Springer

KSME

Journal of Mechanical Science and Technology 27 (9) (2013) 2577~2583
www.springerlink.com/content/1738-494x
DOI 10.1007/s12206-013-0701-y

# Stick-slip conditions in the general motion of a planar rigid body †

Iman Kardan*, Mansour Kabganian, Reza Abiri and Mostafa Bagheri

*Mechanical Engineering Department, Amirkabir University of Technology, 424 Hafez Ave., Tehran, 15875-4413, Iran*

(Manuscript Received June 11, 2012; Revised March 1, 2013; Accepted April 30, 2013)

## Abstract

In this paper we study combined translational and rotational (general) motion of planar rigid bodies in the presence of dry coulomb friction contact. Despite the cases where the body has pure translational/ rotational motion or can be assumed as a point mass, during the general motion, distinct points of the rigid body move in different directions which cause the friction force vector at each point to be different. Therefore, the direction and the magnitude of the overall friction force cannot be intuitively defined. Here the concept of instantaneous center of rotation is used as an effective method to determine the resultant friction force and moment. The main contribution of this paper is to propose novel stick-slip switching conditions for the general in-plane motion of rigid bodies. Simulation results for some combination of external forces are provided and some experimental tests are designed and conducted for practical verification of the proposed stick-slip conditions.

*Keywords*: Dry coulomb friction; General in-plane motion; Stick-slip phenomenon; Friction measurement

## 1. Introduction

Due to nonlinearities and other complexities that taking friction forces into account may introduce in complicated systems, it's a common interest to ignore them, supposing that they are negligible comparing to the other forces. Although this ignorance provides us with some ease, it can degrade the desired performance of the system. Åström [1] showed that dry friction may produce some steady state error in PD-type controllers. On the other hand, limit cycles may occur in PID-type controllers [2]. Recent approaches such as adaptive compensation [3] are also used to suppress the friction effect, which widely depend on the friction formulation.

Hence to improve the performance and also to get the model more similar to the real system, the friction forces should be somehow formulated.

When modeling dry friction, one of the unavoidable matters to be dealt with is the discontinuity induced by the stick-slip motion [4]. Den Hartog [5] and Hong and Liu [6] have studied the motion of a one-dimensional friction oscillator under harmonic excitations with zero-stop, two-stops and multi-stops per cycle. Abdo and Abouelsoud [7] used Liapunov second method to estimate the amplitude of the velocity and displacement of the stick-slip motion of a one-dimensional mass-spring-damper system on a moving belt. Xia [4] proposed a model for investigating the stick-slip motion of a two-dimensional friction oscillator under arbitrary excitations. Duan and Singh [8] proposed a 3-DOF rotational model for a torque converter clutch and investigated the stick-slip conditions for that system. They also studied the forced vibration of a reduced 1-DOF model of their system assuming that the normal load is varying periodically [9]. Theoretical and experimental study of the stick-slip motion of a 2-DOF system with varying normal load is done by Awrejcewicz and Olejnik [10].

In all the aforementioned cases the bodies in focus of study have pure translational/rotational motions. When the body has simultaneous rotational and translational motion, which is commonly called general motion, every point of the body moves in a different direction. Thus in these cases determining the direction and the magnitude of the resultant friction force vector is not straight forward.

Goyal et al. [11-13] presented a rather comprehensive study on how to determine the magnitude and direction of friction force/moment acting on a sliding rigid body in the case of isotropic and anisotropic friction laws. They proposed to use the instantaneous center of rotation to determine the overall friction effect which will be followed in the present work. They also studied the motion of a freely sliding object which comes to rest under the influence of dry friction. Assuming a circular contact area, Zhuravlev [14] has formulated the problem of combined spinning and sliding friction. Kireenkov has used the Zhuravlev theory to study the motion of the bodies with planar dry friction contacts [15]. Kireenkov et al. [16]

*Corresponding author. Tel.: +98 2164543463
E-mail address: i.kardan@aut.ac.ir
† Recommended by Associate Editor Cheolung Cheong
© KSME & Springer 2013

also prepared a test bed for experimental verification of the 2D models of combined friction.

Although the friction modeling for the general motion is provided in these works, they do not study the stick-slip phenomena. In fact there seems to be a lack of literatures which have studied the stick-slip switching conditions in simultaneous rotational and translational motion.

Therefore, utilizing the method of instantaneous center of rotation to determine the overall friction force, the stick-slip conditions in the general motion of a planar rigid body are proposed and experimentally evaluated for the first time in this paper. It should be noted that only rigid bodies with rectangular cross-sections are considered here. However, the presented formulations can be easily extended to any other body shapes.

