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Control of 1D parabolic PDEs with Volterra nonlinearities, Part II: Analysis[☆]

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ABSTRACT

For a class of stabilizing boundary controllers for nonlinear 1D parabolic PDEs introduced in a companion paper, we derive bounds for the gain kernels of our nonlinear Volterra controllers, prove the convergence of the series in the feedback laws, and establish the stability properties of the closed-loop system. We show that the state transformation is at least locally invertible and include an explicit construction for computing the inverse of the transformation. Using the inverse, we show L^2 and H^1 exponential stability and explicitly construct the exponentially decaying closed-loop solutions. We then illustrate the theoretical results on an analytically tractable example.

1. Introduction

For a class of stabilizing boundary controllers for nonlinear 1D parabolic PDEs introduced (in full detail and with examples) in a companion paper (Vazquez & Krstic, 2008), we derive bounds for the gain kernels of our nonlinear Volterra controllers (in the Appendix), prove the convergence of the series in the feedback laws (in Section 4) and establish the stability properties of the closed-loop system (in Section 5). We show that the state transformation is at least locally invertible and include an explicit construction for computing the inverse of the transformation (in Section 6). Using the inverse, we show L^2 and H^1 local exponential stability and explicitly construct the exponentially decaying closed-loop solutions. We then illustrate (in Section 6.1) the theoretical results on an analytically tractable example, introduced in Vazquez and Krstic (2008, Section 5).

2. Preliminaries

Define, as in Vazquez and Krstic (2008), $\xi_0 = x$ and for $i \leq n$, $\hat{\xi}_i^n = (\xi_i, \ldots, \xi_n)$. Let $\mathcal{T}_n(x,\xi) = \{\hat{\xi}_1^n : 0 \leq \xi_n \leq \cdots \leq \xi_1 \leq x \leq 1\}$ and $\mathcal{T}_n = \mathcal{T}_n(1,\xi)$. Define also

$$\prod^n u = \prod_{j=1}^n u(t,\xi_j), \tag{1}$$

$$\int_{\mathcal{T}_n(x,\xi)} f(\hat{\xi}_0^n)\mathrm{d}\hat{\xi}_1^n = \int_0^x \int_0^{\xi_1} \cdots \int_0^{\xi_{n-1}} f(\hat{\xi}_0^n)\mathrm{d}\xi_n \cdots \mathrm{d}\xi_1. \tag{2}$$

We first formalize the concept of convergence of Volterra series with $L^2(\mathcal{T}_n)$ kernels. Consider a Volterra series $F[u]$ with kernels $f_n(\hat{\xi}_0^n)$, i.e.,

$$F[u](t,x) = \sum_{n=1}^{\infty} F_n[u](t,x) = \sum_{n=1}^{\infty} \int_{\mathcal{T}_n(x,\xi)} f_n(\hat{\xi}_0^n) \prod^n u \mathrm{d}\hat{\xi}_1^n. \tag{3}$$

The following definition quantifies the convergence of (3) in $L^2(0,1)$ (in what follows, we will write just L^2 for simplicity).

Definition 2.1. Given (3) with kernels $f_n \in L^2(\mathcal{T}_n)$, we define the radius of convergence ρ as

$$\rho = \left(\limsup_{n\to\infty} \left(\frac{\|f_n\|^2_{L^2(\mathcal{T}_n)}}{n!} \right)^{1/n} \right)^{-1}, \tag{4}$$

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and the gain bound function $f(s):[0,\rho)\to[0,\infty)$ as

$$f(s)=2\sum_{n=1}^{\infty}\frac{n^2\|f_n\|^2_{L^2(\mathcal{T}_n)}}{n!}s^n. \quad (5)$$

Using ρ and f from Definition 2.1 we can state a result that guarantees convergence of the Volterra series (3) in L^2 (a similar result in the L^∞ space for L^∞ kernels is standard (Boyd, Chua, & Desoer, 1984)).

