



Control of 1D parabolic PDEs with Volterra nonlinearities, Part II: Analysis[☆]

Rafael Vazquez^a, Miroslav Krstic^{b,*}

^a Departamento de Ingeniería Aeroespacial, Universidad de Sevilla, 41092 Seville, Spain

^b Department of Mechanical and Aerospace Engineering, University of California San Diego, La Jolla, CA 92093-0411, United States

ARTICLE INFO

Article history:

Received 7 December 2006

Received in revised form

31 March 2008

Accepted 7 April 2008

Available online 2 October 2008

Keywords:

Distributed parameter systems

Stabilization

Nonlinear control

Feedback linearization

Partial differential equations

Lyapunov function

Boundary conditions

ABSTRACT

For a class of stabilizing boundary controllers for nonlinear 1D parabolic PDEs introduced in a companion paper, we derive bounds for the gain kernels of our nonlinear Volterra controllers, prove the convergence of the series in the feedback laws, and establish the stability properties of the closed-loop system. We show that the state transformation is at least locally invertible and include an explicit construction for computing the inverse of the transformation. Using the inverse, we show L^2 and H^1 exponential stability and explicitly construct the exponentially decaying closed-loop solutions. We then illustrate the theoretical results on an analytically tractable example.

© 2008 Elsevier Ltd. All rights reserved.

1. Introduction

For a class of stabilizing boundary controllers for nonlinear 1D parabolic PDEs introduced (in full detail and with examples) in a companion paper (Vazquez & Krstic, 2008), we derive bounds for the gain kernels of our nonlinear Volterra controllers (in the Appendix), prove the convergence of the series in the feedback laws (in Section 4) and establish the stability properties of the closed-loop system (in Section 5). We show that the state transformation is at least locally invertible and include an explicit construction for computing the inverse of the transformation (in Section 6). Using the inverse, we show L^2 and H^1 local exponential stability and explicitly construct the exponentially decaying closed-loop solutions. We then illustrate (in Section 6.1) the theoretical results on an analytically tractable example, introduced in Vazquez and Krstic (2008, Section 5).

2. Preliminaries

Define, as in Vazquez and Krstic (2008), $\xi_0 = x$ and for $i \leq n$, $\hat{\xi}_i^n = (\xi_i, \dots, \xi_n)$. Let $\mathcal{T}_n(x, \xi) = \{\hat{\xi}_1^n : 0 \leq \xi_n \leq \dots \leq \xi_1 \leq x \leq 1\}$ and $\mathcal{T}_n = \mathcal{T}_n(1, \xi)$. Define also

$x \leq 1\}$ and $\mathcal{T}_n = \mathcal{T}_n(1, \xi)$. Define also

$$\prod_{j=1}^n u = \prod_{j=1}^n u(t, \xi_j), \quad (1)$$

$$\int_{\mathcal{T}_n(x, \xi)} f(\hat{\xi}_0^n) d\hat{\xi}_1^n = \int_0^x \int_0^{\xi_1} \dots \int_0^{\xi_{n-1}} f(\hat{\xi}_0^n) d\xi_n \dots d\xi_1. \quad (2)$$

We first formalize the concept of convergence of Volterra series with $L^2(\mathcal{T}_n)$ kernels. Consider a Volterra series $F[u]$ with kernels $f_n(\hat{\xi}_0^n)$, i.e.,

$$\begin{aligned} F[u](t, x) &= \sum_{n=1}^{\infty} F_n[u](t, x) \\ &= \sum_{n=1}^{\infty} \int_{\mathcal{T}_n(x, \xi)} f_n(\hat{\xi}_0^n) \prod_{j=1}^n u d\hat{\xi}_1^n. \end{aligned} \quad (3)$$

The following definition quantifies the convergence of (3) in $L^2(0, 1)$ (in what follows, we will write just L^2 for simplicity).

Definition 2.1. Given (3) with kernels $f_n \in L^2(\mathcal{T}_n)$, we define the radius of convergence ρ as

$$\rho = \left(\limsup_{n \rightarrow \infty} \left(\frac{\|f_n\|_{L^2(\mathcal{T}_n)}^2}{n!} \right)^{1/n} \right)^{-1}, \quad (4)$$

[☆] This work was supported by NSF grant number CMS-0329662. This paper was not presented at any IFAC meeting. This paper was recommended for publication in revised form by Associate Editor Hendrik Nijmeijer under the direction of Editor Hassan K. Khalil.

* Corresponding author. Tel.: +1 858 822 1374; fax: +1 858 822 3107.

E-mail addresses: rvazquez1@us.es (R. Vazquez), krstic@ucsd.edu (M. Krstic).

and the gain bound function $f(s) : [0, \rho) \rightarrow [0, \infty)$ as

$$f(s) = 2 \sum_{n=1}^{\infty} \frac{n^2 \|f_n\|_{L^2(\mathcal{T}_n)}^2}{n!} s^n. \tag{5}$$

Using ρ and f from Definition 2.1 we can state a result that guarantees convergence of the Volterra series (3) in L^2 (a similar result in the L^∞ space for L^∞ kernels is standard (Boyd, Chua, & Desoer, 1984)).

Theorem 1 (Gain Bound Theorem). *Given a Volterra series $F[u]$ as in (3), with kernels $f_n \in L^2(\mathcal{T}_n)$, radius of convergence $\rho > 0$ and gain bound function f , the following results hold.*

- (1) *If $u \in L^2$ verifies that $\|u\|_{L^2}^2 < \rho$, then the integrals and sums in (3) converge (in L^2).*
- (2) *$F[u]$ satisfies $\|F[u]\|_{L^2}^2 \leq f(\|u\|_{L^2}^2)$ and consequently F maps balls of L^2 into balls of L^2 .*

Proof. From definition (3), and using the Cauchy–Schwartz inequality,

$$\begin{aligned} F_n[u]^2 &\leq \|f_n\|_{L^2(\mathcal{T}_n)}^2 \left(\int_{\mathcal{T}_n(x,\xi)} \prod_{i=1}^n u^2 d\hat{\xi}_i^n \right) \\ &= \frac{\|f_n\|_{L^2(\mathcal{T}_n)}^2 \|u\|_{L^2}^{2n}}{n!}, \end{aligned} \tag{6}$$

hence,

$$\begin{aligned} F[u]^2 &= \left(\sum_{n=1}^{\infty} F_n[u] \right)^2 \\ &\leq \left(\sum_{n=1}^{\infty} n^2 F_n[u]^2 \right) \left(\sum_{n=1}^{\infty} \frac{1}{n^2} \right) \\ &\leq 2 \sum_{i=1}^{\infty} \frac{n^2 \|f_n\|_{L^2(\mathcal{T}_n)}^2 \|u\|_{L^2}^{2n}}{n!}, \end{aligned} \tag{7}$$

where we used that $\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6} \leq 2$. Thus we obtain

$$\|F[u]\|_{L^2}^2 = \max_{x \in (0,1)} F[u]^2 \leq 2 \sum_{i=1}^{\infty} \frac{n^2 \|f_n\|_{L^2(\mathcal{T}_n)}^2 \|u\|_{L^2}^{2n}}{n!}. \tag{8}$$

Then from elementary theory of power series and noting that $\lim_{n \rightarrow \infty} \sqrt[n]{n^2} = 1$ and that $\|F[u]\|_{L^2}^2 \leq \|F[u]\|_{L^2}^2$, the result follows. \square

We give now some examples illustrating Theorem 1.

Example 2.1. Let $F[u]$ be a Volterra series with kernels f_n and let C and D be generic positive constants.

- (1) If the kernels f_n verify the uniform bound $\|f_n\|_{L^2(\mathcal{T}_n)}^2 \leq D$, then $\rho = \infty$ and the series is everywhere convergent for $u \in L^2$. We also have that $f(s) = 2s(s+1)D \exp(s)$. Note also that $f(s) \leq 2D(\exp(3s) - 1)$.
- (2) If the kernels f_n grow exponentially as $\|f_n\|_{L^2(\mathcal{T}_n)}^2 \leq DC^n$, then again $\rho = \infty$ and the series is everywhere convergent. We have in this case that $f(s) = 2sC(s+1)D \exp(Cs)$. Note also that $f(s) \leq 2D(\exp(3Cs) - 1)$.
- (3) If the kernels f_n grow as fast as $\|f_n\|_{L^2(\mathcal{T}_n)}^2 \leq n!DC^n$, then $\rho = 1/C$ and the series convergence can only be guaranteed if $\|u\|_{L^2} \leq 1/C$. We have in this case that $f(s) = \frac{2sC(sC+1)D}{(1-sC)^3}$. Note that $f(s) \leq \frac{2D(sC)^2}{(1-sC)^4}$.

Remark 1. Since $\|f_n\|_{L^2(\mathcal{T}_n)}^2 \leq \frac{\|f_n\|_{L^\infty}^2}{n!}$, if $f_n \in L^\infty(\mathcal{T}_n)$, similar results to Theorem 1 can be stated in terms of the L^∞ norms of the f_n 's. Note also that by (8) the L^∞ norm of $F[u]$ is well defined for $u \in L^2$.

3. Control strategy

In the companion paper (Vazquez & Krstic, 2008) we considered the stabilization problem for the plant

$$u_t = u_{xx} + \lambda(x)u + F[u] + uH[u], \tag{9}$$

$$u_x(0) = qu(0), \quad u(1) = U, \tag{10}$$

where $F[u]$ and $H[u]$ are Volterra nonlinearities defined respectively by kernels f_n and h_n , and U the actuation variable. We solved the problem by mapping u into a target system w which verifies

$$w_t = w_{xx} - cw, \tag{11}$$

$$w_x(0) = \bar{q}w(0), \quad w(1) = 0, \tag{12}$$

where $\bar{q} = \max\{0, q\}$. For mapping u into w we use a Volterra transformation

$$w = u - K[u] = u - \sum_{n=1}^{\infty} \int_{\mathcal{T}_n(x,\xi)} k_n(\hat{\xi}_0^n) \prod_{i=1}^n u d\hat{\xi}_i^n, \tag{13}$$

where the kernels k_n in (13) are obtained from the set of PIDEs (40)–(47) in Vazquez and Krstic (2008).

