Chapter 9

9.1

$$
\begin{aligned}
w(x) &= u(x) - \int_0^x k(x,y)u(y)\,dy \\
w_x(x) &= u_x(x) - k(x,x)u(x) - \int_0^x k_x(x,y)u(y)\,dy \\
w_t(x) &= u_t(x) - \int_0^x k(x,y)\left(u_y(y) + g(y)u(0) + \int_0^y f(y,\xi)u(\xi)\,d\xi\right)dy \\
&= u_t(x) - k(x,x)u(x) + k(x,0)u(0) - u(0)\int_0^x k(x,y)g(y)\,dy \\
&\quad + \int_0^x \left(k_y(x,y) - \int_y^x k(x,\xi)f(\xi,y)\,d\xi\right)u(y)\,dy
\end{aligned}
$$

Substituting these expressions in the target system, we get

$$
\begin{aligned}
w_t - w_x &= u(0)\left(g(x) + k(x,0) - \int_0^x k(x,y)g(y)\,dy\right) \\
&\quad + \int_0^x \left(k_x(x,y) + k_y(x,y) + f(x,y) - \int_y^x k(x,\xi)f(\xi,y)\,d\xi\right)u(y)\,dy
\end{aligned}
$$

The expressions in the brackets should be equal to zero which gives the PDE for $k(x,y)$:

$$k_x + k_y = \int_y^x k(x,\xi) f(\xi,y) d\xi - f(x,y)$$
$$k(x,0) = \int_0^x k(x,y) g(y) dy - g(x)\,.$$