This paper is organized as follows. A simple problem of a planar rigid body with 3 DOFs moving under arbitrary external forces in the presence of dry friction is defined in section 2. The case of pure translational motion is studied in section 2.1. For the case of general motion, the method of determining the resultant friction force is presented in section 2.2 and the stick-slip switching conditions are proposed in section 2.3. Section 3 provides the numerical simulation results and the experimental verification method is described in section 4. Finally section 5 concludes the paper.

## 2. A 3-DOF planar rigid body with dry friction contact

A rectangular planar rigid body is considered which is moving under arbitrary external forces as shown in Fig. 1. This body is in dry friction contact with a horizontal surface and has 3 DOFs, two of which being in-plane translations along $x$ and $y$ axes and the other being a rotation about an axis perpendicular to the plane of motion.

Using Newton's second law, one can easily derive the governing equations of the in-plane motion of the body as:

$$\begin{bmatrix} m & 0 & 0 \\ 0 & m & 0 \\ 0 & 0 & J_G \end{bmatrix} \begin{Bmatrix} a_{G_x} \\ a_{G_y} \\ \alpha \end{Bmatrix} = \begin{Bmatrix} F_x + F_{f_x} \\ F_y + F_{f_y} \\ M_G + M_{f_G} \end{Bmatrix} \quad (1)$$

where $m$ is the mass of the body and $J_G$ is its inertia about an axis perpendicular to the plane of motion passing through the center of mass (*CM*). $a_{Gx}$ and $a_{Gy}$ are the linear accelerations of the body along $x$ and $y$ axes and $\alpha$ is its angular acceleration. $F_{fx}$ and $F_{fy}$ are the components of friction force along the corresponding axes and $M_{fG}$ is the friction moment about *CM*.

### *2.1 Pure translational motion*

When the body has pure translational/rotational motion, the friction force/moment vector can be readily determined knowing the velocity vector. If the body is at rest (stick phase), the

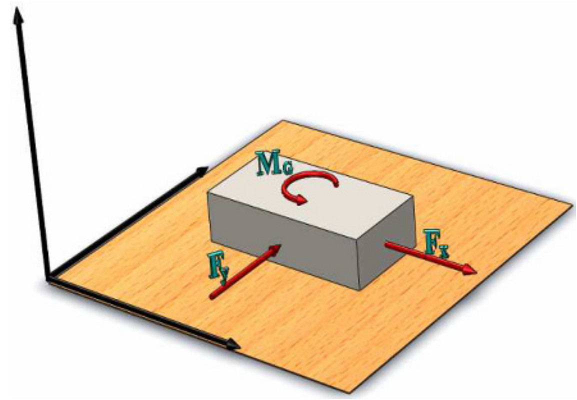


Fig. 1. A planar rigid body under arbitrary excitations.

friction vector equals the net in-plane excitation $\mathbf{F}_{net}$ but in the opposite direction. When the velocity becomes non-zero (slip phase), the friction vector will be in the opposite direction of the velocity vector with a constant magnitude of $\mu_k N$, in which $\mu_k$ is the kinetic friction coefficient and $N$ is the normal contact force. These statements are summarized in Eq. (2).

$$\mathbf{F}_f = \begin{cases} -\mathbf{F}_{net} & \mathbf{V} = \mathbf{0} \\ -\mu_k N \dfrac{\mathbf{V}}{|\mathbf{V}|} & \mathbf{V} \neq \mathbf{0} \end{cases} \quad (2)$$

where $\mathbf{V}$ is the vector of velocity of the body's center of mass. It should be mentioned that for the sake of simplicity two main assumptions are made here:

(1) The body does not take any out of plane rotation. Thus the normal force is constant and considered to be evenly distributed over the contact surface.

(2) The friction coefficients are assumed to be identical allover the contact surface and in all directions.

In this simple case the stick-slip conditions can be easily stated: *The body remains or enters in stick phase if the velocity is zero and Eq. (3) is satisfied*, where $\mu_s$ is the static friction coefficient.

$$|\mathbf{F}_{net}| \le \mu_s N . \quad (3)$$

### *2.2 General motion*

As previously mentioned, during the general motion, at different points of the body the friction force vector has different direction which depends on the direction of the velocity vector of those points. So to determine the resultant friction forces it's necessary to determine the direction of motion of every point of the body.

To do this, we follow the method proposed by Goyal [11] and use the *instantaneous center of rotation (CR)*, about which the body has a pure rotational motion at each given moment.