The control law is determined by (13) at $x = 1$

$$U = \sum_{n=1}^{\infty} \int_{\mathcal{T}_n(1,\xi)} k_n(1, \hat{\xi}_1^n) \prod_{i=1}^n u d\hat{\xi}_i^n. \tag{14}$$

Remark 2. From (13),

$$\begin{aligned} w_x &= u_x - u(x) \sum_{n=1}^{\infty} \int_{\mathcal{T}_n(x,\xi)} k_{n+1}(x, x, \hat{\xi}_1^n) \prod_{i=1}^n u d\hat{\xi}_i^n \\ &\quad - k_1(x, x)u(x) - \sum_{n=1}^{\infty} \int_{\mathcal{T}_n(1,\xi)} k_{nx}(x, \hat{\xi}_1^n) \prod_{i=1}^n u d\hat{\xi}_i^n \\ &= u_x - \bar{k}(x)u(x) - u(x)\bar{K}[u] - \bar{K}[u], \end{aligned} \tag{15}$$

where $\bar{K}[u]$ and $\bar{K}[u]$ are Volterra series in u (not in u_x) with kernels $\bar{k}_n = k_{n+1}(x, x, \hat{\xi}_1^n)$ and $\bar{k}_n = k_{nx}(x, \hat{\xi}_1^n)$. Note that from the boundary condition (Vazquez & Krstic, 2008, (43)), we have that

$$\bar{k} = \hat{q} - \frac{1}{2} \int_0^x \lambda(s)ds, \quad \bar{k}_n = -\frac{1}{2} \int_{\hat{\xi}_1}^x h_n(s, \hat{\xi}_1^n)ds, \tag{16}$$

where $\hat{q} = \min\{0, q\}$. Hence,

$$\begin{aligned} \|w_x\|_{L^2}^2 &\leq 4 \left(\|u_x\|_{L^2}^2 + \|u\|_{L^2}^2 \|\bar{k}\|_{L^\infty}^2 + \|u\|_{L^2}^2 \|\bar{K}[u]\|_{L^2}^2 \right. \\ &\quad \left. + \|\bar{K}[u]\|_{L^2}^2 \right), \end{aligned} \tag{17}$$

which means that the H_1 norm of w can be computed from the H^1 norm of u (note that by Remark 1, $\|\bar{K}[u]\|_{L^2}^2$ is well defined).

In the following sections we study the convergence of (13) and (14) and the properties of the closed-loop system (9), (10) and (14).

4. Convergence analysis for the transformation

In what follows, we make the following very reasonable assumption on the plant kernels.

Assumption 4.1. Let D_h, D_f, ρ_h and ρ_f be positive constants. Then, the following hold.

- (1) $\lambda(x) \in \mathcal{C}^1[0, 1], h_n \in \mathcal{C}^1[\mathcal{T}_n], f_n \in \mathcal{C}^0[\mathcal{T}_n]$.
- (2) The parameter $\lambda(x)$ verifies

$$\max_{x \in [0, 1]} \{|\lambda(x)|\} \leq D_h. \tag{18}$$

- (3) The sequence $h_n(\hat{\xi}_0^n)$ verifies the following bound

$$\max_{(\hat{\xi}_0^n) \in \mathcal{T}_n} \left\{ |h_n(\hat{\xi}_0^n)| + \sum_{i=0}^n |h_{n\hat{\xi}_i}(\hat{\xi}_0^n)| \right\} \leq \frac{n!D_h}{\rho_h^{n-1}}. \tag{19}$$

- (4) The sequence $f_n(\hat{\xi}_0^{n+1})$ verifies the following bound

$$\max_{(\hat{\xi}_0^n) \in \mathcal{T}_n} \left\{ |f_n(\hat{\xi}_0^n)| \right\} \leq \frac{n!D_f}{\rho_f^{n-1}}. \tag{20}$$

In Assumption 4.1, points (3) and (4) quantify the divergence rate bound for plant kernels to ensure convergence of the plant nonlinearities H and F . We also assume the following:

Assumption 4.2. Under the above assumptions, for each n , there exists an $H^1(\mathcal{T}_n)$ solution k_n of the kernel PIDE equations (Vazquez & Krstic, 2008, Equations (40)–(47)).

We next show a result that relates the convergence of the transformation and the feedback law series, respectively (13) and (14), to the convergence of the plant nonlinearities $F[u]$ and $H[u]$.

Theorem 2. Under Assumptions 4.1 and 4.2, the Volterra series in the transformation (13), the control law (14) and the w_x transformation (15) are convergent with radius of convergence

$$\rho_k = \left(\frac{\min\{\rho_f, \rho_h\}}{2} \right)^2 \exp(-2\sqrt{\gamma}), \tag{21}$$

where $\gamma = \max\{1, \|f_1\|_\infty + \|\lambda\|_\infty\}$. Moreover, k_n verifies

$$\|k_n\|_{L^2(\mathcal{T}_n)}^2 \leq (n-1)!4D^2C^{2n-2}e^{2n\sqrt{\gamma}+2\gamma+|\hat{q}|}, \tag{22}$$

$$\|k_{nx}\|_{L^2(\mathcal{T}_n)}^2 \leq n!2D^2C^{2n-2}e^{2n\sqrt{\gamma}+2\gamma+|\hat{q}|}, \tag{23}$$

where $D = D_f + \rho_h D_h + 2((1 + \rho_h)D_h)(|q| + 1) \exp(1 + |\hat{q}|) \sqrt{(1 + \rho_h)^2 D_h^2 + D_f^2}$, $C = \left(\frac{\min\{\rho_f, \rho_h\}}{2}\right)^{-1}$ and $\gamma = 4 \frac{D^2(1+2\sqrt{\gamma})^2}{\gamma^2}$.

See the Appendix for a proof.

Remark 3. In the above theorem, if $q = \infty$ (meaning the plant has a Dirichlet boundary condition at the uncontrolled end), then the above bounds hold setting $q = 0$.

Corollary 4.1. Under the same assumptions of Theorem 2, if the Volterra series nonlinearity of the plant is globally convergent in L^2 , then the transformation Volterra series (13), the control (14) and the w_x transformation (15) converge globally in L^2 as well.

Proof. If the Volterra series nonlinearities of the plant F and H are everywhere convergent, then by the limit (4) being infinity, for any $\epsilon > 0$ (possibly very small), there exists $D_\epsilon > 0$ (possibly very large) such that both f_n and h_n verify

$$\max\{|h_n|, |f_n|\} \leq n!B_\epsilon \epsilon^{n-1}. \tag{24}$$

Hence under the assumptions of Theorem 2, the kernel solution k_n verifies

$$\|k_n\|_{L^2(\mathcal{T}_n)}^2 \leq (n-1)!4D_\epsilon^2 \left(\frac{\epsilon}{2}\right)^{2n-2} e^{2n\sqrt{\gamma_\epsilon}+2\gamma_\epsilon+|\hat{q}|}, \tag{25}$$

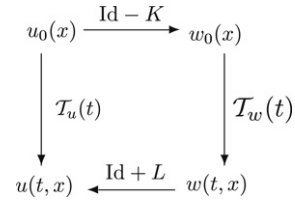


Fig. 1. Commutative diagram for the closed-loop system.

where D_ϵ and γ_ϵ are defined as in Theorem 2 replacing $D_h = D_f = B_\epsilon$ and $\rho_h = \rho_f = 1/\epsilon$, but note that $\gamma = \max\{1, \|f_1\|_\infty + \|\lambda\|_\infty + c\}$ does not depend on ϵ . Then the radius of convergence of the Volterra series defined by k_n is $\rho_k \geq \frac{4 \exp(2\sqrt{\gamma})}{\epsilon^2}$. Since this holds for any positive ϵ , we must have $\rho_k = \infty$. \square

5. Stability analysis

To analyze the behavior of the closed-loop system, we study the invertibility of the change of variables (13). It is natural to seek also a Volterra formulation for this inverse change of variables, which is assumed as having the following form

$$u = w + L[w], \tag{26}$$

which is expanded as

$$u(t, x) = w(t, x) + \sum_{n=1}^{\infty} \int_{\mathcal{T}_n(x, \xi)} l_n(\hat{\xi}_0^n) \prod w d\hat{\xi}_1^n. \tag{27}$$

The existence of this inverse change of variables can be guaranteed employing the theorem for inversion of Volterra series, which is proved in Boyd et al. (1984, Theorem 3.3.1.).

Theorem 3 (Volterra Series Inversion). A Volterra series has a local inverse at the origin if and only if its first (linear) kernel is invertible.

In that context, the word “local” means that a unique Volterra series representation can be found for the inverse transformation, which has the form specified by (27), and whose radius of convergence (in the sense of Definition 2.1 and Theorem 1) is possible finite, even if the transformation is globally convergent.

The direct and inverse transformations give a relation between u and w that can be exploited to obtain properties of u (governed by a complex nonlinear equation) from properties of w (that verify an easy to analyze heat equation). The commutative diagram of Fig. 1 illustrates our strategy. We have denoted the initial conditions for u and w as $u(0, x) = u_0$ and $w(0, x) = w_0$, respectively. In the left, $\mathcal{T}_u(t)$ is the semigroup that governs the behavior of u when the loop is closed, so that $u(t) = \mathcal{T}_u(t)u_0$; its generator can be obtained homogenizing (9) and taking (14) into account. In the right, $\mathcal{T}_w(t)$ is the semigroup generated by the Laplacian operator in (11), so that $w(t) = \mathcal{T}_w(t)w_0$. Above and below are respectively the direct and inverse transformations, $\text{Id} - K$ and $\text{Id} + L$ that relate u and w . We are interested in the properties of u , but direct analysis of $\mathcal{T}_u(t)$ is very difficult—it is generated by a nonlinear operator. Instead, from Fig. 1, we use that $\mathcal{T}_u(t) = (\text{Id} + L) \circ \mathcal{T}_w(t) \circ (\text{Id} - K)$, dividing the analysis into smaller, more tractable pieces. The transformations $\text{Id} + L$ and $\text{Id} - K$ are still nonlinear but time invariant, and are analyzed within the framework of Volterra series, whereas the heat equation semigroup $\mathcal{T}_w(t)$ is linear and simple, producing even explicit solutions. We begin by analyzing \mathcal{T}_w , whose behavior is summarized in the following lemma, which follows from standard estimates for the heat equation (Evans, 1998; Liu, 2003).

Lemma 5.1. Consider the system (11) with boundary conditions (12). Then, the equilibrium $w \equiv 0$ is exponentially stable in the L^2 and H^1 norms, i.e., $\forall t \geq 0$

$$\|w(t)\|_{\mathcal{L}}^2 \leq e^{-t} \|w_0\|_{\mathcal{L}}^2, \tag{28}$$

where \mathcal{L} is either L^2 or H^1 .

Using Lemma 5.1 and the relations illustrated by Fig. 1, we get the following result about the stability properties of the closed-loop system.

Theorem 4. Let Assumptions 4.1 and 4.2 hold and assume that there is an L^2 (resp. H^1) solution u to the closed-loop system (9) with boundary conditions (10) and control law (14). Then, the origin $u \equiv 0$ of the closed-loop system is locally exponentially stable in the L^2 (resp. H^1) norm, i.e., denoting the initial condition for u as $u(0, x) = u_0(x)$, there exists $C_1, C_2 > 0$ such that, if $\|u_0\|_{\mathcal{L}}^2 \leq C_1$, then $\forall t \geq 0$

$$\|u(t)\|_{\mathcal{L}}^2 \leq C_2 e^{-t} \|u_0\|_{\mathcal{L}}^2, \tag{29}$$

where \mathcal{L} is either L^2 or H^1 , and C_1, C_2 depend on the plant parameters, but not on u_0 .

Proof. Under Assumptions 4.1 and 4.2, the transformation (13) exists and converges for $\|u(t)\|_{L^2}^2 \leq \rho_K$, where ρ_K denotes the radius of convergence of the transformation Volterra series. The first kernel of (13) is $Id - K_1$ and constitutes the linear part of the transformation. In Smyshlyaev and Krstic (2004) it is shown that this linear part is always invertible. Hence, using Theorem 3, the whole transformation is locally invertible and the inverse transformation has the form specified by (27). Therefore there exists $\rho_L > 0$ such that, if $\|w(t)\|_{L^2}^2 < \rho_L$, then (27) converges.