Theorem 1 (*Gain Bound Theorem*). *Given a Volterra series $F[u]$ as in (3), with kernels $f_n\in L^2(\mathcal{T}_n)$, radius of convergence $\rho>0$ and gain bound function f, the following results hold.*

(1) *If $u\in L^2$ verifies that $\|u\|^2_{L^2}<\rho$, then the integrals and sums in (3) converge (in L^2).*
(2) *$F[u]$ satisfies $\|F[u]\|^2_{L^2}\le f(\|u\|^2_{L^2})$ and consequently F maps balls of L^2 into balls of L^2.*

Proof. From definition (3), and using the Cauchy–Schwartz inequality,

$$F_n[u]^2 \le \|f_n\|^2_{L^2(\mathcal{T}_n)}\left(\int_{\mathcal{T}_n(x,\xi)}\prod^n u^2 \mathrm{d}\hat{\xi}_1^n\right) = \frac{\|f_n\|^2_{L^2(\mathcal{T}_n)}\|u\|^{2n}_{L^2}}{n!}, \quad (6)$$

hence,

$$F[u]^2=\left(\sum_{n=1}^{\infty}F_n[u]\right)^2 \le \left(\sum_{n=1}^{\infty}n^2F_n[u]^2\right)\left(\sum_{n=1}^{\infty}\frac{1}{n^2}\right) \le 2\sum_{i=1}^{\infty}\frac{n^2\|f_n\|^2_{L^2(\mathcal{T}_n)}\|u\|^{2n}_{L^2}}{n!}, \quad (7)$$

where we used that $\sum_{n=1}^{\infty}\frac{1}{n^2}=\frac{\pi^2}{6}\le 2$. Thus we obtain

$$\|F[u]\|^2_\infty=\max_{x\in(0,1)}F[u]^2\le 2\sum_{i=1}^{\infty}\frac{n^2\|f_n\|^2_{L^2(\mathcal{T}_n)}\|u\|^{2n}_{L^2}}{n!}. \quad (8)$$

Then from elementary theory of power series and noting that $\lim_{n\to\infty}\sqrt[n]{n^2}=1$ and that $\|F[u]\|^2_{L^2}\le\|F[u]\|^2_\infty$, the result follows. □

We give now some examples illustrating Theorem 1.

Example 2.1. Let $F[u]$ be a Volterra series with kernels f_n and let C and D be generic positive constants.

(1) If the kernels f_n verify the uniform bound $\|f_n\|^2_{L^2(\mathcal{T}_n)}\le D$, then $\rho=\infty$ and the series is everywhere convergent for $u\in L^2$. We also have that $f(s)=2s(s+1)D\exp(s)$. Note also that $f(s)\le 2D(\exp(3s)-1)$.
(2) If the kernels f_n grow exponentially as $\|f_n\|^2_{L^2(\mathcal{T}_n)}\le DC^n$, then again $\rho=\infty$ and the series is everywhere convergent. We have in this case that $f(s)=2sC(sC+1)D\exp(Cs)$. Note also that $f(s)\le 2D(\exp(3Cs)-1)$.
(3) If the kernels f_n grow as fast as $\|f_n\|^2_{L^2(\mathcal{T}_n)}\le n!DC^n$, then $\rho=1/C$ and the series convergence can only be guaranteed if $\|u\|_{L^2}\le 1/C$. We have in this case that $f(s)=\frac{2sC(sC+1)D}{(1-sC)^3}$. Note that $f(s)\le\frac{2D(sC)^2}{(1-sC)^4}$.

Remark 1. Since $\|f_n\|^2_{L^2(\mathcal{T}_n)}\le\frac{\|f_n\|^2_\infty}{n!}$, if $f_n\in L^\infty(\mathcal{T}_n)$, similar results to Theorem 1 can be stated in terms of the L^∞ norms of the f_n's. Note also that by (8) the L^∞ norm of $F[u]$ is well defined for $u\in L^2$.

