Automatica 44 (2008) 2778–2790

Contents lists available at ScienceDirect

Automatica

journal homepage: www.elsevier.com/locate/automatica

Control of 1-D parabolic PDEs with Volterra nonlinearities, Part I: Design ☆

Rafael Vazquez [a], Miroslav Krstic [b,*]

[a] Departamento de Ingeniería Aeroespacial, Universidad de Sevilla, 41092 Seville, Spain

[b] Department of Mechanical and Aerospace Engineering, University of California San Diego, La Jolla, CA 92093-0411, United States

ARTICLE INFO

Article history:
Received 7 December 2006
Received in revised form
31 March 2008
Accepted 7 April 2008
Available online 26 September 2008

Keywords:
Distributed parameter systems
Stabilization
Nonlinear control
Feedback linearization
Partial differential equations
Lyapunov function
Boundary conditions

ABSTRACT

Boundary control of nonlinear parabolic PDEs is an open problem with applications that include fluids, thermal, chemically-reacting, and plasma systems. In this paper we present stabilizing control designs for a broad class of nonlinear parabolic PDEs in 1-D. Our approach is a direct infinite dimensional extension of the finite-dimensional feedback linearization/backstepping approaches and employs spatial Volterra series nonlinear operators both in the transformation to a stable linear PDE and in the feedback law. The control law design consists of solving a recursive sequence of linear hyperbolic PDEs for the gain kernels of the spatial Volterra nonlinear control operator. These PDEs evolve on domains $\mathcal{T}_n$ of increasing dimensions $n+1$ and with a domain shape in the form of a "hyper-pyramid", $0 \leq \xi_n \leq \xi_{n-1} \cdots \leq \xi_1 \leq x \leq 1$. We illustrate our design method with several examples. One of the examples is analytical, while in the remaining two examples the controller is numerically approximated. For all the examples we include simulations, showing blow up in open loop, and stabilization for large initial conditions in closed loop. In a companion paper we give a theoretical study of the properties of the transformation, showing global convergence of the transformation and of the control law nonlinear Volterra operators, and explicitly constructing the inverse of the feedback linearizing Volterra transformation; this, in turn, allows us to prove L^2 and H^1 local exponential stability (with an estimate of the region of attraction where possible) and explicitly construct the exponentially decaying closed loop solutions.

1. Introduction

Boundary control of linear parabolic PDEs is a well established subject with extensive literature. On the other hand, boundary control of *nonlinear* parabolic PDEs is still an open problem as far as general classes of systems are concerned, with many applications of interest including fluids, structures, thermal, chemically-reacting, and plasma systems. Past efforts include the book (Christofides, 2001), which solves problems of nonlinear parabolic PDE control but for *inside-the-domain* actuation, rather than with boundary control, and developments to solve the problem of *motion planning* for boundary controlled nonlinear parabolic PDEs (Meurer, 2005) (using flatness and formal power series) and structural systems (Kugi, Thull, & Kuhnen, 2006) (with a flatness/passivity approach).

When attempting to develop general methods for nonlinear PDEs, it is advisable to take a clue from finite dimensional nonlinear systems. Clearly, one should bet on methods that have emerged as successful there. This essentially eliminates (direct) optimal control methods, because of the requirement to solve Hamilton–Jacobi–Bellman PDEs, and leaves feedback linearization/backstepping/Lyapunov approaches (Isidori, 1995; Khalil, 2002; Krstic, Kanellakopoulos, & Kokotovic, 1995; Sepulchre, Jankovic, & Kokotovic, 1997) as candidates for extension to PDEs. The backstepping approach for *linear* PDEs has reached the level of maturity where a systematic design procedure (Smyshlyaev & Krstic, 2004) is available for a broad class of parabolic integro-differential equations in 1-D. This systematic procedure has found many applications (Krstic, Smyshlyaev, & Siranosian, 2006; Vazquez & Krstic, 2006), including even extensions to the Navier–Stokes equations (Vazquez & Krstic, 2007) and to adaptive PDE control (Krstic, 2005; Smyshlyaev & Krstic, 2005), and is the starting point for our *nonlinear* developments here.

Our early nonlinear efforts (Aamo & Krstic, 2004; Boskovic & Krstic, 2001, 2002, 2003) were *discretization*-based and were successful in addressing some applications but in general cannot be expected to converge when the discretization step goes to zero, as shown in Balogh and Krstic (2003).