Denote by $k(s)$ and $l(s)$ the gain bound functions of the direct and inverse Volterra series transformations, $Id - K$ and $Id + L$ respectively, as defined in (5).

From (13), we have that

$$w_0 = u_0 - K[u_0]. \tag{30}$$

Set $C_1 = k^{-1}(\rho_L)/2 < \rho_K$. Hence, if $u_0 \leq C_1$ we get that

$$\|w(t)\|_{L^2}^2 \leq \|w_0\|_{L^2}^2 \leq k(\|u_0\|_{L^2}^2) \leq k(C_1) < \rho_L \tag{31}$$

for all time t . Therefore, the inverse (26) converges and the relations of Fig. 1 hold for all time $t \geq 0$. Set now $C_3 = \frac{k(C_1)}{C_1}$ and $C_4 = \frac{l(C_1 C_3)}{C_1 C_3}$. Then, for $\|u_0\|_{L^2}^2 < C_1$, since $\|w(t)\|_{L^2}^2 \leq C_3 C_1$ and both $k(s)$ and $l(s)$ are class \mathcal{K} functions (Khalil, 2002), we have that

$$\begin{aligned} \|u(t)\|_{L^2}^2 &\leq l(\|w(t)\|_{L^2}^2) \leq C_4 \|w(t)\|_{L^2}^2 \leq C_4 e^{-t} \|w_0\|_{L^2}^2 \\ &\leq C_3 C_4 e^{-t} \|u_0\|_{L^2}^2, \end{aligned} \tag{32}$$

so setting $C_2 = C_3 C_4$, (29) follows for the L^2 norm. To obtain the bound for the H^1 norm we use (15) and (17), and note that

$$\begin{aligned} u_x &= w_x + \bar{k}(x)(w + L[w]) + (w + L[w])\bar{K}[w + L[w]] \\ &\quad - \tilde{K}[w + L[w]]. \end{aligned} \tag{33}$$

Hence u_x can be recovered from w_x when the Volterra series in (33) converge. If $\|u_0\|_{H^1}^2 \leq C_1$, then obviously $\|u_0\|_{L^2}^2 \leq C_1$, and since the radius of convergence of both \bar{K} and \tilde{K} is at least ρ_K , all the series in the right-hand side in (33) converge. Then we use Lemma 5.1 and proceed in the same way as in (32) for the H^1 norm (using the gain bound functions for \bar{K} and \tilde{K}), obtaining possibly a different C_2 ; to get the same C_2 for both L^2 and H^1 we pick the maximum of the two. Then the result follows. \square

Remark 4. In Theorem 4 we have assumed well-posedness of the closed-loop system. For the case $q = \infty$ (Dirichlet boundary condition at $x = 0$), since (11) and (12) are well-posed in H^1 and since (15) and (33) allow proving local equivalence of the H^1 norms of u and w , the assumption can be dropped, provided u_0 verifies some compatibility conditions (Smyshlyaev & Krstic, 2004) (see Proposition 6.2 for an example). For other values of q , (11) and (12) are well-posed in H^2 and this argument is not enough.

Remark 5. Note that (11) can be solved explicitly. This means that, when $\|u_0\|_{\mathcal{L}} < C_1$, u can be obtained explicitly for all times. We give an illustration for the simplest case, when $q = \infty$. Then, u is given as

$$\begin{aligned} u(t, x) &= 2 \sum_{n=1}^{\infty} e^{-\pi^2 n^2 t} \sin(\pi n x) \int_0^1 \sin(\pi n \xi) [u_0(\xi) \\ &\quad - K[u_0](\xi)] d\xi + L \left[2 \sum_{n=1}^{\infty} e^{-\pi^2 n^2 t} \sin(\pi n x) \right. \\ &\quad \left. \times \int_0^1 \sin(\pi n \xi) [u_0(\xi) - K[u_0](\xi)] d\xi \right]. \end{aligned} \tag{34}$$

For other values of q similar formulas can be written.

The constant C_1 for which Theorem 4 holds determines the “basin of attraction” of the equilibrium at the origin for the closed-loop system. Since $C_1 = k^{-1}(\rho_L)$, if ρ_L and some bound on the k_n ’s are known then C_1 can be more precisely quantified. We state a corollary for Theorem 4 for some particular cases, introduced in Example 2.1, that occur frequently in practice.

Corollary 5.1. Let $\rho_K, \rho_L > 0$ denote the radii of convergence of the direct and inverse Volterra transformations, (13) and (27), respectively. Let C and D denote generic positive constants.

- (1) If $\rho_K = \rho_L = \infty$, then Theorem 4 holds globally, i.e., for all $u \in L^2$.
- (2) If the kernels k_n verify $\|k_n\|_{L^2(\mathcal{J}_n)}^2 \leq D$, then $\rho_K = \infty$ and Theorem 4 holds at least for $\|u\|_{L^2}^2 \leq \frac{1}{3} \log(1 + \frac{\rho_L}{2D})$.
- (3) If the kernels k_n grow like $\|k_n\|_{L^2(\mathcal{J}_n)}^2 \leq DC^n$, then $\rho_K = \infty$ and Theorem 4 holds at least for $\|u\|_{L^2}^2 \leq \frac{1}{3C} \log(1 + \frac{\rho_L}{2D})$.
- (4) If the kernels k_n grow as $\|k_n\|_{L^2(\mathcal{J}_n)}^2 \leq n!DC^n$, then $\rho_K = 1/C$ and Theorem 4 holds for

$$\|u_0\|_{L^2}^2 \leq \frac{1}{C} \left(1 + \sqrt{\frac{D}{2\rho_L}} - \sqrt[4]{\frac{D}{2\rho_L}} \sqrt{\sqrt{\frac{D}{2\rho_L}} + 2} \right) > 0.$$

6. Inverse transformation

Theorem 4 depends critically on the inverse transformation and its properties. Next we give explicit formulas that allow computing the inverse from the kernels k_n .

Define l_1 as the unique function that verifies the following well-posed (Smyshlyaev & Krstic, 2004) PIDE

$$\begin{aligned} \partial_{xx} l_1(x, \xi_1) &= \partial_{\xi_1 \xi_1} l_1(x, \xi_1) - \lambda(x) l_1 - f_1(x, \xi_1) \\ &\quad - \int_{\xi_1}^x l_1(s, \xi_1) f_1(x, s) ds, \end{aligned} \tag{35}$$

with boundary conditions

$$l_1(x, x) = \hat{q} - \frac{1}{2} \int_0^x \lambda(s) ds, \tag{36}$$

$$l_{1\xi_1}(x, 0) = q l_1(x, 0). \tag{37}$$

The following result holds.

Proposition 6.1. *The first kernel l_1 of the inverse (26) is given by the solution of (35)–(37), whereas for $n \geq 2$, l_n is given by the following formula*

$$l_n(\hat{\xi}_0^n) = g_n(\hat{\xi}_0^n) + \int_{\xi_1}^x l_1(x, s)g_n(s, \hat{\xi}_1^n)ds. \tag{38}$$

In (38), the g_n are functions defined as follows:

$$g_n = \sum_{\substack{i_1, \dots, i_j \geq 1, j \geq 2 \\ i_1 + \dots + i_j = n}} p_j[k_j; l_{i_1}, \dots, l_{i_j}], \tag{39}$$

where the function $p_j[k_j; l_{i_1}, \dots, l_{i_j}]$ is recursively computed in the following way. Let

$$p_1[k_j; l_{i_1}, \dots, l_{i_j}] = \begin{cases} k_j(\hat{\xi}_0^j) + \int_{\xi_j}^{\xi_{j-1}} k_j(\hat{\xi}_0^{j-1}, s)l_1(s, \xi_j)ds, & i_j = 1, \\ \int_{\xi_j}^{\xi_{j-1}} k_j(\hat{\xi}_0^{j-1}, s)l_{i_j}(s, \hat{\xi}_j^{j+i_j-1})ds, & i_j > 1, \end{cases} \tag{40}$$

and for $1 \leq m \leq j - 1$, $p_{m+1}[k_j; l_{i_1}, \dots, l_{i_j}]$ is computed from $p_m[k_j; l_{i_1}, \dots, l_{i_j}]$ as follows:

$$p_{m+1}[k_j; l_{i_1}, \dots, l_{i_j}] = \begin{cases} p_m[k_j; l_{i_1}, \dots, l_{i_j}] + q_m[p_m], & i_{j-m} = 1, \\ q_m[p_m], & i_{j-m} > 1. \end{cases} \tag{41}$$

In (41),

$$q_m[p_m] = \int_{\xi_{j-m}}^{\xi_{j-m-1}} D_{j-m}^{\alpha_m, i_{j-m}} [l_{i_{j-m}}(s, \hat{\xi}_{\alpha_m}^{\alpha_m+i_{j-m}-1}) \times p_m[k_j; l_{i_1}, \dots, l_{i_j}](\hat{\xi}_0^{j-m-1}, s, \hat{\xi}_{j-m}^{\alpha_m-1})] ds, \tag{42}$$

where $\alpha_m = j - m + \sum_{\beta=j-m+1}^j i_\beta$ and the function $D_{j-m}^{\alpha_m, i_{j-m}}$ is given in Vazquez and Krstic (2008, Equation (53)).

Proof. Consider the transformation (13) and the inverse (27),

$$w = u - K[u], \tag{43}$$

$$u = w + L[w]. \tag{44}$$

We expand (43) as a sum of K_1 (the linear transformation) and the rest of the transformation (which is nonlinear). Then, we get

$$w = u - K_1[u] - \hat{K}[u], \tag{45}$$

where $\hat{K}[u] = \sum_{n=2}^\infty K_n[u]$. Denoting $z = w + \hat{K}[u]$, we get

$$z = u - K_1[u], \tag{46}$$

which is the equation of a linear Volterra transformation; hence, we can use the result of Smyshlyaev and Krstic (2004) to show that it is invertible and explicitly compute the kernel l_1 of the (linear) inverse L_1 , obtaining

$$u = z + L_1[z]. \tag{47}$$

Substituting now the definition of z in (47),

$$u = w + \hat{K}[u] + L_1[w + \hat{K}[u]], \tag{48}$$

and using the linearity of L_1 ,

$$u = w + L_1[w] + \hat{K}[u] + L_1[\hat{K}[u]]. \tag{49}$$

Replace now (44) in (49). Then,

$$u = w + L_1[w] + \hat{K}[w + L[w]] + L_1[\hat{K}[w + L[w]]]. \tag{50}$$

We define $G[w] = \hat{K}[w + L[w]]$, the composition of two Volterra series. Introducing G in (50) and using the definition of the inverse series (44) we get

$$L[w] = L_1[w] + G[w] + L_1[G[w]], \tag{51}$$

which when expanded for each $n \geq 2$, gives (38). The expression for g given by (39)–(42) follows from repeatedly applying Lemma A.1 to the definition of G as the composition of two Volterra series. \square

Remark 6. From (39)–(42) we get that the n th kernel g_n depends only on the kernels k_1, \dots, k_{n-1} and l_1, \dots, l_{n-1} . Hence, Eq. (38) gives a recursive, explicit formula to compute the kernels l_n beginning at $n = 2$ (l_1 is computed directly from (35)–(37)) up to any desired order.

