Chapter 7

7.1 Differentiating the transformation with respect to x, we get

$$w_x(x) = u_x(x) + c_0 u(x)\,,$$

so that

$$w_x(0) = u_x(0) + c_0 u(0) = c_0 w(0)\,,$$

and

$$w_x(1) = u_x(1) + c_0 u(1) = -c_1 \left(u_t(1) + c_0 \int_0^1 u_t(y)\,dy \right) = -c_1 w_t(1)\,.$$

Finally,

$$w_{tt} - w_{xx} = u_{tt} + c_0 \int_0^x u_{yy}(y)\,dy - u_{xx} - c_0 u_x = -c_0 u_x(0) = 0.$$

7.2 Approximate values are $c_0 \approx 21$, $c_1 \approx 0.8$.

7.3

$$w(x) = u(x) + (c_0 + q) \int_0^x \mathrm{e}^{q(x-y)} u(y)\,dy\,,$$

$$\begin{aligned} w_{tt}(x) &= u_{tt}(x) + (c_0+q)\int_0^x \mathrm{e}^{q(x-y)} u_{yy}(y)\,dy \\ &= u_{tt}(x) + (c_0+q)u_x(x) + (c_0+q)\mathrm{e}^{qx} qu(0) + (c_0+q)qu(x) \\ &\quad - (c_0+q)\mathrm{e}^{qx} qu(0) + q^2(c_0+q)\int_0^x \mathrm{e}^{q(x-y)} u(y)\,dy\,, \end{aligned}$$

$$\begin{aligned} w_x(x) &= u_x(x) + (c_0+q)u(x) + q(c_0+q)\int_0^x \mathrm{e}^{q(x-y)} u(y)\,dy\,, \\ w_{xx}(x) &= u_{xx}(x) + (c_0+q)u_x(x) + q(c_0+q)u(x) \\ &\quad + q^2(c_0+q)\int_0^x \mathrm{e}^{q(x-y)} u(y)\,dy\,. \end{aligned}$$

We get

$$\begin{aligned} w_{tt} - w_{xx} &= 0 \\ w_x(0) - c_0 w(0) &= u_x(0) + (c_0+q)u(0) - c_0 u(0) = 0\,, \end{aligned}$$

and $w_x(1) + c_1 w_t(1) = 0$ gives feedback

$$u_x(1) = -c_1 u_t(1) - (c_0+q)u(1) - (c_0+q)\int_0^1 \mathrm{e}^{q(1-y)}[c_1 u_t(y) + qu(y)]dy$$