Chapter 4

4.1 Let us differentiate the transformation:

$$w(x) = u(x) - \int_0^x k(x,y)u(y)\,dy$$
$$w_x(x) = u_x(x) - k(x,x)u(x) - \int_0^x k_x(x,y)u(y)\,dy\,.$$

Setting $x = 1$ we get

$$w_x(1) = u_x(1) - k(1,1)u(1) - \int_0^1 k_x(1,y)u(y)\,dy\,.$$

From the boundary condition $2w_x(1) = -w(1)$ we get

$$u_x(1) = -\frac{1}{2}u(1) + \frac{1}{2}\int_0^1 k(1,y)u(y)\,dy + k(1,1)u(1) + \int_0^1 k_x(1,y)u(y)\,dy\,.$$

We only need to calculate $k_x(1,y)$:

$$k(x,y) = -\lambda x \frac{I_1\left(\sqrt{\lambda(x^2-y^2)}\right)}{\sqrt{\lambda(x^2-y^2)}}$$
$$k_x(x,y) = -\lambda \frac{I_1\left(\sqrt{\lambda(x^2-y^2)}\right)}{\sqrt{\lambda(x^2-y^2)}} - \lambda x^2 \frac{I_2\left(\sqrt{\lambda(x^2-y^2)}\right)}{x^2-y^2}$$

Finally,

$$u_x(1) = -\frac{\lambda+1}{2}u(1) - \lambda \int_0^1 \left[\frac{3I_1\left(\sqrt{\lambda(1-y^2)}\right)}{2\sqrt{\lambda(1-y^2)}} + \frac{I_2\left(\sqrt{\lambda(1-y^2)}\right)}{1-y^2}\right] u(y)\,dy\,.$$

4.2 First we find u_x to match the boundary condition at $x = 0$:

$$u(x) = w(x) + \int_0^x l(x,y)w(y)\,dy$$
$$u_x(x) = w_x(x) + l(x,x)w(x) + \int_0^x l_x(x,y)w(y)\,dy$$

This gives $l(0,0) = 0$. Next step is to find u_t and u_{xx}:

$$\begin{aligned} u_{xx}(x) &= w_{xx}(x) + w(x)\frac{d}{dx}l(x,x) + l(x,x)w_x(x) + l_x(x,x)w(x) \\ &\quad + \int_0^x l_{xx}(x,y)w(y)\,dy \\ u_t(x) &= w_t(x) + \int_0^x l(x,y)w_{yy}(y)\,dy \\ &= w_{xx}(x) + l(x,x)w_x(x) - l_y(x,x)w(x) + l_y(x,0)w(0) \\ &\quad + \int_0^x l_{yy}(x,y)w(y)\,dy \end{aligned}$$

Matching the PDE we get

$$\begin{aligned} l_{xx}(x,y) - l_{yy}(x,y) &= -\lambda l(x,y) \\ l_y(x,0) &= 0 \\ \frac{d}{dx}l(x,x) &= -\frac{\lambda}{2} \end{aligned}$$

Since $l(0,0) = 0$, the last condition can be written as $l(x,x) = -\lambda x/2$. We now see that the PDE for $l(x,y)$ becomes the PDE for $k(x,y)$ from Exercise 4.1 if we set $\lambda \to -\lambda$ and $l(x,y) \to -l(x,y)$. Therefore,

$$\begin{aligned} l(x,y) &= -\left(-(-\lambda)x\frac{I_1\left(\sqrt{-\lambda(x^2-y^2)}\right)}{\sqrt{-\lambda(x^2-y^2)}}\right) \\ &= -\lambda x\frac{I_1\left(j\sqrt{\lambda(x^2-y^2)}\right)}{j\sqrt{\lambda(x^2-y^2)}} \\ &= -\lambda x\frac{J_1\left(\sqrt{\lambda(x^2-y^2)}\right)}{\sqrt{\lambda(x^2-y^2)}} \end{aligned}$$

4.3 First we find w_x to match the boundary condition at $x = 0$:

$$w_x(x) = u_x(x) - k(x,x)u(x) - \int_0^x k_x(x,y)u(y)\,dy$$

This gives $w_x(0) = u_x(0) - k(0,0)u(0) = (-q - k(0,0))u(0)$, so that $k(0,0) =$

$-q$. Next we find the PDE for $k(x,y)$:

$$\begin{aligned} w_{xx}(x) &= u_{xx}(x) - u(x)\frac{d}{dx}k(x,x) - k(x,x)u_x(x) - k_x(x,x)u(x) \\ &\quad - \int_0^x k_{xx}(x,y)u(y)\,dy \\ w_t(x) &= u_t(x) - \int_0^x k(x,y)u_{yy}(y)\,dy \\ &= u_{xx}(x) - k(x,x)u_x(x) + k(x,0)u_x(0) + k_y(x,x)u(x) - k_y(x,0)u(0) \\ &\quad - \int_0^x k_{yy}(x,y)u(y)\,dy \end{aligned}$$

Matching the terms, we get

$$\begin{aligned} k_{xx}(x,y) &= k_{yy}(x,y) \\ k_y(x,0) &= -qk(x,0) \\ \frac{d}{dx}k(x,x) &= 0 \end{aligned}$$