3. Control strategy

In the companion paper (Vazquez & Krstic, 2008) we considered the stabilization problem for the plant

$$u_t=u_{xx}+\lambda(x)u+F[u]+uH[u], \quad (9)$$
$$u_x(0)=qu(0),\qquad u(1)=U, \quad (10)$$

where $F[u]$ and $H[u]$ are Volterra nonlinearities defined respectively by kernels f_n and h_n, and U the actuation variable. We solved the problem by mapping u into a *target system* w which verifies

$$w_t=w_{xx}-cw, \quad (11)$$
$$w_x(0)=\bar{q}w(0),\qquad w(1)=0, \quad (12)$$

where $\bar{q}=\max\{0,q\}$. For mapping u into w we use a Volterra transformation

$$w=u-K[u]=u-\sum_{n=1}^{\infty}\int_{\mathcal{T}_n(x,\xi)}k_n(\hat{\xi}_0^n)\prod^n u\mathrm{d}\hat{\xi}_1^n, \quad (13)$$

where the kernels k_n in (13) are obtained from the set of PIDEs (40)–(47) in Vazquez and Krstic (2008).

The control law is determined by (13) at $x=1$

$$U=\sum_{n=1}^{\infty}\int_{\mathcal{T}_n(1,\xi)}k_n(1,\hat{\xi}_1^n)\prod^n u\mathrm{d}\hat{\xi}_1^n. \quad (14)$$

Remark 2. From (13),

$$w_x = u_x-u(x)\sum_{n=1}^{\infty}\int_{\mathcal{T}_n(x,\xi)}k_{n+1}(x,x,\hat{\xi}_1^n)\prod^n u\mathrm{d}\hat{\xi}_1^n - k_1(x,x)u(x)-\sum_{n=1}^{\infty}\int_{\mathcal{T}_n(1,\xi)}k_{nx}(x,\hat{\xi}_1^n)\prod^n u\mathrm{d}\hat{\xi}_1^n = u_x-\bar{k}(x)u(x)-u(x)\bar{K}[u]-\tilde{K}[u], \quad (15)$$

where $\bar{K}[u]$ and $\tilde{K}[u]$ are Volterra series in u (not in u_x) with kernels $\bar{k}_n=k_{n+1}(x,x,\hat{\xi}_1^n)$ and $\tilde{k}_n=k_{nx}(x,\hat{\xi}_1^n)$. Note that from the boundary condition (Vazquez & Krstic, 2008, (43)), we have that

$$\bar{k}=\hat{q}-\frac{1}{2}\int_0^x\lambda(s)\mathrm{d}s,\qquad \bar{k}_n=-\frac{1}{2}\int_{\xi_1}^x h_n(s,\hat{\xi}_1^n)\mathrm{d}s, \quad (16)$$

where $\hat{q}=\min\{0,q\}$. Hence,

$$\|w_x\|^2_{L^2}\le 4\left(\|u_x\|^2_{L^2}+\|u\|^2_{L^2}\|\bar{k}\|^2_\infty+\|u\|^2_{L^2}\|\bar{K}[u]\|^2_\infty + \|\tilde{K}[u]\|^2_{L^2}\right), \quad (17)$$

which means that the H_1 norm of w can be computed from the H^1 norm of u (note that by Remark 1, $\|\bar{K}[u]\|^2_\infty$ is well defined).

In the following sections we study the convergence of (13) and (14) and the properties of the closed-loop system (9), (10) and (14).

4. Convergence analysis for the transformation

In what follows, we make the following very reasonable assumption on the plant kernels.

Assumption 4.1. Let D_h, D_f, ρ_h and ρ_f be positive constants. Then, the following hold.

(1) $\lambda(x) \in \mathcal{C}^1[0,1]$, $h_n \in \mathcal{C}^1[\mathcal{T}_n]$, $f_n \in \mathcal{C}^0[\mathcal{T}_n]$.
(2) The parameter $\lambda(x)$ verifies

$$\max_{x\in[0,1]} \{|\lambda(x)|\} \leq D_h. \tag{18}$$

(3) The sequence $h_n(\hat{\xi}_0^n)$ verifies the following bound

$$\max_{(\hat{\xi}_0^n)\in\mathcal{T}_n} \left\{ \left|h_n(\hat{\xi}_0^n)\right| + \sum_{i=0}^{n} \left|h_{n\xi_i}(\hat{\xi}_0^n)\right| \right\} \leq \frac{n!D_h}{\rho_h^{n-1}}. \tag{19}$$