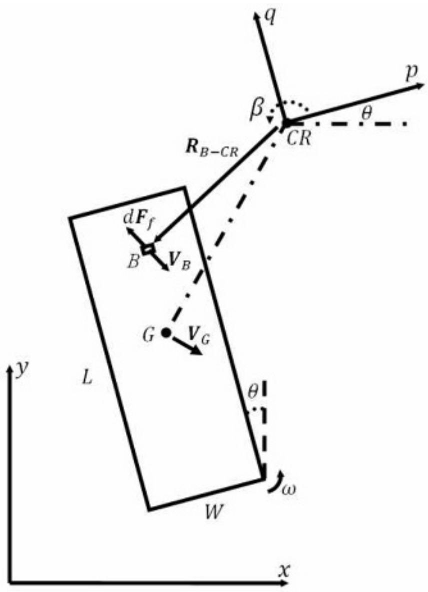


Fig. 2. Inertial and transformed coordinate systems.

Knowing linear and angular velocity of the body's center of mass ($G$), the location of $CR$ is determined as:

$$x_{CR} = \left(\omega \times x_G - V_{G_y}\right)/\omega$$
$$y_{CR} = \left(\omega \times y_G + V_{G_x}\right)/\omega \tag{4}$$

here $(x_G, y_G)$ and $(x_{CR}, y_{CR})$ define the location of center of mass and instantaneous center of rotation respectively, $\omega$ is the angular velocity of the body and $V_{Gx}$ and $V_{Gy}$ are its linear velocities along $x$ and $y$ axes.

Now considering an element with dimensions of $dx$ and $dy$ in the location of point $B$ as shown in Fig. 2, the velocity of this point of the body and the friction force acting on the element can be determined as:

$$\mathbf{V}_B = \boldsymbol{\omega} \times \mathbf{R}_{B-CR}$$
$$d\mathbf{F}_f = -\mu_k \frac{N}{A}\frac{\mathbf{V}_B}{\left|\mathbf{V}_B\right|} dxdy \tag{5}$$

where $\mathbf{R}_{B\text{-}CR}$ is the position of point $P$ relative to $CR$. It's assumed that the normal force is distributed evenly on the contact surface, so that the normal force on each element can be considered to be proportional to its area.

Knowing the friction force at the location of each element, the resultant friction force and moment can be computed by integrating over the entire body.

In order to facilitate the integration process, we propose a coordinate transformation. The coordinate system is translated to the location of $CR$ and rotated as much as $\theta$ which is the body's in-plane rotation. This transformation simplifies the integration limits. The new coordinates of center of mass will become:

$$\begin{Bmatrix} p_G \\ q_G \end{Bmatrix} = R_{CR}\left(\begin{Bmatrix} x_G \\ y_G \end{Bmatrix} - \begin{Bmatrix} x_{CR} \\ y_{CR} \end{Bmatrix}\right) \tag{6}$$

where $p$ and $q$ indicate the components of the transformed coordinate system and $R_{CR}$ is the rotation matrix. Finally the resultant friction components can be computed by integration:

$$F_{f_p} = \iint \left|d\mathbf{F}_f\right| \sin(\beta)$$
$$= \mu_k sign(\omega)\frac{N}{A}\int_{q_G-\frac{L}{2}}^{q_G+\frac{L}{2}}\int_{p_G-\frac{W}{2}}^{p_G+\frac{W}{2}} \frac{p}{\sqrt{p^2+q^2}} dpdq \tag{7}$$

$$F_{f_q} = -\iint \left|d\mathbf{F}_f\right| \cos(\beta)$$
$$= -\mu_k sign(\omega)\frac{N}{A}\int_{q_G-\frac{L}{2}}^{q_G+\frac{L}{2}}\int_{p_G-\frac{W}{2}}^{p_G+\frac{W}{2}} \frac{p}{\sqrt{p^2+q^2}} dpdq \tag{8}$$

$$M_{f_{CR}} = \iint \mathbf{R}_{B-CR} \times d\mathbf{F}_f$$
$$= -\mu_k sign(\omega)\frac{N}{A}\int_{q_G-\frac{L}{2}}^{q_G+\frac{L}{2}}\int_{p_G-\frac{W}{2}}^{p_G+\frac{W}{2}} \sqrt{p^2+q^2}\, dpdq \tag{9}$$

where $F_{fp}$ and $F_{fq}$ are the components of friction force along $p$ and $q$ axes, $M_{fCR}$ is the resultant friction moment about $CR$, and $L$ and $W$ are the dimensions of the body.

