Then the controller is given by (see Exercise 4.1)

$$v_x(1) = -\frac{\lambda_0+1}{2}v(1) - \lambda_0 \int_0^1 \left[\frac{3I_1\left(\sqrt{\lambda_0(1-y^2)}\right)}{2\sqrt{\lambda_0(1-y^2)}} + \frac{I_2\left(\sqrt{\lambda_0(1-y^2)}\right)}{1-y^2} \right] v(y)\,dy\,.$$

Finally, we get back to the original variables:

$$\begin{aligned} u_x(1) &= \left(v_x(1) - \frac{b}{2}v(1)\right)\mathrm{e}^{-\frac{b}{2}} \\ &= -\frac{\lambda_0+b+1}{2}u(1) \\ &\quad - \lambda_0 \int_0^1 \left[\frac{3I_1\left(\sqrt{\lambda_0(1-y^2)}\right)}{2\sqrt{\lambda_0(1-y^2)}} + \frac{I_2\left(\sqrt{\lambda_0(1-y^2)}\right)}{1-y^2} \right] \mathrm{e}^{\frac{b}{2}(y-1)}u(y)\,dy\,. \end{aligned}$$

4.6 Control gain PDE has the solution $k(x,y) = \phi(x-y)$, where the Laplace transform of ϕ is

$$\Phi(s) = \frac{G(s)}{G(s)-s}$$

The Laplace transform of $g(x) = 3\mathrm{e}^{2x}$ is $G(s) = 3/(s-2)$. Therefore,

$$\begin{aligned} \Phi(s) &= \frac{\frac{3}{s-2}}{\frac{3}{s-2}-s} = \frac{-3}{s^2-2s-3} \\ &= \frac{3}{4}\left(\frac{1}{s+1} - \frac{1}{s-3}\right)\,. \end{aligned}$$

Taking the inverse Laplace transform, we get

$$k(x,y) = \phi(x-y) = \frac{3}{4}\left(\mathrm{e}^{-(x-y)} - \mathrm{e}^{3(x-y)}\right)$$

and the controller is

$$u(1) = \frac{3}{4}\int_0^1 \left(\mathrm{e}^{-(1-y)} - \mathrm{e}^{3(1-y)}\right) u(y)\,dy\,.$$

Chapter 5

5.1 The observer is

$$\begin{aligned} \hat{u}_t &= \hat{u}_{xx} + p_1(x)[u_x(1) - \hat{u}_x(1)] \\ \hat{u}_x(0) &= -q\hat{u}(0) \\ \hat{u}(1) &= U + p_{10}[u_x(1) - \hat{u}_x(1)] \end{aligned}$$

We map the error system

$$\begin{aligned}\tilde{u}_t &= \tilde{u}_{xx} - p_1(x)\tilde{u}_x(1)\\ \tilde{u}_x(0) &= -q\tilde{u}(0)\\ \tilde{u}(1) &= -p_{10}\tilde{u}_x(1)\end{aligned}$$

into the target system

$$\begin{aligned}\tilde{w}_t &= \tilde{w}_{xx}\\ \tilde{w}_x(0) &= \tilde{w}(1) = 0\end{aligned}$$

using the transformation

$$\tilde{u}(x) = \tilde{w}(x) - \int_x^1 p(x,y)\tilde{w}(y)\,dy\,.$$

Setting $x = 1$ in the transformation we get $p_{10} = 0$. Since

$$\tilde{u}_x(x) = \tilde{w}_x(x) + p(x,x)\tilde{w}(x) - \int_x^1 p_x(x,y)\tilde{w}(y)\,dy\,,$$

for $x = 0$ we get

$$\begin{aligned}-q\tilde{u}(0) &= p(0,0)\tilde{w}(0) - \int_0^1 p_x(0,y)\tilde{w}(y)\,dy\\ 0 &= (q + p(0,0))\tilde{w}(0) - q\int_0^1 p(0,y)\tilde{w}(y)\,dy - \int_0^1 p_x(0,y)\tilde{w}(y)\,dy\,,\end{aligned}$$

which gives $p(0,0) = -q$ and $p_x(0,y) = -qp(0,y)$. From the error PDE we get

$$\begin{aligned}0 &= \tilde{u}_t(x) - \tilde{u}_{xx}(x) + p_1(x)\tilde{u}_x(1)\\ &= \tilde{w}_t(x) - \int_x^1 p(x,y)w_{yy}(y)\,dy - \tilde{w}_{xx}(x) - \tilde{w}(x)\frac{d}{dx}p(x,x) - p(x,x)\tilde{w}_x(x)\\ &\quad - p_x(x,x)\tilde{w}(x) + \int_x^1 p_{xx}(x,y)\tilde{w}(y)\,dy + p_1(x)\tilde{w}_x(1)\\ &= -p(x,1)w_x(1) - p_x(x,x)\tilde{w}(x) - \int_x^1 p_{yy}(x,y)\tilde{w}(y)\,dy - \tilde{w}(x)\frac{d}{dx}p(x,x)\\ &\quad - p_y(x,x)\tilde{w}(x) + \int_x^1 p_{xx}(x,y)\tilde{w}(y)\,dy + p_1(x)\tilde{w}_x(1)\\ &= (p_1(x) - p(x,1))\tilde{w}_x(1) - 2\tilde{w}(x)\frac{d}{dx}p(x,x)\\ &\quad + \int_x^1 (p_{xx}(x,y) - p_{yy}(x,y))\tilde{w}(y)\,dy\,.\end{aligned}$$

Therefore, $p_1(x) = p(x,1)$ and $p(x,y)$ satisfies the following PDE:

$$\begin{aligned} p_{xx}(x,y) &= p_{yy}(x,y) \\ p_x(0,y) &= -qp(0,y) \\ \frac{d}{dx}p(x,x) &= 0\,. \end{aligned}$$

Since $p(0,0) = -q$, the last condition is equivalent to $p(x,x) = -q$. We look for the solution in the form $p(x,y) = \phi(y-x)$. Then ϕ satisfies the ODE

$$\begin{aligned} \phi'(y) &= q\phi(y) \\ \phi(0) &= -q\,, \end{aligned}$$

which has the solution $\phi(y) = -q\mathrm{e}^{qy}$. Finally,

$$p_1(x) = p(x,1) = \phi(1-x) = -q\mathrm{e}^{q(1-x)}\,.$$

5.2 Taking the Laplace transform we get

$$\begin{aligned} su(x,s) &= u''(x,s) \\ u'(0,s) &= -qu(0,s) \\ u(1,s) &= U(s)\,. \end{aligned}$$

General solution of this ODE is

$$u(x,s) = A\sinh(\sqrt{s}x) + B\cosh(\sqrt{s}x)\,,$$

and the constants A and B are found from the boundary conditions:

$$\sqrt{s}A = -qB, \qquad A\sinh(\sqrt{s}) + B\cosh(\sqrt{s}) = U(s)\,,$$

which gives

$$-qB\sinh(\sqrt{s}) + B\sqrt{s}\cosh(\sqrt{s}) = \sqrt{s}U(s)\,.$$

Finally,

$$u(0,s) = B = \frac{\sqrt{s}}{\sqrt{s}\cosh(\sqrt{s}) - q\sinh(\sqrt{s})}U(s)\,.$$