(4) The sequence $f_n(\hat{\xi}_0^{n+1})$ verifies the following bound

$$\max_{(\hat{\xi}_0^n)\in\mathcal{T}_n} \left\{ \left|f_n(\hat{\xi}_0^n)\right| \right\} \leq \frac{n!D_f}{\rho_f^{n-1}}. \tag{20}$$

In Assumption 4.1, points (3) and (4) quantify the divergence rate bound for plant kernels to ensure convergence of the plant nonlinearities H and F. We also assume the following:

Assumption 4.2. Under the above assumptions, for each n, there exists an $H^1(\mathcal{T}_n)$ solution k_n of the kernel PIDE equations (Vazquez & Krstic, 2008, Equations (40)–(47)).

We next show a result that relates the convergence of the transformation and the feedback law series, respectively (13) and (14), to the convergence of the plant nonlinearities $F[u]$ and $H[u]$.

Theorem 2. *Under Assumptions 4.1 and 4.2, the Volterra series in the transformation (13), the control law (14) and the w_x transformation (15) are convergent with radius of convergence*

$$\rho_k = \left(\frac{\min\{\rho_f, \rho_h\}}{2}\right)^2 \exp\left(-2\sqrt{\gamma}\right), \tag{21}$$

where $\gamma = \max\{1, \|f_1\|_\infty + \|\lambda\|_\infty\}$. Moreover, k_n verifies

$$\|k_n\|^2_{L^2(\mathcal{T}_n)} \leq (n-1)!4D^2C^{2n-2}e^{2n\sqrt{\gamma}+2\Upsilon+|\hat{q}|}, \tag{22}$$

$$\|k_{nx}\|^2_{L^2(\mathcal{T}_n)} \leq n!2D^2C^{2n-2}e^{2n\sqrt{\gamma}+2\Upsilon+|\hat{q}|}, \tag{23}$$

where $D = D_f + \rho_h D_h + 2((1+\rho_h)D_h)(|q|+1)\exp(1+|\hat{q}|)\sqrt{(1+\rho_h)^2D_h^2 + D_f^2}$, $C = \left(\frac{\min\{\rho_f,\rho_h\}}{2}\right)^{-1}$ and $\Upsilon = 4\frac{D^2(1+2\sqrt[4]{\gamma})^2}{\gamma^2}$.

See the Appendix for a proof.

Remark 3. In the above theorem, if $q = \infty$ (meaning the plant has a Dirichlet boundary condition at the uncontrolled end), then the above bounds hold setting $q = 0$.

Corollary 4.1. *Under the same assumptions of Theorem 2, if the Volterra series nonlinearity of the plant is globally convergent in L^2, then the transformation Volterra series (13), the control (14) and the w_x transformation (15) converge globally in L^2 as well.*

Proof. If the Volterra series nonlinearities of the plant F and H are everywhere convergent, then by the limit (4) being infinity, for any $\epsilon > 0$ (possibly very small), there exists $D_\epsilon > 0$ (possibly very large) such that both f_n and h_n verify

$$\max\{|h_n|, |f_n|\} \leq n!B_\epsilon \epsilon^{n-1}. \tag{24}$$

Hence under the assumptions of Theorem 2, the kernel solution k_n verifies

$$\|k_n\|^2_{L^2(\mathcal{T}_n)} \leq (n-1)!4D_\epsilon^2 \left(\frac{\epsilon}{2}\right)^{2n-2} e^{2n\sqrt{\gamma_\epsilon}+2\Upsilon_\epsilon+|\hat{q}|}, \tag{25}$$

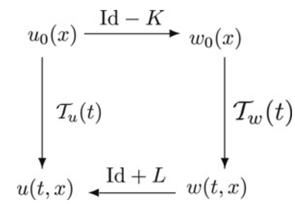

Fig. 1. Commutative diagram for the closed-loop system.