Now the components should be transformed to inertial coordinates and the friction moment should be taken about the center of mass:

$$\begin{Bmatrix} F_{f_x} \\ F_{f_y} \end{Bmatrix} = R_{CR}^{\ T}\begin{Bmatrix} F_{f_p} \\ F_{f_q} \end{Bmatrix}$$
$$M_{f_G} = M_{f_{CR}} + F_{f_x}\left(y_G - y_{CR}\right) - F_{f_y}\left(x_G - x_{CR}\right). \tag{10}$$

Finally the accelerations are computed using Eq. (1) and by integrating them, velocity and position of the body in the next time step are defined. According to Eq. (4) as $\omega$ goes to zero, position of $CR$ shifts to infinity, implicating that the body has a pure translational motion. The method to deal with this case is discussed in section 2.1.

### *2.3 Stick-slip switching conditions*

The conditions of switching between stick and slip phases in a general motion of a planar rigid body are derived in this section. By definition, using $CR$ to describe the motion of a rigid body combines the rotational and translational motions into a pure rotation about *CR*. Therefore while every one of linear or angular velocities is non-zero, the body remains in slip phase. In order to enter or remain in stick phase, all the velocities should be zero and the external forces should be below a maximum. To determine this upper limit the concept of *instantaneous center of zero acceleration* is employed, which is a point with zero acceleration at each instant and is abbreviated hereafter as *CA*.

It is obvious that during the stick phase all the accelerations are zero and therefore, using the *CA* seems not to be applicable here. But we use the *CA* to determine the stick-slip switching conditions by making an assumption such that:

*In the stick phase and under the influence of external forces, the body tends to move in the direction that it would move in the absence of friction.*

This assumption could be justified regarding the fact that friction does not cause any motion by itself and just tend to counteract the relative motion of bodies in contact.

Assuming that there is no friction, the linear and angular accelerations of the body can be calculated as:

$$\begin{Bmatrix} a_{xi} \\ a_{yi} \\ \alpha_i \end{Bmatrix} = \begin{bmatrix} m & 0 & 0 \\ 0 & m & 0 \\ 0 & 0 & J_G \end{bmatrix}^{-1} \begin{Bmatrix} F_x \\ F_y \\ M_G \end{Bmatrix} \tag{11}$$

where the subscript *i* indicates that the quantities are computed assuming a frictionless contact. The location of *CA* is then determined using Eq. (12).

$$x_{CA} = \frac{\alpha_i \times x_G - a_{yi}}{\alpha_i}, \quad y_{CA} = \frac{\alpha_i \times y_G + a_{xi}}{\alpha_i}. \tag{12}$$

By setting the origin of transformed coordinate system to *CA*, the maximum moment that static friction can exert about *CA* could be defined as:

$$\left(M_{f_{CA}}\right)_{max} = \iint \mathbf{R}_{B-CA} \times d\mathbf{F}_f = -\mu_s sign(\alpha_i)\frac{N}{A}\int_{q_{Gi}-\frac{L}{2}}^{q_{Gi}+\frac{L}{2}}\int_{p_{Gi}-\frac{W}{2}}^{p_{Gi}+\frac{W}{2}} \sqrt{p^2+q^2}\,dpdq \tag{13}$$

$$\begin{Bmatrix} p_{Gi} \\ q_{Gi} \end{Bmatrix} = R_{CA}\left(\begin{Bmatrix} x_G \\ y_G \end{Bmatrix} - \begin{Bmatrix} x_{CA} \\ y_{CA} \end{Bmatrix}\right) \tag{14}$$

where $R_{CA}$ is the rotation matrix between inertial coordinate system and the transformed one located at *CA*.

Finally the stick condition can be stated as: *The body will remain or enter in stick phase, if all the velocities of the body are zero and the resultant moment of external forces about CA is less than the maximum moment which static friction can apply*, or in the other words:

$$M_{CA} \le \left(M_{f_{CA}}\right)_{max}. \tag{15}$$

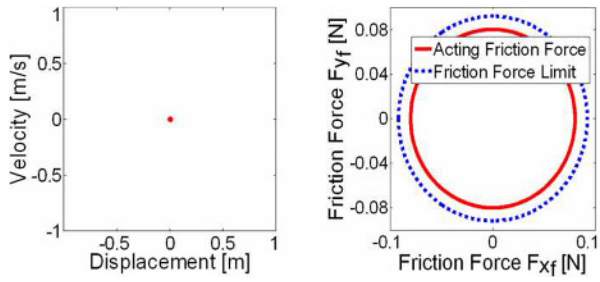


Fig. 3. No-slip responses.