6.1. Analytical example

For the analytic example of Vazquez and Krstic (2008, Section 5), we have $K[u] = K_2[u] = \hat{K}[u]$ and $l_1 = 0$ because $k_1 = f_1 = 0$. These facts greatly simplify the formulas for l_n in Proposition 6.1. We have that $l_2 = k_2$ and for $n > 2$,

$$l_n = \int_{\xi_2}^{\xi_1} k_2(x, \xi_1, s)l_{n-1}(s, \xi_2, \dots, \xi_n)ds + \sum_{i=2}^{n-2} \int_{\xi_1}^x D_1^{n-i+1, i} \left[l_i(\sigma, \hat{\xi}_{n-i+1}^n) \times \left(\int_{\xi_1}^\sigma k_2(x, \sigma, s)l_{n-i}(s, \hat{\xi}_1^{n-i})ds \right) \right] d\sigma + \int_{\xi_1}^x D_1^{2, n-1} [k_2(x, s, \xi_1)l_{n-1}(s, \hat{\xi}_2^n)] ds. \tag{52}$$

Using formula (52) and symbolical software, we explicitly find the first three kernels as $l_1 = 0$ and

$$l_2 = \xi_1 \xi_2 (x - \xi_1)(x - \xi_2), \tag{53}$$

$$l_3 = \xi_1 \xi_2 \xi_3 \left[(2x - \xi_2 - \xi_3) \left(\frac{\xi_1^5 - x^5}{5} + \frac{x^4 - \xi_1^4}{4} (x + \xi_1) + \frac{\xi_1^3 - x^3}{3} x \xi_1 \right) + (x(\xi_2 + \xi_3) - \xi_2 \xi_3) \left(\frac{x^4 + \xi_1^4}{4} + \frac{\xi_1^3 - x^3}{3} (x + \xi_1) + \frac{x^2 + \xi_1^2}{2} x \xi_1 \right) + (x - \xi_1) \times \left(\frac{\xi_2^5 - x^5}{5} + \frac{x^4 + \xi_2^4}{4} (x + \xi_2 + \xi_3) + \frac{\xi_2^3 - x^3}{3} \times (x(\xi_2 + \xi_3) + \xi_2 \xi_3) + \frac{x^2 + \xi_2^2}{2} x \xi_2 \xi_3 \right) \right]. \tag{54}$$

Using (52) we can study the convergence of the inverse Volterra series for the example. First we analyze the growth of the kernels.

Lemma 6.1. *For l_n defined as in (53) and (52), it holds that for $n \geq 2$,*

$$|l_n(x, \xi_1, \dots, \xi_n)| \leq n! \frac{1}{16^{n-1}} x^{5n-6}. \tag{55}$$

Proof. For $n = 2$, the claim of (55) is true since $l_2 = k_2 \leq \frac{1}{16} x^4$ as we found in Vazquez and Krstic (2008, Equation (75)). We now assume (55) for $n - 1, n - 2, \dots, 2$ and prove it holds for $n \geq 3$.

Taking absolute values in (52), using Vazquez and Krstic (2008, Equation (53)) and (55) for $n - 1, n - 2, \dots, 2$, we get

$$|l_n| \leq \frac{x^4}{16} \left(\frac{(n-1)!}{16^{n-2}} \int_0^x s^{5n-11} ds + \sum_{i=2}^{n-2} \binom{n}{i} \frac{(n-i)!i!}{16^{n-i-1+i-1}} \int_0^x s^{5(n-i)-6} ds \right) \times \int_0^x s^{5i-6} ds + \frac{n(n-1)!}{16^{n-2}} \int_0^x s^{5n-11} ds, \tag{56}$$

where the binomial coefficients come from using Vazquez and Krstic (2008, Remark 5). Hence,

$$\begin{aligned} |l_n| &\leq n! \frac{x^4}{16^{n-1}} \left(\frac{n+1}{n} \int_0^x s^{5n-11} ds + \sum_{i=2}^{n-2} \int_0^x s^{5i-6} ds \int_0^x s^{5(n-i)-6} ds \right) \\ &\leq n! \frac{x^4}{16^{n-1}} \left(\frac{n+1}{n(5n-10)} x^{5n-10} + x^{5(n-i)-5+5i-5} \right) \\ &\quad \times \left(\sum_{i=2}^{n-2} \frac{1}{(5(n-i)-5)(5i-5)} \right) \\ &\leq n! \frac{x^{5n-6}}{16^{n-1}} \left(\frac{n+1}{n(5n-10)} + \frac{1}{5n-10} \right) \\ &\quad \times \left(\sum_{i=2}^{n-2} \left(\frac{1}{(5(n-i)-5)} + \frac{1}{5i-5} \right) \right) \\ &\leq n! \frac{x^{5n-6}}{16^{n-1}} \left(\frac{n+1}{n(5n-10)} + \frac{n-3}{5n-10} \left(\frac{1}{5} + \frac{1}{5} \right) \right) \\ &\leq n! \frac{x^{5n-6}}{16^{n-1}} \left(\frac{5+n(n+2)}{5n(5n-10)} \right), \tag{57} \end{aligned}$$

and since $5 + n(n + 2) \leq 5n(5n - 10)$ for $n \geq 3$, inequality (55) follows for n . \square

We now state the result of Theorem 4 for the example, illustrating how to prove well-posedness for Dirichlet boundary conditions.

Proposition 6.2. Consider the closed-loop plant (9) and (10) where F is given by Vazquez and Krstic (2008, (71)–(73)) and control law (Vazquez & Krstic, 2008, (70)). Let $u_0 \in H^1(0, 1)$ be the initial condition for u , verifying the compatibility conditions

$$u_0(0) = 0, \quad u_0(1) = \frac{1}{2} \left(\int_0^1 \xi(x - \xi)u(\xi)d\xi \right)^2. \tag{58}$$

Then, there is a unique solution $u(t, x)$ such that $u \in L^2((0, \infty), H^1(0, 1))$ and the origin $u \equiv 0$ of the closed-loop system is locally exponentially stable in the L^2 and H^1 norm, i.e., there exists $C_2 > 0$ such that, if $\|u_0\|_{\mathcal{L}}^2 \leq 32$, then $\forall t \geq 0$

$$\|u(t)\|_{\mathcal{L}}^2 \leq C_2 e^{-t} \|u_0\|_{\mathcal{L}}^2, \tag{59}$$

where \mathcal{L} is either L^2 or H^1 and $C_2 > 0$ does not depend on u_0 . Moreover, we can write the closed-loop solution for $u(t, x)$ as

$$u = 2 \sum_{n=1}^{\infty} e^{-\pi^2 n^2 t} \sin(\pi n x) \int_0^1 \sin(\pi n \xi) \left[u_0(\xi) - \frac{1}{2} \times \left(\int_0^\xi \eta(\xi - \eta)u_0(\eta)d\eta \right)^2 \right] d\xi + L \left[2 \sum_{n=1}^{\infty} e^{-\pi^2 n^2 t} \right]$$

$$\times \sin(\pi n x) \int_0^1 \sin(\pi n \xi) \left[u_0(\xi) - \frac{1}{2} \left(\int_0^\xi \eta(\xi - \eta) \times u_0(\eta)d\eta \right)^2 \right] d\xi \Big]. \tag{60}$$

Proof. Using Lemma 6.1, since $x \leq 1$ we get that, for all $n, |l_n| \leq \frac{n!}{16^{n-1}}$. Hence, from Definition 2.1 and Example 2.1, we have that for the inverse Volterra series defined by $l_2 = k_2$ and (52), the radius of convergence is $\rho_L = 16^2 = 256$. The gain bound function for the transformation (13), since k is finite, can be written as $k(s) = 2s + s^2/32$. Using ρ_L and $k(s)$, and proceeding as in Corollary 5.1, we get the L^2 and H^1 results of Theorem 4 for initial conditions u_0 verifying $\|u_0\|_{\mathcal{L}}^2 \leq C_1 = k^{-1}(\rho_L)/2 = 32$. Moreover, (58) implies that $w_0(0) = w_0(1) = 0$ and since $w_0 \in H^1$, Eq. (11) has a unique solution w in $L^2((0, \infty), H^1(0, 1))$ (Evans, 1998, Theorems 3 and 4, pages 356–358) (in fact more regularity is obtained, but we do not need it). Using that

$$w_x = u_x - \frac{1}{2} \left(\int_0^x (2x - \xi)u(t, \xi)d\xi \right)^2, \tag{61}$$

$$u_x = w_x + \frac{1}{2} \left(\int_0^x (2x - \xi)L[w](t, \xi)d\xi \right)^2, \tag{62}$$

are valid ($L[w]$ converges) if $\|u\|_{L^2}^2 \leq 32$ (which is also implied in the $\mathcal{L} = H^1$ case if $\|u\|_{H^1}^2 \leq 32$), then u also has an $L^2((0, \infty), H^1(0, 1))$ solution. The explicit solutions are obtained solving the heat equation. \square

As simulation results in Vazquez and Krstic (2008, Section 6) show for the example, where $u_0 = 400x(1 - x)$ implying that $\|u_0\|_{L^2}^2 \approx 5000$, the result is far from limited to such a small neighborhood of the origin. This illustrates the rather conservative nature of Theorem 4.

7. Conclusions and open problems

The efforts on nonlinear boundary control of PDEs of parabolic type have so far resulted primarily in negative results—results that show that control cannot prevent finite time blow-up. While in this paper we formulate the first general framework in which the problem is tractable, there are some parts of the analysis that could be filled in with additional details, though their engineering relevance is minor.

In our formulation, finding the controller’s Volterra kernels is the main design task. In Vazquez and Krstic (2008) we have derived the set of equations that the kernels need to verify, a recursive set of linear hyperbolic PDEs on domains of increasing dimension and decreasing volume, with moving boundaries. In Vazquez and Krstic (2008, Section 5) we present a particular solution in detail, and then we show numerical examples in Vazquez and Krstic (2008, Section 6). However, beyond numerical evidence we have not provided any general well-posedness proof for the kernel equations and just assumed it. Nevertheless in Section 4 we derived a priori estimates to show that the existence of an H^1 solution to the kernel equations is enough to define a convergent Volterra series in the transformation and the control feedback law.

In Section 5 we provided a result of L^2 and H^1 exponential stability. We have not pursued the study of stability in higher regularity functional spaces, like the H^2 space—which would be useful to establish well-posedness of the closed-loop system. Such spaces are endowed with norms whose study under our framework requires some manipulation (term-by-term second-order differentiation) of the transformation Volterra series. Before

justifying such an operation we need more insight into the regularity of the kernel equation solutions.

The stability result is *local* in nature because it relies critically on the properties of the inverse transformation (27). Even if the transformation (13) is globally convergent in L^2 , it is not possible to generically guarantee that it has an everywhere defined inverse. We have shown through an example that this result is too conservative and there is room for improvement. However, there are plants falling in the class of (9) that are not globally stabilizable (see Vazquez and Krstic (2008, Section 3.2 and Section 6.2)) and therefore a global stability result is not possible for the whole class. Thus, to obtain a global result, we need to refine (9) and identify a subclass of systems for which the Volterra series transformation (13) is globally invertible. This implies, by Corollary 5.1, that those plants are globally stabilized by feedback law (14).

