\hat{w}$:

$$\begin{aligned}\hat{w}_{xx}(x,t) &= \hat{g}v_{xx}(x,t)+\eta_{xx}(x,t)+\hat{g}^2v(x,t)+\hat{g}\eta(x,t)\\ &\quad+\hat{g}^2\int_0^x \alpha(x-\xi)(\hat{g}v(\xi,t)+\eta(\xi,t))\,\mathrm{d}\xi\end{aligned} \tag{A3}$$

$$\begin{aligned}\hat{w}_t(x,t) &= \hat{g}v_t(x,t)+\eta_t(x,t)-\hat{g}u(0,t)+\beta(x)u(0,t)+\hat{g}^2v(x,t)+\hat{g}\eta(x,t)\\ &\quad-\beta(x)(\hat{g}v(0,t)+\eta(0,t))+\dot{\hat{g}}v(x,t)+\dot{\hat{g}}\int_0^x \hat{g}\alpha(x-\xi)v(\xi,t)\,\mathrm{d}\xi\\ &\quad+\frac{\dot{\hat{g}}}{2\hat{g}}\int_0^x [(x-\xi)\beta(x-\xi)+\hat{g}\alpha(x-\xi)](\hat{g}v(\xi,t)+\eta(\xi,t))\,\mathrm{d}\xi\end{aligned} \tag{A4}$$

In (A4) we integrated by parts twice (α and β are defined in (49)). Subtracting (A3) from (A4) and using the inverse transformation (66), we get

$$\begin{aligned}\hat{w}_t(x,t) &= \hat{w}_{xx}(x,t)+\beta(x)\hat{e}(0,t)+\dot{\hat{g}}v(x,t)+\dot{\hat{g}}\int_0^x \hat{g}\alpha(x-\xi)v(\xi,t)\,\mathrm{d}\xi\\ &\quad+\dot{\hat{g}}\int_0^x [(x-\xi)\beta(x-\xi)+\hat{g}\alpha(x-\xi)]\left(\hat{w}(\xi,t)-\hat{g}\int_0^\xi(\xi-y)\hat{w}(y,t)\,\mathrm{d}y\right)\mathrm{d}\xi\end{aligned} \tag{A5}$$

Changing the order of integration in the double integral (last term in (A5)), computing the internal integral, and gathering all the terms together, we obtain (46). □

Lemma 5
Transformation (94) maps (68)–(69), (93) into system (95)–(97).

Proof
First, we verify boundary conditions (96), (97):

$$\hat{w}_x(0,t) = -\hat{q}(u(0,t)-\hat{q}v(0,t)-\eta(0,t)) = -\hat{q}\hat{e}(0,t) \tag{A6}$$

$$\hat{w}(1,t) = u(1,t)+\int_0^1 \hat{q}\mathrm{e}^{\hat{q}(x-\xi)}(\hat{q}v(\xi,t)+\eta(\xi,t))\,\mathrm{d}\xi = u_1 \tag{A7}$$

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Next, we compute the derivatives of $\hat{w}$:

$$\hat{w}_{xx}(x,t)=\hat{q}v_{xx}(x,t)+\eta_{xx}(x,t)+\hat{q}^2v_x(x,t)+\hat{q}\eta_x(x,t)+\hat{q}^3v(x,t)+\hat{q}^2\eta(x,t) + \int_0^x \hat{q}^3 \mathrm{e}^{\hat{q}(x-\xi)}(\hat{q}v(\xi,t)+\eta(\xi,t))\,\mathrm{d}\xi \tag{A8}$$

$$\begin{aligned}\hat{w}_t(x,t)=&\,\hat{q}v_t(x,t)+\eta_t(x,t)+\hat{q}^2v_x(x,t)-\hat{q}^2\mathrm{e}^{\hat{q}x}v_x(0,t)+\hat{q}^3v(x,t)+\hat{q}^2\eta(x,t)+\hat{q}\eta_x(x,t)\\ &-\hat{q}^3\mathrm{e}^{\hat{q}x}v(0,t)-\hat{q}^2\mathrm{e}^{\hat{q}x}\eta(0,t)+\int_0^x \hat{q}^3\mathrm{e}^{\hat{q}(x-\xi)}(\hat{q}v(\xi,t)+\eta(\xi,t))\,\mathrm{d}\xi+\dot{\hat{q}}v(x,t)\\ &+\dot{\hat{q}}\int_0^x \hat{q}\mathrm{e}^{\hat{q}(x-\xi)}v(\xi,t)\,\mathrm{d}\xi+\dot{\hat{q}}\int_0^x(1+\hat{q}(x-\xi))\mathrm{e}^{\hat{q}(x-\xi)}(\hat{q}v(\xi,t)+\eta(\xi,t))\,\mathrm{d}\xi\end{aligned} \tag{A9}$$

In (A9) we integrated by parts twice. Subtracting (A8) from (A9) and using the inverse transformation

$$\hat{q}v(x,t)+\eta(x,t)=\hat{w}(x,t)-\hat{q}\int_0^x \hat{w}(y,t)\,\mathrm{d}y \tag{A10}$$

we get

$$\begin{aligned}\hat{w}_t(x,t)=&\,\hat{w}_{xx}(x,t)+\hat{q}^2\mathrm{e}^{\hat{q}x}\hat{e}(0,t)+\dot{\hat{q}}v(x,t)+\dot{\hat{q}}\int_0^x \hat{q}\mathrm{e}^{\hat{q}(x-\xi)}v(\xi,t)\,\mathrm{d}\xi\\ &+\dot{\hat{q}}\int_0^x(1+\hat{q}(x-\xi))\mathrm{e}^{\hat{q}(x-\xi)}\left(\hat{w}(\xi,t)-\hat{q}\int_0^\xi \hat{w}(y,t)\,\mathrm{d}y\right)\mathrm{d}\xi\end{aligned} \tag{A11}$$

Changing the order of integration in the double integral (the last term in (A11)) and computing the internal integral, we obtain (95). □

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