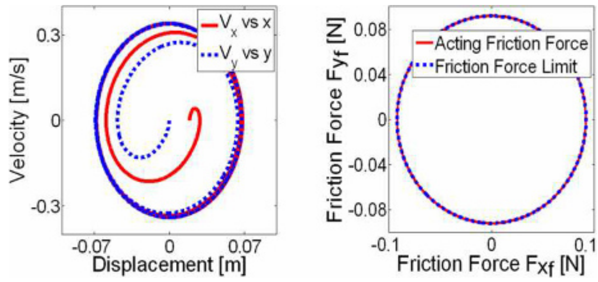


Fig. 4. Zero-stop responses.

## 3. Simulation results

### *3.1 Pure translational motion*

Xia [4] states that when the excitations along x and y directions are sinusoidal with a difference of $\pi/2$ in their phases but identical amplitudes and frequencies, the response will have a circular orbit. In fact in this case the resultant external force vector will have a constant magnitude. If the magnitude is greater than friction limit force, the body will always slip (zero-stop). Otherwise, no motion will occur (no-slip). Therefore in this case, stick-slip motion won't happen.

To assess these statements, the values of parameters are chosen as: $\mu_s = \mu_k = 0.4$, $L = 0.03$ *[m]*, $W = 0.01$ *[m]*, $m = 0.0234$ *[kg]*, $J_G = 1.95e{-6}$ *[kg.m²]*, and two different sets of external forces are applied to the body.

The first set of excitations is chosen to have an amplitude of *0.08 [N]* with $M_G = 0$, $F_x = 0.08\cos(5t)$, and $F_y = 0.08\cos(5t+\pi/2)$. As can be seen from Fig. 3, in this case the magnitude of the applied force is less than the friction limit and no slip occurs.

The second set of excitations has an amplitude of *0.1 [N]* with $M_G = 0$, $F_x = 0.1\cos(5t)$, and $F_y = 0.1\cos(5t+\pi/2)$. In this case the magnitude of the applied force always remains greater than the friction limit and the body never enters the stick phase (Fig. 4).

As shown in Fig. 5, a case of multi-stop per cycle is also simulated using the same value of parameters but a different set of excitations with $M_G = 0$, $F_x = 0.08\cos(5t)$, and $F_y = 0.073\sin(3t)$.

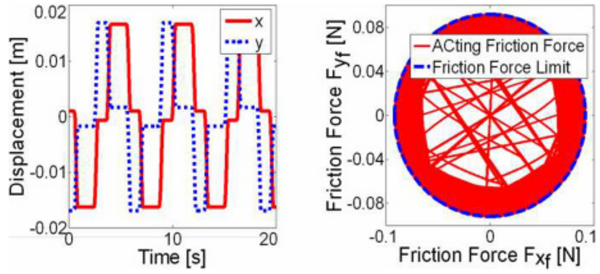


Fig. 5. Four-stop responses.

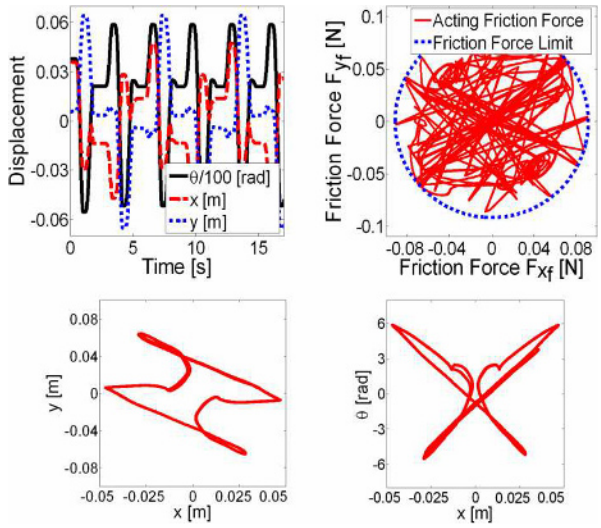


Fig. 6. The general in-plane responses.

It is notable that the initial position and orientation of the body is chosen in such a way that makes the responses symmetric about the origin.

### 3.2 General motion

To impose a general motion to the body, the excitations $M_G = 0.0007cos(4t+\pi/4)$, $F_x = 0.07cos(5t)$, and $F_y = 0.06sin(3t)$ are applied with the same values of parameters as before. Responses are illustrated in Fig. 6.

Simulations results indicate that although for the pure translational motions the magnitude of the resultant friction force equals the value of $\mu_k N$ during the slip phase, this no longer holds for the case of general motion. However this magnitude always remains within the circle of friction limit as expected. The results also reaffirm that stick phase happens simultaneously for all the degrees-of-freedom.

## 4. Experimental validation

In order to practically validate the presented friction modeling, some definite forces and torques should be applied to the body in a way that a general motion is imposed. Then the body's displacements should be measured and compared to the simulation results.