Our approach is a direct infinite dimensional extension of the finite-dimensional feedback linearization/backstepping approaches and employs spatial Volterra series nonlinear operators both in the state transformation to a stable linear PDE and in the

☆ This work was supported by NSF grant number CMS-0329662. This paper was not presented at any IFAC meeting. This paper was recommended for publication in revised form by Associate Editor Hendrik Nijmeijer, under the direction of Editor Hassan K. Khalil.

* Corresponding author. Tel.: +1 858 822 1374; fax: +1 858 822 3107.
E-mail addresses: rvazquez1@us.es (R. Vazquez), krstic@ucsd.edu (M. Krstic).

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doi:10.1016/j.automatica.2008.04.013

feedback law. The control law design consists of solving a recursive sequence of linear hyperbolic PDEs for the gain kernels of the spatial Volterra nonlinear control operator. These PDEs evolve on domains $\mathcal{T}_n$ of increasing dimensions $n+1$ and with a domain shape in the form of a "hyper-pyramid," $0 \le \xi_n \le \xi_{n-1} \cdots \le \xi_1 \le x \le 1$. We illustrate our design method with several examples. One of the examples is analytical, while in the remaining two examples the controller is numerically approximated. For all the examples we include simulations, showing blow up in open loop, and stabilization for large initial conditions in closed loop.

In a companion paper (Vazquez & Krstic, 2008) we study the properties of the transformation, showing global convergence of the transformation and control law nonlinear Volterra operators, and including an explicit construction of the inverse of the feedback linearizing Volterra transformation for both the general case and the analytical example; this, in turn, allows us to prove local L^2 and H^1 exponential stability (with an estimate of the region of attraction where possible) and explicitly construct the exponentially decaying closed loop solutions.

This paper solves the open problem 5.1 in the *Unsolved Problems* volume (Balogh & Krstic, 2003).

2. Class of systems under study

We study the following class of parabolic systems,

$$u_t(t,x) = u_{xx}(t,x) + \lambda(x)u(t,x) + F[u](t,x) + uH[u](t,x), \quad (1)$$

for $x \in (0,1)$, with the following boundary conditions

$$u_x(t,0) = qu(t,0), \qquad u(t,1) = U(t), \quad (2)$$

where $U(t)$ is the control input and $F[u]$ and $H[u]$ are Volterra series nonlinearities as explained below. In (2), q is a number that can take any value. The particular cases $q = 0$ and $q = \infty$ can be used to model, respectively, Neumann and Dirichlet boundary conditions at $x = 0$. For simplicity we consider a Dirichlet boundary condition at $x = 1$, but different boundary conditions at the controlled end can be accommodated in our design. In the sequel, we will omit time and space dependency of the state when possible.

Define $\xi_0 = x$ and for any $i \le n$, $\hat{\xi}_i^n = (\xi_i, \ldots, \xi_n)$. Let $\mathcal{T}_n(x,\xi) = \left\{\hat{\xi}_1^n : 0 \le \xi_n \le \cdots \le \xi_1 \le x\right\}$ and $\mathcal{T}_n = \mathcal{T}_n(1,\xi)$. Define also

$$\prod^i u = \prod_{j=1}^i u(t,\xi_j), \qquad \prod^{i,k} u = \prod_{\substack{j=1\\ j\ne k}}^i u(t,\xi_j), \quad (3)$$

$$\int_{\mathcal{T}_n(x,\xi)} f(\hat{\xi}_0^n)\mathrm{d}\hat{\xi}_1^n = \int_0^x \int_0^{\xi_1} \cdots \int_0^{\xi_{n-1}} f(\hat{\xi}_0^n)\mathrm{d}\xi_n \ldots \mathrm{d}\xi_1. \quad (4)$$

A Volterra series is defined as a functional (i.e., a function that depends on another function), and has the form

$$F[u](t,x) = \sum_{n=1}^{\infty} F_n[u](t,x), \quad (5)$$

where the notation $F_n[u](t,x)$ emphasizes the fact that each $F_n[u]$ is defined as a functional of $u(t,x)$ and also depends on t and x. The precise definition of each term is, using the notation of (3) and (4),

$$F_n[u](t,x) = \int_{\mathcal{T}_n(x,\xi)} f_n(\hat{\xi}_0^n) \prod^i u \,\mathrm{d}\hat{\xi}_1^n, \quad (6)$$

where f_n is called the n-th Volterra (triangular) kernel.