where D_ϵ and Υ_ϵ are defined as in Theorem 2 replacing $D_h = D_f = B_\epsilon$ and $\rho_h = \rho_f = 1/\epsilon$, but note that $\gamma = \max\{1, \|f_1\|_\infty + \|\lambda\|_\infty + c\}$ does not depend on ϵ. Then the radius of convergence of the Volterra series defined by k_n is $\rho_k \geq \frac{4\exp(2\sqrt{\gamma})}{\epsilon^2}$. Since this holds for any positive ϵ, we must have $\rho_k = \infty$. □

5. Stability analysis

To analyze the behavior of the closed-loop system, we study the invertibility of the change of variables (13). It is natural to seek also a Volterra formulation for this inverse change of variables, which is assumed as having the following form

$$u = w + L[w], \tag{26}$$

which is expanded as

$$u(t,x) = w(t,x) + \sum_{n=1}^{\infty} \int_{\mathcal{T}_n(x,\xi)} l_n(\hat{\xi}_0^n) \prod^n w \mathrm{d}\hat{\xi}_1^n. \tag{27}$$

The existence of this inverse change of variables can be guaranteed employing the theorem for inversion of Volterra series, which is proved in Boyd et al. (1984, Theorem 3.3.1.).

Theorem 3 (*Volterra Series Inversion*). *A Volterra series has a local inverse at the origin if and only if its first (linear) kernel is invertible.*

In that context, the word "local" means that a unique Volterra series representation can be found for the inverse transformation, which has the form specified by (27), and whose radius of convergence (in the sense of Definition 2.1 and Theorem 1) is possible finite, even if the transformation is globally convergent.

The direct and inverse transformations give a relation between u and w that can be exploited to obtain properties of u (governed by a complex nonlinear equation) from properties of w (that verify an easy to analyze heat equation). The commutative diagram of Fig. 1 illustrates our strategy. We have denoted the initial conditions for u and w as $u(0,x) = u_0$ and $w(0,x) = w_0$, respectively. In the left, $\mathcal{T}_u(t)$ is the semigroup that governs the behavior of u when the loop is closed, so that $u(t) = T_u(t)u_0$; its generator can be obtained homogenizing (9) and taking (14) into account. In the right, $\mathcal{T}_w(t)$ is the semigroup generated by the Laplacian operator in (11), so that $w(t) = \mathcal{T}_w(t)w_0$. Above and below are respectively the direct and inverse transformations, Id − K and Id + L that relate u and w. We are interested in the properties of u, but direct analysis of $\mathcal{T}_u(t)$ is very difficult—it is generated by a nonlinear operator. Instead, from Fig. 1, we use that $\mathcal{T}_u(t) = (\mathrm{Id}+L)\circ\mathcal{T}_w(t)\circ(\mathrm{Id}-K)$, dividing the analysis into smaller, more tractable pieces. The transformations Id + L and Id − K are still nonlinear but time invariant, and are analyzed within the framework of Volterra series, whereas the heat equation semigroup $\mathcal{T}_w(t)$ is linear and simple, producing even explicit solutions. We begin by analyzing $\mathcal{T}_w$, whose behavior is summarized in the following lemma, which follows from standard estimates for the heat equation (Evans, 1998; Liu, 2003).

Lemma 5.1. *Consider the system (11) with boundary conditions (12). Then, the equilibrium $w \equiv 0$ is exponentially stable in the L^2 and H^1 norms, i.e., $\forall t \geq 0$*

$$\|w(t)\|_{\mathcal{L}}^2 \leq e^{-t}\|w_0\|_{\mathcal{L}}^2, \tag{28}$$

where $\mathcal{L}$ is either L^2 or H^1.

Using Lemma 5.1 and the relations illustrated by Fig. 1, we get the following result about the stability properties of the closed-loop system.