Appendix

The next technical result, proved in Vazquez and Krstic (2008), is heavily used throughout the text.

Lemma A.1. *The following two identities hold.*

$$\int_{\mathcal{T}_n(x,\xi)} f_n(\hat{\xi}_0^n) d\hat{\xi}_1^n = \int_{\mathcal{T}_{n-1}(x,\xi)} \int_{\xi_m}^{\xi_{m-1}} f_n(\hat{\xi}_0^{m-1}, s, \hat{\xi}_m^{n-1}) ds d\hat{\xi}_1^{n-1}, \tag{A.1}$$

$$\int_{\mathcal{T}_n(x,\xi)} f_n(\hat{\xi}_0^n) \int_{\mathcal{T}_m(\xi_j,\sigma)} g_m(\xi_j, \hat{\sigma}_1^m) d\hat{\sigma}_1^m d\hat{\xi}_1^n = \int_{\mathcal{T}_{n+m}(x,\xi)} D_j^{n,m}[f_n(\hat{\xi}_0^n)g_m(\xi_j, \hat{\xi}_{n+1}^{n+m})] d\hat{\xi}_1^{n+m}, \tag{A.2}$$

where $D_j^{n,m}$ was defined in Vazquez and Krstic (2008, Equation (53)).

A.1. Proof of Theorem 2

Define the (x -dependent) $L^2(\mathcal{T}_n)$ norm of $k_n(\hat{\xi}_0^n)$ as

$$\|k_n(x)\|_{L^2(\mathcal{T}_n(x))}^2 = \int_{\mathcal{T}_n(x,\xi)} k_n^2(\hat{\xi}_0^n) d\hat{\xi}_1^n \tag{A.3}$$

For simplicity we will write $\|k_n(x)\|_{L^2(\mathcal{T}_n)}$.

The proof of the theorem requires a number of technical results. The first result is used to get a simpler expression for the number of terms in the right-hand side of the kernel PIDE equation.

Lemma A.2. *For $n \geq m \geq 0$, we have that*

$$\sum_{j=0}^{n-m} \binom{m+j}{j} = \frac{(n+1)!}{(m+1)!(n-m)!}. \tag{A.4}$$

Proof. We have that

$$\begin{aligned} \sum_{j=0}^{n-m} \binom{m+j+1}{j} &= \binom{m+1}{0} + \sum_{j=1}^{n-m} \binom{m+j+1}{j} \\ &= 1 + \sum_{j=1}^{n-m} \binom{m+j}{j} + \sum_{j=1}^{n-m} \binom{m+j}{j-1} \\ &= \sum_{j=0}^{n-m} \binom{m+j}{j} + \sum_{j=0}^{n-m-1} \binom{m+j+1}{j}, \end{aligned} \tag{A.5}$$

where we have used the fact that

$$\binom{n+1}{k+1} = \binom{n}{k+1} + \binom{n}{k}. \tag{A.6}$$

Hence, from (A.5) we get

$$\begin{aligned} \sum_{j=0}^{n-m} \binom{m+j}{j} &= \sum_{j=0}^{n-m} \binom{m+j+1}{j} - \sum_{j=0}^{n-m-1} \binom{m+j+1}{j} \\ &= \binom{n+1}{n-m}, \end{aligned} \tag{A.7}$$

and the result follows. \square

The next result allows estimating the various norms arising in the proof of the theorem.

Lemma A.3. *The following estimates hold.*

(1) *For D defined in Vazquez and Krstic (2008, Equation (53)), we have that for any function g_n ,*

$$\begin{aligned} \|D_j^{n,m}[g_n f_m](x)\|_{L^2(\mathcal{T}_{n+m})}^2 &\leq x^m \frac{\|f_m\|_{\infty}^2}{m!} \frac{(m+n-j)!}{m!(n-j)!} \|g_n(x)\|_{L^2(\mathcal{T}_n)}^2. \end{aligned} \tag{A.8}$$

(2) *For B defined in Vazquez and Krstic (2008, Equation (54)), we have that*

$$\begin{aligned} \|B_n^m[k_{n-m+1}, f_m](x)\|_{L^2(\mathcal{T}_n)}^2 &\leq x^m \frac{(n+1)!(n-m+1)}{(m+1)!(n-m)!} \frac{\|f_m\|_{\infty}^2}{m!} \\ &\quad \times \|k_{n-m+1}(x)\|_{L^2(\mathcal{T}_{n-m+1})}^2. \end{aligned} \tag{A.9}$$

(3) *For C defined in Vazquez and Krstic (2008, Equation (55)), we have that*

$$\begin{aligned} \|C_n^m[k_{n-m}, h_m](x)\|_{L^2(\mathcal{T}_n)}^2 &\leq \frac{x^m n!(n-m)}{(m+1)!(n-m-1)!} \frac{\|h_m\|_{\infty}^2}{m!} \\ &\quad \times \|k_{n-m}(x)\|_{L^2(\mathcal{T}_{n-m})}^2. \end{aligned} \tag{A.10}$$

Proof. Using the definition of D , we have that

$$\begin{aligned} \|D_j^{n,m}[g_n f_m](x)\|_{L^2(\mathcal{T}_{n+m})}^2 &= \int_{\mathcal{T}_{n+m}(x,\xi)} D_j^{n,m}[g_n f_m]^2(\hat{\xi}_0^{n+m-1}) d\hat{\xi}_1^{n+m} \\ &= \int_{\mathcal{T}_{n+m}(x,\xi)} \left(\sum_{\hat{\gamma}_1^{n-j+m} \in P_{n-j}(\hat{\xi}_{j+1}^{n+m})} g_n(\hat{\xi}_0^j, \hat{\gamma}_1^{n-j}) f_m(\xi_j, \hat{\gamma}_{n-j+1}^{n+m-j}) \right)^2 \\ &\quad \times d\hat{\xi}_1^{n+m}. \end{aligned} \tag{A.11}$$

Since $P_{n-j}(\hat{\xi}_{j+1}^{n+m})$ has $\frac{(n+m-j)!}{m!(n-j)!}$ elements and as $(\sum_{k=1}^n p_k)^2 \leq n \sum_{k=1}^n p_k^2$, (A.11) yields

$$\begin{aligned} \|D[g_n f_m](x)\|_{L^2(\mathcal{T}_{n+m})}^2 &\leq \frac{(n+m-j)!}{m!(n-j)!} \\ &\quad \times \int_{\mathcal{T}_{n+m}(x,\xi)} \sum_{\hat{\gamma}_1^{n-j+m} \in P_{n-j}(\hat{\xi}_{j+1}^{n+m})} \left(g_n^2(\hat{\xi}_0^j, \hat{\gamma}_1^{n-j}) f_m^2(\xi_j, \hat{\gamma}_{n-j+1}^{n+m-j}) \right) \\ &\quad \times d\hat{\xi}_1^{n+m} \\ &= \frac{(n+m-j)!}{m!(n-j)!} \int_{\mathcal{T}_{n+m}(x,\xi)} D_j^{n,m}[g_n^2 f_m^2](\hat{\xi}_0^{n+m}) d\hat{\xi}_1^{n+m}. \end{aligned} \tag{A.12}$$

From Lemma A.1

$$\begin{aligned} & \int_{\mathcal{T}_{n+m}(x,\xi)} D_j^{n,m} [g_n^2 f_m^2] (\hat{\xi}_0^{n+m}) d\hat{\xi}_1^{n+m} \\ &= \int_{\mathcal{T}_n(x,\xi)} g_n^2 (\hat{\xi}_0^n) \left(\int_{\mathcal{T}_m(\xi_j,\sigma)} f_m^2 (\xi_j, \hat{\sigma}_1^m) d\hat{\sigma}_1^m \right) d\hat{\xi}_1^n \\ &\leq \frac{\|f_m\|_\infty^2 x^m}{m!} \int_{\mathcal{T}_n(x,\xi)} g_n^2 (\hat{\xi}_0^n) d\hat{\xi}_1^n, \end{aligned} \tag{A.13}$$

hence

$$\begin{aligned} & \|D_j^{n,m} [g_n f_m](x)\|_{L^2(\mathcal{T}_{n+m})}^2 \\ &\leq \frac{(n+m-j)!}{m!(n-j)!} \frac{\|f_m\|_\infty^2 x^m}{m!} \|g_n(x)\|_{L^2(\mathcal{T}_n)}^2, \end{aligned} \tag{A.14}$$

which gives (A.8). For (A.9), and using Lemma A.1, we get

$$\begin{aligned} & \|B_n^m [k_{n-m+1}, f_m](x)\|_{L^2(\mathcal{T}_n)}^2 \\ &= \int_{\mathcal{T}_n(x,\xi)} \left(\sum_{j=1}^{n-m+1} \int_{\xi_j}^{\xi_{j-1}} D_j^{n-m+1,m} \left[k_{n-m+1} (\hat{\xi}_0^{j-1}, s, \hat{\xi}_{n-m}^j) f_m (s, \hat{\xi}_{n-m+1}^n) \right] ds \right)^2 d\hat{\xi}_1^n \\ &\leq (n-m+1) \sum_{j=1}^{n-m+1} \int_{\mathcal{T}_n(x,\xi)} \int_{\xi_j}^{\xi_{j-1}} D_j^{n-m+1,m} \\ &\quad \times \left[k_{n-m+1} (\hat{\xi}_0^{j-1}, s, \hat{\xi}_{n-m}^j) f_m (s, \hat{\xi}_{n-m+1}^n) \right]^2 ds d\hat{\xi}_1^n \\ &= (n-m+1) \sum_{j=1}^{n-m+1} \int_{\mathcal{T}_{n+1}(x,\xi)} D_j^{n-m+1,m} \\ &\quad \times \left[k_{n-m+1} (\hat{\xi}_0^{n-m+1}) f_m (\xi_j, \hat{\xi}_{n-m}^{n+1}) \right]^2 d\hat{\xi}_1^{n+1} \\ &= (n-m+1) \sum_{j=1}^{n-m+1} \|D_j^{n-m+1,m} \\ &\quad \times [k_{n-m+1} f_m](x)\|_{L^2(\mathcal{T}_{n+1})}^2. \end{aligned} \tag{A.15}$$

Using then (A.8) we get

$$\begin{aligned} & \|B_n^m [k_{n-m+1}, f_m](x)\|_{L^2(\mathcal{T}_n)}^2 \\ &\leq (n-m+1) \left(\sum_{j=1}^{n-m+1} \frac{(n+1-j)!}{m!(n-m+1-j)!} \right) \\ &\quad \times x^m \frac{\|f_m\|_\infty^2}{m!} \|k_{n-m+1}(x)\|_{L^2(\mathcal{T}_{n-m+1})}^2. \end{aligned} \tag{A.16}$$

Applying Lemma A.2 on the sum in (A.16), we obtain

$$\begin{aligned} & \sum_{j=1}^{n-m+1} \frac{(n+1-j)!}{m!(n-m+1-j)!} = \sum_{j=0}^{n-m} \frac{(m+j)!}{m!j!} \\ &= \sum_{j=0}^{n-m} \binom{m+j}{j} \\ &= \frac{(n+1)!}{(m+1)!(n-m)!}, \end{aligned} \tag{A.17}$$

hence,

$$\begin{aligned} & \|B_n^m [k_{n-m+1}, f_m](x)\|_{L^2(\mathcal{T}_n)}^2 \leq x^m \frac{(n+1)!(n-m+1)}{(m+1)!(n-m)!} \frac{\|f_m\|_\infty^2}{m!} \\ &\quad \times \|k_{n-m+1}(x)\|_{L^2(\mathcal{T}_{n-m+1})}^2. \end{aligned} \tag{A.18}$$

The estimate for C is obtained in the same way as the estimate for B. The result then follows. \square

Remark 7. Since $I[k_n, f_1] = B_n^1[k_n, f_1]$, we get

$$\|I[k_n, f_1](x)\|_{L^2(\mathcal{T}_n)}^2 \leq \frac{n^2(n+1)}{2} \|f_1\|_{\xi_\infty}^2 \|k_n(x)\|_{L^2(\mathcal{T}_n)}^2. \tag{A.19}$$

The next lemma is useful to treat the Robin boundary condition.