Volterra series (Volterra, 1959) are widely known and studied in the control literature (Boyd, Chua, & Desoer, 1984; Isidori, 1995; Lamnabhi-Lagarrigue, 1996; Sastry, 1999). They are causal functionals (Fliess, 1981) that represent the general solution for nonlinear equations, generalizing the convolution solution for linear systems. An excellent exposition on Volterra series can be found in Rugh (1981).

In the sequel, we will omit time and/or space dependency of the state when possible.

3. Motivating examples

We give two examples of nonlinear plants that fall into the class of systems of Section 2.

3.1. Coupled nonlinear plant

Consider the following nonlinear plant

$$u_t = u_{xx} + \mu v, \quad (7)$$

$$0 = v_{xx} + \omega^2 v + uv + u, \quad (8)$$

where μ and ω are plant parameters, with boundary conditions

$$u(0) = v(0) = 0, \quad (9)$$

$$u(1) = U, \qquad v(1) = V, \quad (10)$$

where $U(t)$ and $V(t)$ are actuation variables.

This kind of plant *structure*, consisting of an evolution equation (Eq. (7), parabolic in this case) coupled with an static equation (Eq. (8), elliptic in this case), arises in some relevant applications, for example fluid mechanics (Vazquez & Krstic, 2007), structural problems (Krstic et al., 2006), or singularly perturbed problems in thermal fluid convection (Vazquez & Krstic, 2006) or chemical reactors (Boskovic & Krstic, 2002).

To obtain a plant equation in the class of (1), we solve for v in terms of u from (8). Define

$$v = \sum_{n=1}^{\infty} v_n, \qquad V = \sum_{n=1}^{\infty} V_n, \quad (11)$$

where v_1 verifies

$$0 = v_{1xx} + \omega^2 v_1 + u, \quad (12)$$

and for $n > 1$, v_n verifies

$$0 = v_{nxx} + \omega^2 v_n + uv_{n-1}, \quad (13)$$

with boundary conditions, for each n,

$$v_n(0) = 0, \qquad v_n(1) = V_n. \quad (14)$$

Since V in (10) is one of our two control inputs, we are free to choose V_n in any meaningful way if the series for V in (11) converges and the solution for (12)–(14) also makes the series for v in (11) convergent.

In this case, it is possible to solve (12) and (13) explicitly. Denoting $v_0 = 1$, we get the following recursive solution for $n \ge 1$

$$v_n = -\int_0^x \frac{\sin(\omega(x-\xi))}{\omega} v_{n-1}(\xi)u(\xi)\mathrm{d}\xi + \frac{\sin(\omega x)}{\sin\omega} \times \left(V_n + \int_0^1 \frac{\sin(\omega(1-\xi))}{\omega} v_{n-1}u(\xi)\mathrm{d}\xi\right). \quad (15)$$

Set the control law V as follows.

$$V_n = -\int_0^1 \frac{\sin(\omega(1-\xi))}{\omega} v_{n-1}u(\xi)\mathrm{d}\xi. \quad (16)$$

The reason to choose this particular control law is to get a spatially strict-feedback (Krstic et al., 1995) solution, i.e., a solution that is causal in space, meaning that $v(x)$ only depends on values of $u(\xi)$

for $0 \leq \xi \leq x$. This is a requirement of the backstepping method and was used in Vazquez and Krstic (2007), Krstic et al. (2006) and Vazquez and Krstic (2006) to control *linear* plants structurally similar to (7) and (8).

With this control law the solution to the Eqs. (12) and (13) is

$$v_n = -\int_0^x \frac{\sin(\omega(x-\xi))}{\omega} v_{n-1}(\xi)u(\xi)\mathrm{d}\xi. \quad (17)$$

We can solve the recursion in (17), getting

$$v_n = \frac{(-1)^n}{\omega^n}\int_{\mathcal{T}_n(x,\xi)}\prod_{j=1}^n \left[\sin\left(\omega(\xi_{j-1}-\xi_j)\right)u(\xi_j)\right]\mathrm{d}\hat{\xi}_1^n. \quad (18)$$

Plugging (18) into (16), we obtain a general formula for V_n as follows:

$$V_n = \frac{(-1)^n}{\omega^n}\int_{\mathcal{T}_n}\sin(\omega(1-\xi_1))u(\xi_1) \times \prod_{j=1}^{n-1}\left[\sin\left(\omega(\xi_j-\xi_{j+1})\right)u(\xi_{j+1})\right]\mathrm{d}\hat{\xi}_1^n. \quad (19)$$