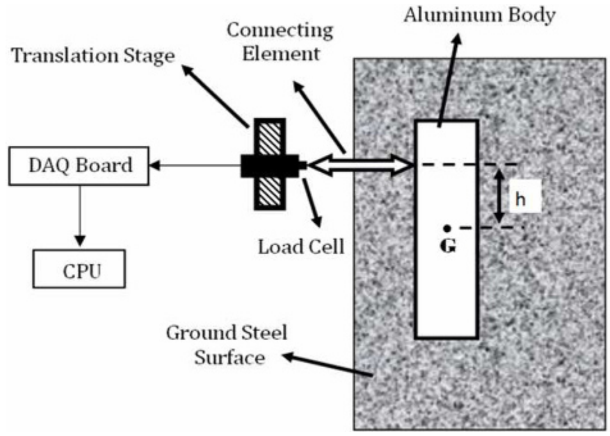


Fig. 7. Schematic overview of the setup.

This procedure is almost impossible in practice because of the difficulties related to applying a desired force to a moving body. So the whole presented friction formulation could not be verified experimentally. But evaluation of the stick-slip switching condition, which is the main contribution of this work, is possible by conducting a test as follows.

It is known that a point force which its line of action does not pass through the center of mass can be replaced by the same force at the center of mass and a moment. The magnitude of the moment is a function of the eccentricity and the magnitude of the applied force. So, if a body at rest is subjected to an increasing eccentric point force, its initial motion will be a general motion. By measuring the size and knowing the location of the applied force, the required force to impose an initial general motion to the rest body can be measured practically. The proposed stick-slip switching condition can be verified by comparing the measured force to the theoretically computed one.

It is evident that in this test only the magnitude of the applied force at the beginning of the motion is measured and therefore just the condition of switching from stick to slip phase is verified. However according to Eq. (15) it is clear that the validity of condition of entering the stick phase (switching from slip to stick phase) can be directly concluded from the validity of the condition of switching from stick to slip phase. Therefore it can be declared that by running this test we can verify the whole presented stick-slip switching conditions.

To perform the described test an experimental setup is prepared (Figs. 7 and 8). The setup includes a load cell *SS2* by *Sherborne Sensors* mounted on a 2-DOF motorized translation stage by *Standa*. The load cell signal is transferred to a PC through a *DAQ* board *PCI 1716* by *Advantech*.

As illustrated in Fig. 9, two L-shaped links are prepared to transfer force between the body and the load cell. The one with a flat end is used in measuring the friction coefficients and the other one is used to apply eccentric point forces to the body. The body is a $30\times10\times10$ $[mm^3]$ aluminum rectangular cube which is placed on a ground steel surface.

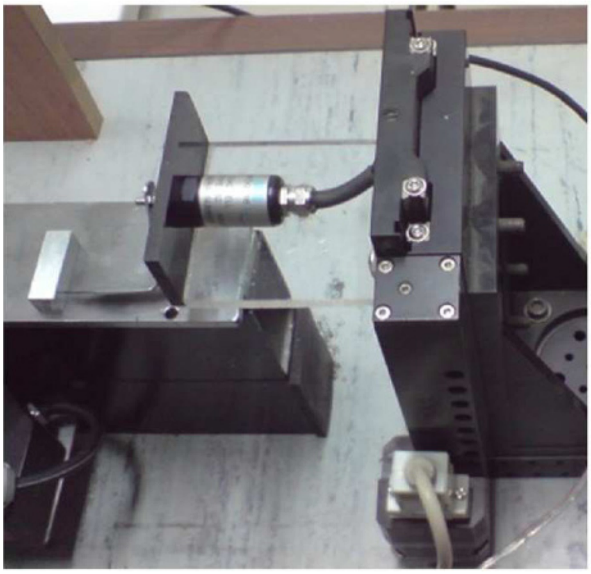

Fig. 8. The prepared experimental setup.

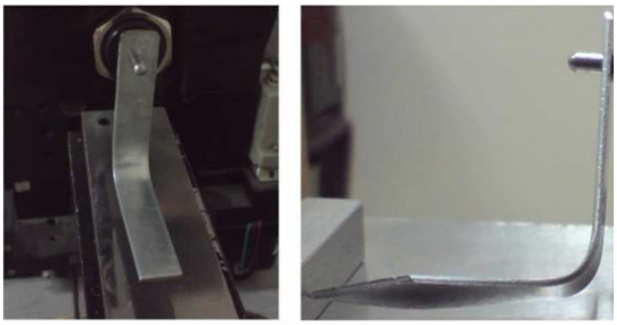

Fig. 9. L-shaped connecting links.

