Chapter 9

9.1

\[ w(x) = u(x) - \int_0^x k(x, y)u(y) \, dy \]

\[ w_x(x) = u_x(x) - k(x, x)u(x) - \int_0^x k_x(x, y)u(y) \, dy \]

\[ w_t(x) = u_t(x) - \int_0^x k(x, y) \left( u_y(y) + g(y)u(0) + \int_0^y f(y, \xi)u(\xi) \, d\xi \right) \, dy \]

\[ = u_t(x) - k(x, x)u(x) + k(x, 0)u(0) - u(0) \int_0^x k(x, y)g(y) \, dy \]

\[ + \int_0^x \left( k_u(x, y) - \int_y^x k(x, \xi) f(\xi, y) \, d\xi \right) u(y) \, dy \]

Substituting these expressions in the target system, we get

\[ w_t - w_x = u(0) \left( g(x) + k(x, 0) - \int_0^x k(x, y)g(y) \, dy \right) \]

\[ + \int_0^x \left( k_x(x, y) + k_y(x, y) + f(x, y) - \int_y^x k(x, \xi) f(\xi, y) \, d\xi \right) u(y) \, dy \]
The expressions in the brackets should be equal to zero which gives the PDE for $k(x, y)$:

\[ k_x + k_y = \int_y^x k(x, \xi)f(\xi, y)d\xi - f(x, y) \]

\[ k(x, 0) = \int_0^x k(x, y)g(y)dy - g(x) . \]