Assuming that $u(t,x) \in L^2(0,1)$, both series in (11) converge in L^2 since using that $|\sin(\omega)/\omega| \leq 1$, one can bound $\|v_n\|_{L^2}^2$ as follows.

$$\begin{aligned}\|v_n\|_{L^2}^2 &= \int_0^1 v_n^2(x)\mathrm{d}x \\ &\leq \left|\frac{\sin(\omega)}{\omega}\right|^{2n}\int_0^1\left(\int_{\mathcal{T}_n(x,\xi)}\prod^i u\mathrm{d}\hat{\xi}_1^n\right)^2\mathrm{d}x \\ &\leq \left|\frac{\sin(\omega)}{\omega}\right|^{2n}\frac{1}{n!^2}\int_0^1\left(\int_0^x u^2(\xi_1)\mathrm{d}\xi_1\right)^n\mathrm{d}x \\ &\leq \frac{\|u\|_{L^2}^{2n}}{n!^2}. \end{aligned} \quad (20)$$

Hence, using the Cauchy–Schwartz inequality in ℓ_2,

$$\begin{aligned}\|v\|_{L^2}^2 &= \int_0^1\left(\sum_{n=1}^\infty v_n(x)\right)^2\mathrm{d}x \leq \left(\sum_{n=1}^\infty n^2\|v_n\|_{L^2}^2\right)\left(\sum_{n=1}^\infty\frac{1}{n^2}\right) \\ &\leq 2\sum_{n=1}^\infty\frac{\|u\|_{L^2}^{2n}}{(n-1)!^2}, \end{aligned} \quad (21)$$

where we used $\sum_{n=1}^\infty \frac{1}{n^2} = \pi^2/6 \leq 2$. Thus, $\|v\|_{L^2}^2 \leq 2\|u\|_{L^2}^2 \mathrm{e}^{\|u\|_{L^2}^2}$. Similarly $|V| \leq 2\|u\|_{L^2}^2 \exp\left(\|u\|_{L^2}^2\right)$.

Plugging the solution for v into (7), we reach

$$u_t = u_{xx} + \sum_{n=1}^\infty\int_{\mathcal{T}_n(x,\xi)} f_n(\hat{\xi}_0^n)\prod^i u\,\mathrm{d}\hat{\xi}_1^n, \quad (22)$$

where $f_n = \mu\frac{(-1)^n}{\omega^n}\prod_{j=1}^n \sin\left(\omega(\xi_{j-1}-\xi_j)\right)$, an autonomous system in u with boundary conditions

$$u(0) = 0, \qquad u(1) = U, \quad (23)$$

and now the problem is reduced to designing U such that the above system is guaranteed to be stable in L^2.

Eq. (22) is a particular example of (1) with $\lambda = H = 0$, and $q = \infty$.

3.2. Parabolic semilinear equation

Consider the plant

$$v_t = v_{xx} + f(v), \quad (24)$$

where $f(v)$ is a nonlinear function *analytic at the origin*, verifying $f(0) = 0$, with boundary conditions

$$v(0) = 0, \qquad v_x(1) = U, \quad (25)$$

where U is the actuation variable.

To cast (24) into the form of (1) we differentiate (24) in x, getting

$$v_{xt} = v_{xxx} + f'(v)v_x. \quad (26)$$

Call $u = v_x$. Then, $v = \int_0^x u(\xi)\mathrm{d}\xi$ and (26) yields

$$u_t = u_{xx} + uf'\left(\int_0^x u(\xi)\mathrm{d}\xi\right), \quad (27)$$

with boundary conditions

$$u_x(0) = 0, \qquad u(1) = U. \quad (28)$$

The boundary condition at 0 was obtained evaluating (24) at $x = 0$ and using (25) and $f(0) = 0$. Expanding f' in its Taylor series at the origin, and calling

$$\lambda = f'(0), \quad (29)$$

$$h_n = f^{(n+1)}(0), \quad n \geq 1, \quad (30)$$

we can write (27) as

$$u_t = u_{xx} + \lambda u + u\sum_{n=1}^\infty\frac{h_n}{n!}\left(\int_0^x u(\xi)\mathrm{d}\xi\right)^n, \quad (31)$$

and since

$$\left(\int_0^x u(\xi)\mathrm{d}\xi\right)^n = n!\int_{\mathcal{T}_n(x,\xi)}\prod^i u\,\mathrm{d}\hat{\xi}_1^n, \quad (32)$$

we get

$$u_t = u_{xx} + \lambda u + u\sum_{n=1}^\infty\int_{\mathcal{T}_n(x,\xi)} h_n\prod^i u\,\mathrm{d}\hat{\xi}_1^n, \quad (33)$$

with boundary conditions (28). Eq. (33) falls in the class of (1) with $F = 0$, $q = 0$, and λ and H given by (29) and (30). Note that stability of u in the L^2 norm implies stability of v in the H^1 norm, as $u(0) = 0$.