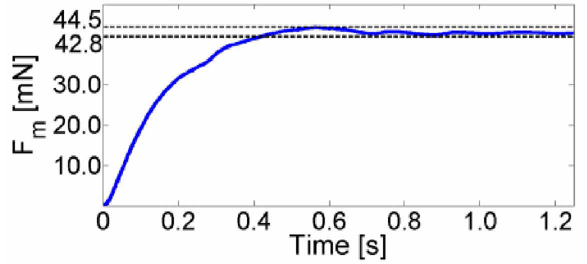


Fig. 10. A sample of the measured force for pure translation motion.

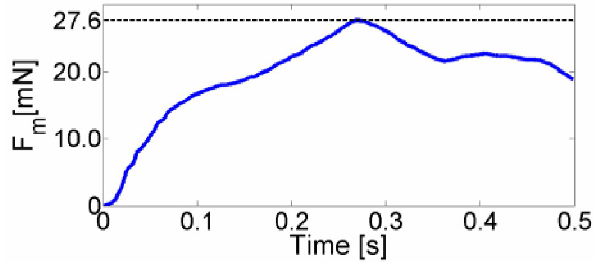


Fig. 11. A sample of the measured eccentric force for h = 10 mm.

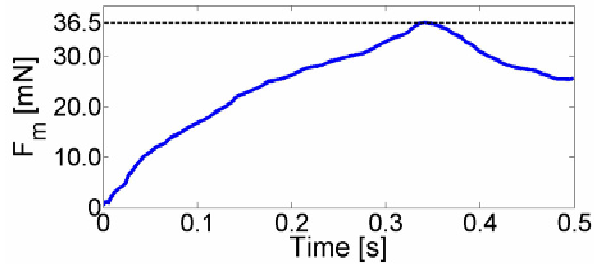


Fig. 12. A sample of the measured eccentric force for h = 5 mm.

As the first step of the validation procedure the friction parameters should be determined practically. The motorized stage moves the load cell with a constant speed and by using the flat-end link a pure translational motion is imposed to the body. As the link touches the body, because of the elastic property of the load cell tip, the applied force begins to increase. This increase continues till the exerted force overcomes the static friction and the body starts to move. At this moment the measured force decreases rapidly and for the rest of the motion gains an almost fixed value with small fluctuations.

Before starting to move and while moving with a constant speed the body has zero acceleration, so it can be claimed that the highest value of the measured force equals the maximum value of the static friction $\mu_s N$ and its final fixed value equals the kinetic friction $\mu_k N$.

A sample of the applied force is shown in Fig. 10. Knowing the weight of the body, the friction coefficients are calculated to be $\mu_s = 0.55$ and $\mu_k = 0.52$.

To evaluate the proposed stick-slip switching conditions for the general motion, the sharp-end link is used to apply an eccentric force to the body. As shown in Fig. 7 the force is exerted perpendicular to the body's side with a distance of $h$ millimeter from the center of mass. Similar to the previous case the load cell is moved with a constant speed and when the link touches the body, the measured force increases rapidly till the motion is started. The measured force for two different values of $h = 10$ *[mm]* and $h = 5$ *[mm]* is depicted in Figs. 11 and 12, respectively.

A comparison of experimental and theoretical forces which are required for initiating a general motion is provided in Table 1, in which the theoretical forces are calculated as follows: The Inertial coordinate system is defined such that $\theta = 0$ and the external force lies along $x$ direction. Therefore $a_{yi} = 0$ and $R_{CA} = I$ where $I$ is the identity matrix. Then knowing mass and moment of inertia of the body, the accelerations $a_{xi}$ and $\alpha_i$ are computed using Eq. (11) and the location of $CA$ is defined through Eq. (12). Finally $(M_{fCA})_{max}$ is determined using Eq. (13) and the external force which can produce such moment about $CA$ is considered as the theoretical force.

The good agreement between theoretical and experimental results indicates that the proposed model predicts the switching conditions very well. The results also show that by increasing the eccentricity, as can be inferred intuitively, the

Table 1. Comparison of experimental and theoretical force for initiating a general motion.

| Eccentricity [mm] | Theoretical force [mN] | Measured force [mN] | Relative frror % |
|---|---|---|---|
| h = 5 | 34.7 | 36.5 | 4.93 |
| h = 10 | 26.2 | 27.6 | 5.07 |

body starts to move with less effort. It should be also noted that because of the complexity of the imposed motion no analysis can be done about the changes in the force measured after the motion is started.