Lemma A.4. Let $n \geq 1, q > 0, k_n(\hat{\xi}_0^n) \in H^1(\mathcal{T}_n)$. Then

$$\begin{aligned} & q \int_{\mathcal{T}_{n-1}(x,\xi)} k_n^2(\hat{\xi}_0^{n-1}, 0) d\hat{\xi}_1^{n-1} \leq q \int_{\mathcal{T}_{n-1}(x,\xi)} k_n^2(x, x, \hat{\xi}_1^{n-1}) d\hat{\xi}_1^{n-1} \\ &\quad + q^2 \|k_n(x)\|_{L^2(\mathcal{T}_n)}^2 + \sum_{j=1}^n \|k_{n\xi_j}^2(x)\|_{L^2(\mathcal{T}_n)}. \end{aligned} \tag{A.20}$$

Proof. By the fundamental theorem of calculus,

$$\begin{aligned} & q \int_{\mathcal{T}_{n-1}(x,\xi)} k_n^2(\hat{\xi}_0^{n-1}, 0) d\hat{\xi}_1^{n-1} \\ &\quad - q \int_{\mathcal{T}_{n-1}(x,\xi)} k_n^2(\hat{\xi}_0^{n-1}, \xi_{n-1}) d\hat{\xi}_1^{n-1} \\ &= -q \int_{\mathcal{T}_n(x,\xi)} \partial_{\xi_n} k_n^2(\hat{\xi}_0^n) d\hat{\xi}_1^n \\ &= -2q \int_{\mathcal{T}_n(x,\xi)} k_n k_{n\xi_n}(\hat{\xi}_0^n) d\hat{\xi}_1^n. \end{aligned} \tag{A.21}$$

Similarly, using Lemma A.1, for $j = 0, \dots, n-2$,

$$\begin{aligned} & q \int_{\mathcal{T}_{n-1}(x,\xi)} k_n^2(\hat{\xi}_0^j, \xi_{j+1}, \hat{\xi}_{j+1}^{n-1}) d\hat{\xi}_1^{n-1} \\ &\quad - q \int_{\mathcal{T}_{n-1}(x,\xi)} k_n^2(\hat{\xi}_0^j, \xi_j, \hat{\xi}_{j+1}^{n-1}) d\hat{\xi}_1^{n-1} \\ &= -q \int_{\mathcal{T}_n(x,\xi)} \partial_{\xi_j} k_n^2(\hat{\xi}_0^n) d\hat{\xi}_1^n \\ &= -2q \int_{\mathcal{T}_n(x,\xi)} k_n k_{n\xi_{j+1}}(\hat{\xi}_0^n) d\hat{\xi}_1^n, \end{aligned} \tag{A.22}$$

hence

$$\begin{aligned} & q \int_{\mathcal{T}_{n-1}(x,\xi)} k_n^2(\hat{\xi}_0^{n-1}, 0) d\hat{\xi}_1^{n-1} \\ &\quad - q \int_{\mathcal{T}_{n-1}(x,\xi)} k_n^2(x, x, \hat{\xi}_1^{n-1}) d\hat{\xi}_1^{n-1} \\ &= -2q \int_{\mathcal{T}_n(x,\xi)} k_n \left(\sum_{j=1}^n k_{n\xi_j} \right) (\hat{\xi}_0^n) d\hat{\xi}_1^n \\ &\leq q^2 \|k_n(x)\|_{L^2(\mathcal{T}_n)}^2 + \sum_{j=1}^n \|k_{n\xi_j}^2(x)\|_{L^2(\mathcal{T}_n)}, \end{aligned} \tag{A.23}$$

and the result follows. \square

The next lemma is used to get a precise estimate of the kernel growth.

Lemma A.5. Let $n > 2, x \in [0, 1]$ and $\gamma > 0$. Then

$$\int_0^x \xi^{\frac{n-1}{2}} \exp(\gamma\xi) d\xi \leq \frac{x^{\frac{n}{2}} \exp(\gamma x)}{\sqrt{n+1}\sqrt{D}}. \tag{A.24}$$

Proof. Using the Cauchy–Schwartz inequality,

$$\begin{aligned} \int_0^x \xi^{\frac{n-1}{2}} \exp(\gamma \xi) \, d\xi &\leq \frac{x^{\frac{n}{2}} \exp(\gamma x)}{\sqrt{n+1}\sqrt{D}} \\ &\leq \sqrt{\left(\int_0^x \xi^{n-1} \, d\xi\right) \left(\int_0^x \exp\left(\frac{\gamma}{2}\xi\right) \, d\xi\right)} \\ &= \sqrt{\frac{x^n (\exp(\frac{\gamma}{2}x) - 1)}{2n\gamma}} \\ &\leq \frac{x^{\frac{n}{2}} \exp(\gamma x)}{\sqrt{n+1}\sqrt{\gamma}}, \end{aligned} \tag{A.25}$$

where we have used that $n + 1 \leq 2n$. Hence the result follows. \square

The next result is the main ingredient in the proof of **Theorem 2**.

Proposition A.1. Let $g_n(x) \geq 0$ be a sequence of differentiable functions defined for $n \geq 1, x \in [0, 1]$, and verifying $g_n(0) = 0$. Assume the following estimate holds for $n \geq 1$,

$$\begin{aligned} \frac{d}{dx} g_n &\leq (nB + E)g_n(x) + C^{n-1}D\sqrt{\frac{x^n n!}{2}} \\ &\quad + \frac{\sqrt{(n+1)!}}{B} \sum_{m=2}^n C^{m-1} D g_{n-m+1}(x) \\ &\quad \times \sqrt{\frac{x^m}{m(n-m+1)!} \left(1 + \frac{2}{\sqrt{(n+1)x}}\right)}, \end{aligned} \tag{A.26}$$

where $B, C, E > 0$ and $D \geq B^2$. Then, $g_n(x)$ verifies the following bound:

$$g_n(x) \leq \sqrt{n!} D C^{n-1} x^{n/2} \exp(((nB + E + \gamma)x)), \tag{A.27}$$

where $\gamma = 4 \frac{D^2(1+2\sqrt{B})^2}{B^4} > 1$.

Proof. We prove the claim by complete induction. For $n = 1$, the bound for g_1 is not dependent on other g_n 's:

$$\frac{d}{dx} g_1(x) \leq (B + E)g_1 + D\sqrt{\frac{x}{2}}. \tag{A.28}$$

Using the comparison principle (Khalil), since $g_1(0) = 0$,

$$\begin{aligned} g_1(x) &\leq D \frac{1}{\sqrt{2}} \int_0^x \sqrt{\xi} \exp(((B + E)(x - \xi))) \, d\xi \\ &\leq D\sqrt{x} \exp((B + E)x) \\ &\leq D\sqrt{x} \exp((B + E + \gamma)x), \end{aligned} \tag{A.29}$$

so the result follows for $n = 1$. For $n \geq 2$, we assume that the claim holds for g_j if $j = 1, \dots, n - 1$. Then $g'_n(x)$ is bounded as follows

$$\begin{aligned} \frac{d}{dx} g_n(x) &\leq (nB + E)g_n(x) + C^{n-1}D\sqrt{\frac{x^n n!}{2}} \\ &\quad + \frac{\sqrt{(n+1)!}}{B} \sum_{m=2}^n C^{n-1} D^2 \sqrt{(n-m+1)!} \\ &\quad \times e^{(B(n-m+1)+E+\gamma)x} \sqrt{\frac{x^{n+1}}{m(n-m+1)!}} \\ &\quad \times \left(1 + \frac{2}{\sqrt{(n+1)x}}\right) \\ &= nB g_n(x) + C^{n-1}D\sqrt{\frac{x^n n!}{2}} + C^{n-1} \frac{D^2}{B} \end{aligned}$$

$$\begin{aligned} &\times \sqrt{(n+1)!} \sqrt{x^{n+1}} e^{(\gamma+E)x} \sum_{m=2}^n e^{B(n-m+1)x} \\ &\times \frac{1}{\sqrt{m}} \left(1 + \frac{2}{\sqrt{(n+1)x}}\right). \end{aligned} \tag{A.30}$$

Now, denote $z = \exp(Bx)$. Note that the sum in the last line of (A.30) can be written as

$$\begin{aligned} \sum_{m=2}^n \exp(B(n-m+1)x) \frac{1}{\sqrt{m}} &= \sum_{m=2}^n \frac{z^{n-m+1}}{\sqrt{m}} \\ &\leq \frac{1}{\sqrt{2}} \sum_{m=1}^{n-1} z^m \\ &= \frac{1}{\sqrt{2}} \frac{z^n - z}{z - 1}. \end{aligned} \tag{A.31}$$

Since $z = \exp(Bx)$, we have that $z - 1 \geq Bx$, which implies that

$$\begin{aligned} \sum_{m=2}^n \exp(B(n-m+1)x) &\leq \frac{\exp(Bnx) - \exp(Bx)}{\exp(Bx) - 1} \\ &\leq \frac{\exp(Bnx)}{Bx}. \end{aligned} \tag{A.32}$$

Similarly, we can also write

$$\begin{aligned} \sum_{m=2}^n \exp(B(n-m+1)x) &\leq \sqrt{\sum_{m=2}^n e^{B(n-m+1)x}} \sqrt{\sum_{m=2}^n e^{B(n-m+1)x}} \\ &\leq \sqrt{\frac{\exp(Bnx)}{Bx}} \sqrt{(n-1) \exp(B(n-1)x)} \\ &= \sqrt{\frac{n+1}{Bx}} \exp(Bnx). \end{aligned} \tag{A.33}$$

We use (A.33) for the part of (A.30) affected by $\frac{2}{\sqrt{(n+1)x}}$, and (A.32) for the rest. Then (A.30) yields

$$\begin{aligned} \frac{d}{dx} g_n(x) &\leq (nB + E)g_n(x) + C^{n-1}D\sqrt{\frac{x^n n!}{2}} \\ &\quad + C^{n-1} \frac{D^2}{B^2} \sqrt{(n+1)!} \sqrt{x^{n-1}} \\ &\quad \times \frac{1}{\sqrt{2}} e^{Bnx+Ex+\gamma x} (1 + 2\sqrt{B}). \end{aligned} \tag{A.34}$$