Since $k(0,0) = -q$, the last condition is equivalent to $k(x,x) = -q$. The general solution of this PDE has the form $k(x,y) = \phi(x-y) + \psi(x+y)$, where ϕ and ψ are arbitrary functions. Since $k(x,x)$ is just a constant, without loss of generality we can set $\psi \equiv 0$. We get the following ODE for ϕ:

$$\begin{aligned} \phi'(x) &= q\phi(x) \\ \phi(0) &= -q\,, \end{aligned}$$

which has the solution $\phi(x) = -q\mathrm{e}^{qx}$. Therefore, $k(x,y) = -q\mathrm{e}^{q(x-y)}$ and the controller is

$$u(1) = -q\int_0^1 \mathrm{e}^{q(1-y)}u(y)\,dy$$

4.4 The solution of the target system is

$$w(x,t) = 2\sum_{n=0}^{\infty} \mathrm{e}^{-\sigma_n^2 t}\cos(\sigma_n x)\int_0^1 \cos(\sigma_n x)\,w_0(x)dx\,,$$

where $\sigma_n = \pi/2 + \pi n$. First we express $w_0(x)$ in terms of $u_0(x)$:

$$\begin{aligned} w_0(x) &= u_0(x) - \int_0^x k(x,y)u(y)\,dy \\ &= u_0(x) + q\int_0^x \mathrm{e}^{q(x-y)}u_0(y)\,dy \end{aligned}$$

We get

$$\begin{aligned}\int_0^1 \cos(\sigma_n x)\, w_0(x)dx &= \int_0^1 \cos(\sigma_n x)\, u_0(x)dx \\ &\quad + q\int_0^1 \cos(\sigma_n x)\int_0^x \mathrm{e}^{q(x-y)} u_0(y)\, dy\, dx \\ &= \int_0^1 u_0(x)\left(\cos(\sigma_n x) + q\int_x^1 \mathrm{e}^{q(y-x)}\cos(\sigma_n y)\, dy\right) dx \\ &= \int_0^1 f(x)u_0(x)\, dx\,,\end{aligned}$$

where

$$f(x) = \frac{\sigma_n^2\cos(\sigma_n x) + q\sigma_n\left[(-1)^n \mathrm{e}^{q(1-x)} - \sin(\sigma_n x)\right]}{\sigma_n^2 + q^2}\,.$$

We now use the inverse transformation to derive the solution for u:

$$\begin{aligned}u(x,t) &= w(x,t) - q\int_0^x w(y,t)\, dy \\ &= 2\sum_{n=0}^{\infty} \mathrm{e}^{-\sigma_n^2 t}\left(\cos(\sigma_n x) - q\int_0^x \cos(\sigma_n y)\ dy\right)\int_0^1 f(x)u_0(x)\, dx \\ &= 2\sum_{n=0}^{\infty} \mathrm{e}^{-\sigma_n^2 t}\left(\cos(\sigma_n x) - \frac{q}{\sigma_n}\sin(\sigma_n x)\right)\int_0^1 f(x)u_0(x)\, dx \\ &= 2\sum_{n=0}^{\infty} e^{-\sigma_n^2 t}\left(\sigma_n\cos(\sigma_n x) - q\sin(\sigma_n x)\right) \\ &\qquad \times \int_0^1 \frac{\sigma_n\cos(\sigma_n \xi) - q\sin(\sigma_n x) + (-1)^n q e^{q(1-x)}}{\sigma_n^2 + q^2} u_0(x)\, dx\,.\end{aligned}$$

4.5 Let $v(x) = u(x)\mathrm{e}^{\frac{b}{2}x}$, then we have

$$\begin{aligned}v_t &= v_{xx} + \left(\lambda - \frac{b^2}{4}\right)v \\ v_x(0) &= 0\end{aligned}$$

Denote $\lambda_0 = \lambda - b^2/4$ and let us choose the target system

$$\begin{aligned}w_t &= w_{xx} \\ w_x(0) &= 0 \\ w_x(1) &= -\frac{1}{2}w(1)\,.\end{aligned}$$

Then the controller is given by (see Exercise 4.1)

$$\begin{aligned} v_x(1) = &-\frac{\lambda_0+1}{2}v(1) \\ &-\lambda_0\int_0^1\left[\frac{3I_1\left(\sqrt{\lambda_0(1-y^2)}\right)}{2\sqrt{\lambda_0(1-y^2)}}+\frac{I_2\left(\sqrt{\lambda_0(1-y^2)}\right)}{1-y^2}\right]v(y)\,dy\,. \end{aligned}$$

Finally, we get back to the original variables:

$$\begin{aligned} u_x(1) &= \left(v_x(1)-\frac{b}{2}v(1)\right)\mathrm{e}^{-\frac{b}{2}} \\ &= -\frac{\lambda_0+b+1}{2}u(1) \\ &\quad -\lambda_0\int_0^1\left[\frac{3I_1\left(\sqrt{\lambda_0(1-y^2)}\right)}{2\sqrt{\lambda_0(1-y^2)}}+\frac{I_2\left(\sqrt{\lambda_0(1-y^2)}\right)}{1-y^2}\right]\mathrm{e}^{\frac{b}{2}(y-1)}u(y)\,dy\,. \end{aligned}$$

4.6 Control gain PDE has the solution $k(x,y) = \phi(x-y)$, where the Laplace transform of ϕ is

$$\Phi(s) = \frac{G(s)}{G(s)-s}$$

The Laplace transform of $g(x) = 3\mathrm{e}^{2x}$ is $G(s) = 3/(s-2)$. Therefore,

$$\begin{aligned} \Phi(s) &= \frac{\frac{3}{s-2}}{\frac{3}{s-2}-s} = \frac{-3}{s^2-2s-3} \\ &= \frac{3}{4}\left(\frac{1}{s+1}-\frac{1}{s-3}\right)\,. \end{aligned}$$

Taking the inverse Laplace transform, we get

$$k(x,y) = \phi(x-y) = \frac{3}{4}\left(\mathrm{e}^{-(x-y)}-\mathrm{e}^{3(x-y)}\right)$$

and the controller is

$$u(1) = \frac{3}{4}\int_0^1\left(\mathrm{e}^{-(1-y)}-\mathrm{e}^{3(1-y)}\right)u(y)\,dy\,.$$