Theorem 4. *Let Assumptions 4.1 and 4.2 hold and assume that there is an L^2 (resp. H^1) solution u to the closed-loop system (9) with boundary conditions (10) and control law (14). Then, the origin $u \equiv 0$ of the closed-loop system is locally exponentially stable in the L^2 (resp. H^1) norm, i.e., denoting the initial condition for u as $u(0, x) = u_0(x)$, there exists $C_1, C_2 > 0$ such that, if $\|u_0\|_{\mathcal{L}}^2 \leq C_1$, then $\forall t \geq 0$*

$$\|u(t)\|_{\mathcal{L}}^2 \leq C_2 e^{-t}\|u_0\|_{\mathcal{L}}^2, \tag{29}$$

where $\mathcal{L}$ is either L^2 or H^1, and C_1, C_2 depend on the plant parameters, but not on u_0.

Proof. Under Assumptions 4.1 and 4.2, the transformation (13) exists and converges for $\|u(t)\|_{L^2}^2 \leq \rho_K$, where ρ_K denotes the radius of convergence of the transformation Volterra series. The first kernel of (13) is $\mathrm{Id} - K_1$ and constitutes the linear part of the transformation. In Smyshlyaev and Krstic (2004) it is shown that this linear part is *always* invertible. Hence, using Theorem 3, the whole transformation is locally invertible and the inverse transformation has the form specified by (27). Therefore there exists $\rho_L > 0$ such that, if $\|w(t)\|_{L^2}^2 < \rho_L$, then (27) converges.

Denote by $k(s)$ and $l(s)$ the gain bound functions of the direct and inverse Volterra series transformations, $Id - K$ and $Id + L$ respectively, as defined in (5).

From (13), we have that

$$w_0 = u_0 - K[u_0]. \tag{30}$$

Set $C_1 = k^{-1}(\rho_L)/2 < \rho_K$. Hence, if $u_0 \leq C_1$ we get that

$$\|w(t)\|_{L^2}^2 \leq \|w_0\|_{L^2}^2 \leq k(\|u_0\|_{L^2}^2) \leq k(C_1) < \rho_l \tag{31}$$

for all time t. Therefore, the inverse (26) converges and the relations of Fig. 1 hold for all time $t \geq 0$. Set now $C_3 = \frac{k(C_1)}{C_1}$ and $C_4 = \frac{l(C_1 C_3)}{C_1 C_3}$. Then, for $\|u_0\|_{L^2}^2 < C_1$, since $\|w(t)\|_{L^2}^2 \leq C_3 C_1$ and both $k(s)$ and $l(s)$ are class $\mathcal{K}$ functions (Khalil, 2002), we have that

$$\begin{aligned}\|u(t)\|_{L^2}^2 &\leq l(\|w(t)\|_{L^2}^2) \leq C_4\|w(t)\|_{L^2}^2 \leq C_4 e^{-t}\|w_0\|_{L^2}^2 \\ &\leq C_3 C_4 e^{-t}\|u_0\|_{L^2}^2,\end{aligned} \tag{32}$$

so setting $C_2 = C_3 C_4$, (29) follows for the L^2 norm. To obtain the bound for the H^1 norm we use (15) and (17), and note that

$$\begin{aligned}u_x &= w_x + \bar{k}(x)(w + L[w]) + (w + L[w])\bar{K}[w + L[w]] \\ &\quad - \tilde{K}[w + L[w]].\end{aligned} \tag{33}$$

Hence u_x can be recovered from w_x when the Volterra series in (33) converge. If $\|u_0\|_{H^1}^2 \leq C_1$, then obviously $\|u_0\|_{L^2}^2 \leq C_1$, and since the radius of convergence of both $\bar{K}$ and $\tilde{K}$ is at least ρ_K, all the series in the right-hand side in (33) converge. Then we use Lemma 5.1 and proceed in the same way as in (32) for the H^1 norm (using the gain bound functions for $\bar{K}$ and $\tilde{K}$), obtaining possibly a different C_2; to get the same C_2 for both L^2 and H^1 we pick the maximum of the two. Then the result follows. □

Remark 4. In Theorem 4 we have assumed well-posedness of the closed-loop system. For the case $q = \infty$ (Dirichlet boundary condition at $x = 0$), since (11) and (12) are well-posed in H^1 and since (15) and (33) allow proving local equivalence of the H^1 norms of u and w, the assumption can be dropped, provided u_0 verifies some compatibility conditions (Smyshlyaev & Krstic, 2004) (see Proposition 6.2 for an example). For other values of q, (11) and (12) are well-posed in H^2 and this argument is not enough.