## 5. Conclusions

General motion of a planar rigid body in the presence of dry friction is formulated and stick-slip switching conditions are proposed. Although the external forces can be of any form, simulation results for some combinations of harmonic excitations are provided. The results show that the presented formulation effectively models the cases of zero-stop, one-stop and multi-stops per cycle in all the 1, 2 and 3-DOF in-plane motions.

Some tests are designed and conducted for practical validation of the proposed stick-slip conditions. An experimental setup is prepared for applying eccentric forces to the body and measuring the force which initiates a general motion. The comparison of theoretical and experimental results reveals the effectiveness of the proposed model in predicting the conditions of switching between stick and slip phases.

## References

[1] K. J. Åström, Control of systems with friction, *proceeding of the 4th international conference on motion and vibration control*, Zurich, Switzerland (1998) 25-32.

[2] H. Olsson and K. J. Åström, Friction generated limit cycles, *IEEE Transactions on Control and Systems Technology*, 9 (4) (2001) 629-636.

[3] V. Erfanian and M. Kabganian, Adaptive trajectory control and dynamic friction compensation for a flexible-link robot, *Journal of Mechanics*, 26 (2010) 205-217.

[4] F. Xia, Modelling of a two-dimensional coulomb friction oscillator, *Journal of Sound and Vibration*, 265 (2003) 1063-1074.

[5] J. P. Den Hartog, Forced vibrations with combined coulomb and viscous friction, *Transactions of the American Society of Mechanical Engineers*, 53 (1931) 107-115.

[6] H. K. Hong and C. S. Liu, Coulomb friction oscillator: modelling and responses to harmonic loads and base excitations, *Journal of Sound and Vibration*, 229 (2000) 1171-1192.

[7] J. Abdo and A. A. Abouelsoud, Analytical approach to estimate amplitude of stick-slip oscillations, *Journal of Theoretical and Applied Mechanics*, 49 (4) (2011) 971-986.

[8] C. Duan and R. Singh, Stick-slip behavior in torque converter clutch, SAE Transactions, *Journal of Passenger Car: Mechanical Systems*, 116 (2005) 2785-2795.

[9] C. Duan and R. Singh, Forced vibrations of a torsional oscillator with coulomb friction under a periodically varying normal load, *Journal of Sound and Vibration*, 325 (2009) 499-506.

[10] J. Awrejcewicz and P. Olejnik, Occurrence of stick-slip phenomenon, *Journal of Theoretical and Applied Mechanics*, 45 (1) (2007) 33-40.

[11] S. Goyal, Planar sliding of a rigid body with dry friction: limit surfaces and moment function, *Ph.D. Thesis,* Cornell University, Ithaca, *NY, USA*, (1989).

[12] S. Goyal, A. Ruina and J. Papadopoulos, Planar sliding with dry friction. Part1. limit surface and moment function, *Wear*, 143 (1991) 307-330.

[13] S. Goyal, A. Ruina and J. Papadopoulos, Planar sliding with dry friction. Part2. dynamics of motion, *Wear*, 143 (1991) 331-352.

[14] V. G. Zhuravlev, The model of dry friction in the problem of the rolling of rigid bodies, *J. Appl. Math. Mech*, 62 (5) (1998) 705-710.

[15] A. A. Kireenkov, Combined model of sliding and rolling friction in dynamics of bodies on a rough plane, *Mechanics of Solids*, 43 (3) (2008) 412-425.

[16] A. A. Kireenkov, S. V. Semendyaev and V. F. Filatov, Experimental study of coupled two-dimensional models of sliding and spinning friction, *Mechanics of Solids*, 45 (6) (2010) 921-930.

**Iman Kardan** received his B.Sc. and M.Sc. degree in Mechanical Engineering from Amirkabir University of Technology (Tehran Polytechnic), Tehran, Iran in 2008 and 2011, respectively. Since 2008, he has been a research assistant in the System Dynamics and Control Research Laboratory, mechanical engineering department, Amirkabir University of Technology. His research interest is in the area of robotics, microrobotics, control and smart materials.

**Mansour Kabganian** received his B.Sc. from Amirkabir University of Technology (Tehran Polytechnic), Tehran, Iran, in 1975, M.Sc. from University of Tarbiat Modarres, Tehran, in 1988, and Ph.D. from the University of Ottawa, Ottawa, Canada, in 1995, all in mechanical engineering. He joined the Department of Mechanical Engineering at Amirkabir University of Technology in 1988, where he is a Professor and the Head of the System Dynamics and Control Research Laboratory. He is also the Head of the Space Dynamics and Control Research Center, which he established in 2009. His main research interests are control, stability, and dynamical systems such as spacecrafts, vehicles, and microrobotics.