Remark 1. For the open-loop plant (24), finite-time blow up instabilities are likely to occur when $f(u)$ is superlinear. This was first studied in a classical paper (Fujita, 1966) for power-like nonlinearities, and has been the subject of systematic study in subsequent years (see the reviews Levine (1990) and Deng and Levine (2000)). More recently the question of *controllability* of these kind of equations has been considered. For superlinear functions which grow faster than $|u|\log^2(1+|u|)$ lack of global controllability is proved in Fernandez-Cara and Zuazua (2000). Therefore, for many nonlinearities $f(v)$ only *local* or restricted results can be achieved; for example in Coron and Trelat (2004) boundary control is used to move between sets of steady states for plants with superlinear nonlinearities.

4. Control strategy

The objective is to find a Volterra feedback law $U(t)$, so the controlled system (1) and (2) is stable. To achieve that, a new *target system* PDE is introduced in the form

$$w_t = w_{xx}, \tag{34}$$

with homogeneous boundary conditions

$$w_x(0) = \bar{q}w(0), \qquad w(1) = 0, \tag{35}$$

where $\bar{q} = \max\{0, q\}$. The plant (34) and (35) is an L^2 and H^1 exponentially stable system by standard results from linear PDE theory.

The idea of the method is to transform (1) into (34). For this we use a change of variables based on a Volterra series,

$$w = u - K[u] = u - \sum_{n=1}^{\infty} K_n[u]. \tag{36}$$

Evaluating (36) at $x = 1$ and using (2) and (35), we arrive at the control law

$$U = \sum_{n=1}^{\infty} K_n[u](1). \tag{37}$$

Therefore, the control is computed from the the Volterra kernels that define (36).

Remark 2. Some of the right hand side terms in (1) might be helpful for stability, for instance, a negative reaction term, i.e., $\lambda(x) \leq 0$ for all x, or the Volterra nonlinearity resulting from $-v^3$ in (24). Those terms could be kept in the target system (34), with only minor modifications in the kernel equations that follows.

We assume the series in (36) can be differentiated term by term.[1] Substituing (36) into (34) we get

$$\frac{\partial}{\partial t}\left(u - \sum_{n=1}^{\infty} K_n[u]\right) = \frac{\partial^2}{\partial x^2}\left(u - \sum_{n=1}^{\infty} K_n[u]\right). \tag{38}$$

Using (1) for u_t in (38) the following equation is obtained:

$$\lambda(x)u + \sum_{n=1}^{\infty} F_n[u] + u\sum_{n=1}^{\infty} H_n[u] = \sum_{n=1}^{\infty}\left(\frac{\partial}{\partial t}K_n[u] - \frac{\partial^2}{\partial x^2}K_n[u]\right). \tag{39}$$

From (39), we can obtain a set of of partial integro-differential equations (PIDEs) for the kernels k_i that define the control (37). While the details of the derivation are presented in the Appendix, the PIDE verified by the n-th order kernel is given by

$$\partial_{xx}k_n = \sum_{i=1}^{n}\left(\partial_{\xi_i\xi_i}k_n + \lambda(\xi_i)k_n\right) + \sum_{m=1}^{n-1} C_n^m[k_{n-m}, h_m] - f_n + I_n[k_n, f_1] + \sum_{m=2}^{n} B_n^m[k_{n-m+1}, f_m]. \tag{40}$$

The functions B_n^m, C_n^m and I_n in (40) have an involved definition that requires additional notation and the introduction of some intermediate functions. Hence for clarity we first finish stating and discussing the kernel equations and then introduce the concepts towards the precise definition of B_n^m, C_n^m and I_n, which is given respectively in (54), (55) and (56).