Integrating and since $g_n(0) = 0$,

$$\begin{aligned} g_n(x) &\leq C^{n-1} \sqrt{\frac{n!}{2}} \left[D \int_0^x e^{(Bn+E)(x-\xi)} \xi^{n/2} \, d\xi + \frac{D^2}{B^2} \right. \\ &\quad \times \sqrt{n+1} (1 + 2\sqrt{B}) \int_0^x e^{(Bn+E)x} \xi^{\frac{n-1}{2}} e^{\gamma \xi} \, d\xi \left. \right] \\ &\leq C^{n-1} D \sqrt{\frac{n!}{2}} e^{(Bn+E)x} \int_0^x \xi^{n/2} \, d\xi \\ &\quad + C^{n-1} \sqrt{(n+1)!} \frac{D^2}{B^2 \sqrt{2}} (1 + 2\sqrt{B}) e^{(Bn+E)x} \\ &\quad \times \int_0^x \xi^{\frac{n-1}{2}} e^{\gamma \xi} \, d\xi \\ &\leq C^{n-1} D \sqrt{\frac{n!}{2}} x^{n/2+1} \frac{2}{n+2} e^{(Bn+E)x} \\ &\quad + C^{n-1} \sqrt{(n+1)!} \frac{D^2}{B^2 \sqrt{2}} (1 + 2\sqrt{B}) e^{(Bn+E)x} \end{aligned}$$

$$\begin{aligned} & \times \frac{x^n}{\sqrt{n+1}} \frac{e^{\gamma \xi}}{2 \frac{D(1+2\sqrt{B})}{B^2}} \\ & \leq \sqrt{n!} x^{n/2} C^{n-1} D e^{Bnx+Ex+\gamma x} \frac{1}{\sqrt{2}} \left(\frac{2}{n+2} + \frac{1}{2} \right) \\ & \leq \sqrt{n!} x^{n/2} C^{n-1} D e^{Bnx+Ex+\gamma x}, \end{aligned} \tag{A.35}$$

where we have used Lemma A.5 and the definition of $\gamma = 4 \frac{D^2(1+2\sqrt{B})^2}{B^4}$. This completes the proof. \square

Proposition A.2. Let Assumptions 4.1 and 4.2 hold. Define for each $n \geq 1$ the function $\psi_n(\xi_0^n)$ as the solution of the wave equation

$$\psi_{nxx} = \sum_{i=1}^n \psi_{n\xi_i \xi_i}, \tag{A.36}$$

with boundary conditions

$$\psi_n(x, x, \xi_2^n) = \hat{\phi}_n(x, \xi_2^n), \tag{A.37}$$

$$\psi_{n\xi_{i-1}}(\xi_0^n) \Big|_{\xi_{i-1}=\xi_i} = \psi_{n\xi_i}(\xi_0^n) \Big|_{\xi_{i-1}=\xi_i}, \quad i = 2, \dots, n, \tag{A.38}$$

$$\psi_{n\xi_n}(\xi_0^{n-1}, 0) = q\psi_n(\xi_0^{n-1}, 0), \tag{A.39}$$

where $\hat{\phi}_1(x) = \hat{q} - 1/2 \int_0^x \lambda(s) ds$, and for $n \geq 2$, $\hat{\phi}_n(x, \xi_2^n) = -1/2 \int_{\xi_2}^x h_{n-1}(s, \xi_2^n)$. Then

$$\begin{aligned} & \|\psi_n(x)\|_{L^2(\mathcal{T}_n)}^2 + \|\psi_{nxx}(x)\|_{L^2(\mathcal{T}_n)}^2 \\ & \leq 4(\rho_h + 1)D_h^2 x^n (1 + |q|)^2 e^{1+|\hat{q}|} \frac{(n-1)!}{\rho_h^{2(n-1)}}, \end{aligned} \tag{A.40}$$

and

$$\|\varphi_n\|_{L^2(\mathcal{T}_n)}^2 \leq D_\varphi^2 \left(\frac{1}{\rho_\varphi} \right)^{2(n-1)} n! x^n, \tag{A.41}$$

where $\varphi_n = -\sum_{i=1}^n \lambda(\xi_i) \psi_n - I_n[\psi_n, f_1] - c\psi_n - \sum_{m=2}^n B_n^m[\psi_{n-m+1}, f_m] - \sum_{m=1}^{n-1} C_n^m[\psi_{n-m}, h_m]$ and $\rho_\varphi = \frac{\min\{\rho_f, \rho_h\}}{2}$, $D_\varphi = 2((\rho_h + 1)D_h)(|q| + 1) \exp(1 + |\hat{q}|) \sqrt{(\rho_h + 1)^2 D_h^2 + D_f^2}$.

Proof. Under Assumption 4.2, there is an H_1 solution to (A.36)–(A.39). Consider

$$\begin{aligned} L_{\psi_n}(x) &= \int_{\mathcal{T}_n(x, \xi)} \frac{(1 + \hat{q}^2)\psi_n^2 + \psi_{nx}^2 + \sum_{j=1}^n \psi_{n\xi_j}^2}{2} d\hat{\xi}_1^n \\ &+ \frac{q}{2} \int_{\mathcal{T}_{n-1}(x, \xi)} \psi_n^2(x, \hat{\xi}_1^{n-1}, 0) d\hat{\xi}_1^{n-1} \\ &+ \frac{|\hat{q}|}{2} \int_{\mathcal{T}_{n-1}(x, \xi)} \hat{\phi}_n^2(x, \hat{\xi}_1^{n-1}) d\hat{\xi}_1^{n-1}. \end{aligned} \tag{A.42}$$

By Lemma A.4, $L_{\psi_n}(x) \geq 0$. It is straightforward to show (see Proof of Theorem 2), using (A.36)–(A.39), that for $n \geq 2$,

$$\begin{aligned} L'_{\psi_n}(x) &= (1 + \hat{q}^2) \int_{\mathcal{T}_n(x, \xi)} \psi_n \psi_{nxx}(x, \hat{\xi}_1^n) d\hat{\xi}_1^n \\ &+ |\hat{q}| \int_{\mathcal{T}_{n-1}(x, \xi)} \hat{\phi}_n \hat{\phi}_{nxx}(x, \hat{\xi}_1^{n-1}) d\hat{\xi}_1^{n-1} \\ &+ \int_{\mathcal{T}_{n-1}(x, \xi)} \frac{(1 + \hat{q}^2)\hat{\phi}_n^2 + \hat{\phi}_{nx}^2 + \sum_{j=2}^n \hat{\phi}_{n\xi_j}^2}{2} (x, \hat{\xi}_2^n) d\hat{\xi}_2^n \end{aligned}$$

$$\begin{aligned} & + \frac{q}{2} \int_{\mathcal{T}_{n-2}(x, \xi)} \hat{\phi}_n^2(x, \hat{\xi}_1^{n-2}, 0) d\hat{\xi}_1^{n-2} \\ & + \frac{|\hat{q}|}{2} \int_{\mathcal{T}_{n-2}(x, \xi)} \hat{\phi}_n^2(x, x, \hat{\xi}_1^{n-2}) d\hat{\xi}_1^{n-2} \\ & \leq (1 + |\hat{q}|)L_{\psi_n}(x) + (n-1)! D_h^2 \left(\frac{1}{\rho_h} \right)^{2(n-2)} \\ & \times \left(\frac{x^{n-1}(1 + |\hat{q}|)^2 n + 2}{(n-1)!} + \frac{|q|}{2} \frac{x^{n-2}}{(n-2)!} \frac{x^2}{8} \right) \\ & \leq (1 + |\hat{q}|)L_{\psi_n}(x) + (n-1)! x^{n-1} D_h^2 \\ & \times \left(\frac{1}{\rho_h} \right)^{2(n-2)} (1 + |\hat{q}|)^2 (1 + |q|) \frac{n+2}{8}. \end{aligned} \tag{A.43}$$

Since $L_{\psi_n}(0) = 0$, integrating (A.43) and using that $\|\psi_n(x)\|_{L^2(\mathcal{T}_n)}^2 + \|\psi_{nxx}(x)\|_{L^2(\mathcal{T}_n)}^2 \leq 2L_{\psi_n}(x)$ we get (A.40) for $n \geq 2$. For $n = 1$, (A.40) follows similarly.

Finally, to get the estimate on φ we use (A.40) and Lemma A.3 in the definition of φ , obtaining (A.41). This finishes the proof. \square

Now we prove the main theorem.

Proof of Theorem 2. We obtain the estimates (22) by a Lyapunov method. First, we homogenize the equation defining $\hat{k}_n(\xi_0^n) = k_n(\xi_0^n) - \psi_n(\xi_0^n)$, where ψ_n was defined in Proposition A.2.

Then the kernels \hat{k}_n verify (Vazquez & Krstic, 2008, Equations (40)–(47)) with the equation in Vazquez and Krstic (2008, Equation (43)) replaced by $\hat{k}_n(x, x, \xi_2^n) = 0$ and an additional right-hand-side term, φ_n , as defined in Proposition A.2, in Vazquez and Krstic (2008, Equation (40)). For simplicity, we drop hats and define $\gamma = \max\{1, \|f_1\|_\infty + \|\lambda\|_\infty\}$.

Consider, for each $n \geq 1$, the following Lyapunov function

$$\begin{aligned} L_n(x) &= \int_{\mathcal{T}_n(x, \xi)} \frac{(n^2 \gamma + \hat{q}^2)k_n^2 + k_{nx}^2 + \sum_{j=1}^n k_{n\xi_j}^2}{2} d\hat{\xi}_1^n \\ &+ \frac{q}{2} \int_{\mathcal{T}_{n-1}(x, \xi)} k_n^2(x, \hat{\xi}_1^{n-1}, 0) d\hat{\xi}_1^{n-1}, \end{aligned} \tag{A.44}$$

which is positive definite by application of Lemma A.4. Note that (A.44) is equivalent to the H^1 norm in $\mathcal{T}_n(x)$:

$$\begin{aligned} L_n(x) &= \frac{n^2 \gamma \|k_n(x)\|_{L^2(\mathcal{T}_n)}^2 + \|k_{nx}(x)\|_{L^2(\mathcal{T}_n)}^2}{2} \\ &+ \frac{\sum_{j=1}^n \|k_{n\xi_j}(x)\|_{L^2(\mathcal{T}_n)}^2}{2}. \end{aligned} \tag{A.45}$$

If $q = \infty$ (Dirichlet boundary condition at $x = 0$), then set $q = 0$ in (A.44) and the rest of the proof.