Remark 5. Note that (11) can be solved explicitly. This means that, when $\|u_0\|_{\mathcal{L}} < C_1$, u can be obtained explicitly for all times. We give an illustration for the simplest case, when $q = \infty$. Then, u is given as

$$\begin{aligned}u(t, x) &= 2\sum_{n=1}^{\infty} e^{-\pi^2 n^2 t}\sin(\pi n x)\int_0^1 \sin(\pi n \xi)\,[u_0(\xi) \\ &\quad - K[u_0](\xi)]\,d\xi + L\left[2\sum_{n=1}^{\infty} e^{-\pi^2 n^2 t}\sin(\pi n x)\right. \\ &\quad \left.\times \int_0^1 \sin(\pi n \xi)\,[u_0(\xi) - K[u_0](\xi)]\,d\xi\right].\end{aligned} \tag{34}$$

For other values of q similar formulas can be written.

The constant C_1 for which Theorem 4 holds determines the "basin of attraction" of the equilibrium at the origin for the closed-loop system. Since $C_1 = k^{-1}(\rho_L)$, if ρ_L and some bound on the k_n's are known then C_1 can be more precisely quantified. We state a corollary for Theorem 4 for some particular cases, introduced in Example 2.1, that occur frequently in practice.

Corollary 5.1. *Let $\rho_K, \rho_L > 0$ denote the radii of convergence of the direct and inverse Volterra transformations, (13) and (27), respectively. Let C and D denote generic positive constants.*

(1) *If $\rho_K = \rho_L = \infty$, then Theorem 4 holds globally, i.e., for all $u \in L^2$.*
(2) *If the kernels k_n verify $\|k_n\|_{L^2(\mathcal{T}_n)}^2 \leq D$, then $\rho_K = \infty$ and Theorem 4 holds at least for $\|u\|_{L^2}^2 \leq \frac{1}{3}\log\left(1 + \frac{\rho_L}{2D}\right)$.*
(3) *If the kernels k_n grow like $\|k_n\|_{L^2(\mathcal{T}_n)}^2 \leq DC^n$, then $\rho_K = \infty$ and Theorem 4 holds at least for $\|u\|_{L^2}^2 \leq \frac{1}{3C}\log\left(1 + \frac{\rho_L}{2D}\right)$.*
(4) *If the kernels k_n grow as $\|k_n\|_{L^2(\mathcal{T}_n)}^2 \leq n!DC^n$, then $\rho_K = 1/C$ and Theorem 4 holds for*

$$\|u_0\|_{L^2}^2 \leq \frac{1}{C}\left(1 + \sqrt{\frac{D}{2\rho_L}} - \sqrt[4]{\frac{D}{2\rho_L}}\sqrt{\sqrt{\frac{D}{2\rho_L}} + 2}\right) > 0.$$

6. Inverse transformation

Theorem 4 depends critically on the inverse transformation and its properties. Next we give explicit formulas that allow computing the inverse from the kernels k_n.

Define l_1 as the unique function that verifies the following well-posed (Smyshlyaev & Krstic, 2004) PIDE

$$\begin{aligned}\partial_{xx} l_1(x, \xi_1) &= \partial_{\xi_1\xi_1} l_1(x, \xi_1) - \lambda(x) l_1 - f_1(x, \xi_1) \\ &\quad - \int_{\xi_1}^x l_1(s, \xi_1) f_1(x, s)\,ds,\end{aligned} \tag{35}$$

with boundary conditions

$$l_1(x, x) = \hat{q} - \frac{1}{2}\int_0^x \lambda(s)\,ds, \tag{36}$$

$$l_{1\xi_1}(x, 0) = q l_1(x, 0). \tag{37}$$