The solution to the PIDE (40) needs to satisfy the following boundary conditions. For $n = 1$,

$$k_1(x, x) = \hat{q} - \frac{1}{2}\int_0^x \lambda(s)\mathrm{d}s, \tag{41}$$

$$k_{1\xi_1}(x, 0) = qk_1(x, 0), \tag{42}$$

where $\hat{q} = \min\{0, q\}$, while for $n \geq 2$,

$$k_n(x, x, \hat{\xi}_2^n) = -\frac{1}{2}\int_{\xi_2}^{x} h_{n-1}(s, \hat{\xi}_2^n)\mathrm{d}s, \tag{43}$$

$$k_{nx}(x, x, \hat{\xi}_2^n) = -\frac{1}{4}\left(3h_{n-1}(\xi_2, \hat{\xi}_2^n) + h_{n-1}(x, \hat{\xi}_2^n)\right) + \frac{1}{2}\int_{\xi_2}^{x} \phi_n(s, \hat{\xi}_2^n)\mathrm{d}s \quad , \tag{44}$$

$$k_{n\xi_{i-1}}(\hat{\xi}_0 n)|_{\xi_{i-1}=\xi_i} = k_{n\xi_i}(\hat{\xi}_0^n)|_{\xi_{i-1}=\xi_i}, \quad i = 2, \ldots, n, \tag{45}$$

$$k_{n\xi_n}(\hat{\xi}_0^{n-1}, 0) = qk_n(\hat{\xi}_0^{n-1}, 0), \tag{46}$$

which are of mixed kind. In (44), the function ϕ_n is defined as

$$\phi_n = \left[\sum_{i=2}^{n}\partial_{\xi_i\xi_i}k_n + \sum_{i=1}^{n}\lambda(\xi_i)k_n + \sum_{m=1}^{n-1}C_n^m[k_{n-m}, h_m] + \sum_{m=2}^{n}B_n^m[k_{n-m+1}, f_m] + I_n[k_n, f_1] - f_n\right]_{x=\xi_1}. \tag{47}$$

Eq. (40) is a hyperbolic PIDE, for each k_n, evolving in the interior of the domain $\mathcal{T}_{n+1}$, which is a "hyper-pyramid" of dimension $n+1$ (in particular, a triangle for $n = 1$, and a pyramid for $n = 2$). Note that, by (32), the volume of $\mathcal{T}_{n+1}$ decreases rapidly as the dimension n increases, as given by the following formula:

$$\mathrm{Vol}\,(\mathcal{T}_{n+1}) = \frac{1}{(n+1)!}. \tag{48}$$

Remark 3. The domain $\mathcal{T}_{n+1}$ has $n + 2$ "sides" (which belong to n-dimensional hyperplanes) on its boundary. These are

$$\mathcal{R}_0 = \{\hat{\xi}_0^n : 0 < \xi_n < \cdots < \xi_1 < x = 1\}, \tag{49}$$

$$\mathcal{R}_1 = \{\hat{\xi}_0^n : 0 < \xi_n < \cdots < \xi_1 = x < 1\}, \tag{50}$$

$$\mathcal{R}_i = \{\hat{\xi}_0^n : 0 < \xi_n < \cdots < \xi_i = \xi_{i-1} < \cdots < \xi_1 < x < 1\}, \quad i = 2, \ldots, n \tag{51}$$

$$\mathcal{R}_{n+1} = \{\hat{\xi}_0^n : 0 = \xi_n < \cdots < \xi_1 < x < 1\}. \tag{52}$$

The boundary conditions (43) and (44) are non-homogeneous and given on $\mathcal{R}_1$. Note that the bracket in the definition of ϕ_n in (47), which is needed for (44), is evaluated at $x = \xi_1$ and thus can be computed from (43), without needing to know the kernel k_n a priori (this is explicitly illustrated next with the formula for ϕ_2 in (68)). The boundary condition (45) is given on $\mathcal{R}_i$, for $i = 2, \ldots, n$ and represents the value of the normal derivative of k_n in the boundary $\mathcal{R}_i$, hence it is of Neumann type. The boundary condition (46) is of Robin type and given on $\mathcal{R}_{n+1}$. The value of the function k_n on $\mathcal{R}_0$ is what needs to be found for the control law (37).

Remark 4. Eq. (40) with its boundary conditions can be reinterpreted as a wave equation in spacetime. If one thinks of x as time (time-like variable) and the other variables $\xi_1, \xi_2, \ldots, \xi_n$ as space coordinates (space-like variables), then the domain can be seen as a n-dimensional hyper-pyramid in $\mathbb{R}^n$ that *grows* (linearly in "time" x), with boundaries $\mathcal{R}_1, \mathcal{R}_2, \ldots, \mathcal{R}_{n+1}$ that are also

[1] This assumption requires uniform convergence of the transformation Volterra series which is shown in Vazquez and Krstic (2008, Theorem 2).