Taking the x derivative of (A.44), we get

$$\begin{aligned} L'_n(x) &= (n^2 \gamma + \hat{q}^2) \int_{\mathcal{T}_n(x, \xi)} k_n k_{nxx} d\hat{\xi}_1^n \\ &+ \int_{\mathcal{T}_n(x, \xi)} \left(k_{nx} k_{nxx} + \sum_{j=1}^n k_{n\xi_j} k_{n\xi_j x}(x, \hat{\xi}_1^n) \right) d\hat{\xi}_1^n \\ &+ \int_{\mathcal{T}_{n-1}(x, \xi)} \frac{\gamma k_n^2(x, x, \hat{\xi}_1^{n-1}) + k_{nx}^2(x, x, \hat{\xi}_1^{n-1})}{2} \\ &+ \frac{\sum_{j=1}^n k_{n\xi_j}^2(x, x, \hat{\xi}_1^{n-1})}{2} d\hat{\xi}_1^{n-1} \end{aligned}$$

$$\begin{aligned}
 &+ q \int_{\mathcal{J}_{n-1}(x,\xi)} k_n k_{nx}(x, \hat{\xi}_1^{n-1}, 0) d\hat{\xi}_1^{n-1} \\
 &+ \frac{q}{2} \int_{\mathcal{J}_{n-2}(x,\xi)} k_n^2(x, x, \hat{\xi}_1^{n-2}, 0) d\hat{\xi}_1^{n-2}. \tag{A.46}
 \end{aligned}$$

From the boundary conditions, $k_n^2(x, x, \hat{\xi}_1^{n-1}) = 0$, so $k_{n\xi_j}^2(x, x, \hat{\xi}_1^{n-1}) = 0$ for $j \geq 2$, and $k_{nx}(x, x, \hat{\xi}_1^{n-1}) = -k_{n\xi_1}(x, x, \hat{\xi}_1^{n-1})$. Hence, the third line of (A.46) is greatly simplified as follows.

$$\begin{aligned}
 &\int_{\mathcal{J}_{n-1}(x,\xi)} \frac{(n^2\gamma + \hat{q}^2)k_n^2(x, x, \hat{\xi}_1^{n-1}) + k_{nx}^2(x, x, \hat{\xi}_1^{n-1})}{2} \\
 &+ \frac{\sum_{j=1}^n k_{n\xi_j}^2(x, x, \hat{\xi}_1^{n-1})}{2} d\hat{\xi}_1^{n-1} \\
 &= \int_{\mathcal{J}_{n-1}(x,\xi)} k_{nx}^2(x, x, \hat{\xi}_1^{n-1}) d\hat{\xi}_1^{n-1}. \tag{A.47}
 \end{aligned}$$

Using the kernel PIDE equation for the first term in the second line of (A.46) we get

$$\begin{aligned}
 &\int_{\mathcal{J}_n(x,\xi)} k_{nx} k_{nxx}(x, \hat{\xi}_1^n) d\hat{\xi}_1^n \\
 &= \sum_{j=1}^n \int_{\mathcal{J}_n(x,\xi)} k_{nx} k_{n\xi_j \xi_j}(x, \hat{\xi}_1^n) d\hat{\xi}_1^n \\
 &+ \int_{\mathcal{J}_n(x,\xi)} \left(\sum_{j=1}^n \lambda(\xi_j) \right) k_{nx} k(x, \hat{\xi}_1^n) d\hat{\xi}_1^n \\
 &- \int_{\mathcal{J}_n(x,\xi)} (k_{nx} f_n + k_{nx} \varphi_n)(x, \hat{\xi}_1^n) d\hat{\xi}_1^n \\
 &+ \int_{\mathcal{J}_n(x,\xi)} k_{nx} I[k_n, f_1](x, \hat{\xi}_1^n) d\hat{\xi}_1^n \\
 &+ \sum_{m=2}^n \int_{\mathcal{J}_n(x,\xi)} k_{nx} B_n^m [k_{n-m+1}, f_m](x, \hat{\xi}_1^n) d\hat{\xi}_1^n \\
 &+ \sum_{m=1}^{n-1} \int_{\mathcal{J}_n(x,\xi)} k_{nx} C_n^m [k_{n-m}, h_m](x, \hat{\xi}_1^n) d\hat{\xi}_1^n. \tag{A.48}
 \end{aligned}$$

Now the first integral in the second line of (A.48) can be expressed as

$$\begin{aligned}
 &\sum_{j=1}^n \int_{\mathcal{J}_n(x,\xi)} k_{nx} k_{n\xi_j \xi_j}(x, \hat{\xi}_1^n) d\hat{\xi}_1^n \\
 &= \sum_{j=1}^{n-1} \int_{\mathcal{J}_{n-1}(x,\xi)} \int_{\xi_j}^{\xi_{j-1}} k_{nx} k_{n\xi_j \xi_j}(x, \hat{\xi}_1^{j-1}, s, \hat{\xi}_j^{n-1}) ds d\hat{\xi}_1^{n-1} \\
 &+ \int_{\mathcal{J}_n(x,\xi)} k_{nx} k_{n\xi_n \xi_n}(x, \hat{\xi}_1^n) d\hat{\xi}_1^n. \tag{A.49}
 \end{aligned}$$

Integrating by parts in ξ_j in (A.49),

$$\begin{aligned}
 &\sum_{j=1}^n \int_{\mathcal{J}_n(x,\xi)} k_{nx} k_{n\xi_j \xi_j}(x, \hat{\xi}_1^n) d\hat{\xi}_1^n \\
 &= - \sum_{j=1}^{n-1} \int_{\mathcal{J}_{n-1}(x,\xi)} \int_{\xi_j}^{\xi_{j-1}} k_{nx \xi_j} k_{n\xi_j}(x, \hat{\xi}_1^{j-1}, s, \xi_j^{n-1}) ds d\hat{\xi}_1^{n-1} \\
 &- \int_{\mathcal{J}_n(x,\xi)} k_{nx \xi_n} k_{n\xi_n}(x, \hat{\xi}_1^n) d\hat{\xi}_1^n \\
 &+ \sum_{j=1}^{n-1} \int_{\mathcal{J}_{n-1}(x,\xi)} \left(k_{nx} k_{n\xi_j}(x, \hat{\xi}_1^{j-1}, \xi_{j-1}, \hat{\xi}_j^{n-1}) \right.
 \end{aligned}$$

$$\begin{aligned}
 &\left. - k_{nx} k_{n\xi_j}(x, \hat{\xi}_1^{j-1}, \xi_j, \hat{\xi}_j^{n-1}) \right) d\hat{\xi}_1^{n-1} \\
 &+ \int_{\mathcal{J}_n(x,\xi)} \left(k_{nx} k_{n\xi_n}(x, \hat{\xi}_1^{n-1}, \xi_{n-1}) \right. \\
 &\left. - k_{nx} k_{n\xi_n}(x, \hat{\xi}_1^{n-1}, 0) \right) d\hat{\xi}_1^{n-1} \\
 &= - \sum_{j=1}^n \int_{\mathcal{J}_n(x,\xi)} k_{nx \xi_j} k_{n\xi_j}(x, \hat{\xi}_1^n) d\hat{\xi}_1^n \\
 &+ \int_{\mathcal{J}_{n-1}(x,\xi)} k_{nx} k_{n\xi_1}(x, x, \hat{\xi}_1^{n-1}) d\hat{\xi}_1^{n-1} \\
 &+ \sum_{j=1}^{n-1} \int_{\mathcal{J}_{n-1}(x,\xi)} \left(k_{nx} k_{n\xi_{j+1}}(x, \hat{\xi}_1^j, \xi_j, \hat{\xi}_j^{n-1}) \right. \\
 &\left. - k_{nx} k_{n\xi_j}(x, \hat{\xi}_1^{j-1}, \xi_j, \hat{\xi}_j^{n-1}) \right) d\hat{\xi}_1^{n-1} \\
 &- \int_{\mathcal{J}_{n-1}(x,\xi)} k_{nx} k_{n\xi_n}(x, \hat{\xi}_1^{n-1}, 0) d\hat{\xi}_1^{n-1}, \tag{A.50}
 \end{aligned}$$

and using the Neumann boundary conditions for $\xi_j = \xi_{j+1}$ and the Robin boundary conditions for $\xi_n = 0$, we get

$$\begin{aligned}
 &\sum_{j=1}^n \int_{\mathcal{J}_n(x,\xi)} k_{nx} k_{n\xi_j \xi_j}(x, \hat{\xi}_1^n) d\hat{\xi}_1^n \\
 &= - \sum_{j=1}^n \int_{\mathcal{J}_n(x,\xi)} k_{nx \xi_j} k_{n\xi_j}(x, \hat{\xi}_1^n) d\hat{\xi}_1^n \\
 &- \int_{\mathcal{J}_{n-1}(x,\xi)} k_{nx}^2(x, x, \hat{\xi}_1^{n-1}) d\hat{\xi}_1^{n-1} \\
 &- q \int_{\mathcal{J}_{n-1}(x,\xi)} k_{nx} k_n(x, \hat{\xi}_1^{n-1}, 0) d\hat{\xi}_1^{n-1}, \tag{A.51}
 \end{aligned}$$

where we have used again that $k_{n\xi_1}(x, x, \hat{\xi}_1^{n-1}) = -k_{nx}(x, x, \hat{\xi}_1^{n-1})$. Then some terms cancel out, leaving

$$\begin{aligned}
 \frac{d}{dx} L_n &= (n^2\gamma + \hat{q}^2) \int_{\mathcal{J}_n(x,\xi)} k_n k_{nx}(x, \hat{\xi}_1^n) d\hat{\xi}_1^n \\
 &+ \int_{\mathcal{J}_n(x,\xi)} \left(\sum_{j=1}^n \lambda(\xi_j) \right) k_{nx} k(x, \hat{\xi}_1^n) d\hat{\xi}_1^n \\
 &+ \int_{\mathcal{J}_n(x,\xi)} (k_{nx} \varphi_n(x, \hat{\xi}_1^n)) d\hat{\xi}_1^n \\
 &+ \int_{\mathcal{J}_n(x,\xi)} k_{nx} I[k_n, f_1](x, \hat{\xi}_1^n) d\hat{\xi}_1^n \\
 &+ \sum_{m=2}^n \int_{\mathcal{J}_n(x,\xi)} k_{nx} B_n^m [k_{n-m+1}, f_m](x, \hat{\xi}_1^n) d\hat{\xi}_1^n \\
 &+ \sum_{m=1}^{n-1} \int_{\mathcal{J}_n(x,\xi)} k_{nx} C_n^m [k_{n-m}, h_m](x, \hat{\xi}_1^n) d\hat{\xi}_1^n \\
 &- \int_{\mathcal{J}_n(x,\xi)} k_{nx} f_n(x, \hat{\xi}_1^n) d\hat{\xi}_1^n, \tag{A.52}
 \end{aligned}$$

and using Cauchy–Schwartz and Young’s inequalities, and the definition of γ (note that if $f_1 = 0$ then $I[k_n, f_1] = 0$),

$$\begin{aligned}
 \frac{dL_n}{dx} &\leq (n\sqrt{\gamma} + |\hat{q}|) \int_{\mathcal{J}_n(x,\xi)} \frac{(n^2\gamma + \hat{q}^2)k_n^2 + k_{nx}^2}{2} d\hat{\xi}_1^n \\
 &+ \sqrt{\int_{\mathcal{J}_n(x,\xi)} k_{nx}^2 d\hat{\xi}_1^n} \left(\|f_n\|_\infty \sqrt{\frac{x^n}{n!}} + \sqrt{\int_{\mathcal{J}_n(x,\xi)} \varphi_n^2 d\hat{\xi}_1^n} \right)
 \end{aligned}$$

