Chapter 3

3.1 Let \( u(x,t) = X(x)T(t) \), then

\[
\frac{\dot{T}(t)}{T(t)} = \frac{X''(x) + \lambda X(x)}{X(x)} = \sigma,
\]

so that \( T(t) = T(0)e^{\sigma t} \) and \( X(x) \) satisfies the following ODE:

\[
X''(x) + (\lambda - \sigma)X(x) = 0
\]

with boundary conditions \( X'(0) = 0 \) and \( X(1) = 0 \). General solution of this ODE is

\[
X(x) = A \sin \sqrt{\lambda - \sigma} x + B \cos \sqrt{\lambda - \sigma} x.
\]

Boundary conditions give \( A = 0 \) and

\[\cos \sqrt{\lambda - \sigma} = 0 \Rightarrow \lambda - \sigma = \left(\frac{\pi}{2} + \pi n\right)^2, \quad n = 0, 1, 2, \ldots\]

Therefore we have \((C_n = T(0)B)\)

\[
u(x,t) = \sum_{n=0}^{\infty} C_n e^{\left[\lambda - (\frac{\pi}{2} + \pi n)^2\right] t} \cos \left[\left(\frac{\pi}{2} + \pi n\right) x \right]
\]

The constants \( C_n \) are determined from the initial condition:

\[
u_0(x) \cos \left[\left(\frac{\pi}{2} + \pi m\right) x \right] = \sum_{n=0}^{\infty} C_n \cos \left[\left(\frac{\pi}{2} + \pi n\right) x \right] \cos \left[\left(\frac{\pi}{2} + \pi m\right) x \right]
\]

\[\int_0^1 \nu_0(x) \cos \left[\left(\frac{\pi}{2} + \pi m\right) x \right] dx = \frac{1}{2} C_m.
\]

Finally,

\[
u(x,t) = 2 \sum_{n=0}^{\infty} e^{\left[\lambda - (\frac{\pi}{2} + \pi n)^2\right] t} \cos \left[\left(\frac{\pi}{2} + \pi n\right) x \right] \int_0^1 \nu_0(x) \cos \left[\left(\frac{\pi}{2} + \pi n\right) x \right] dx
\]

The PDE is unstable when

\[\lambda - \left(\frac{\pi}{2} + \pi n\right)^2 > 0
\]

for at least one \( n \), which gives \( \lambda > \pi^2/4 \).

3.2 Let \( u(x,t) = X(x)T(t) \), then

\[
\frac{\dot{T}(t)}{T(t)} = \frac{X''(x)}{X(x)} = \sigma,
\]
so that \( T(t) = T(0)e^{\sigma t} \) and \( X(x) \) satisfies the following ODE:

\[
X''(x) - \sigma X(x) = 0
\]

with boundary conditions \( X'(0) = -qX(0) \) and \( X(1) = 0 \). General solution of this ODE is

\[
X(x) = A \sinh \sqrt{\sigma}x + B \cosh \sqrt{\sigma}x.
\]

From the boundary condition at \( x = 0 \) we get \( \sqrt{\sigma}A = -qB \). Then \( X(1) = 0 \) gives

\[
-q \sinh \sqrt{\sigma} + \sqrt{\sigma} \cosh \sqrt{\sigma} = 0.
\]

This equation cannot be solved in closed form, but we still can find the critical value of \( q \) by letting \( \sigma \to 0 \):

\[
q_0 = \lim_{\sigma \to 0} \frac{\sqrt{\sigma} \cosh \sqrt{\sigma}}{\sinh \sqrt{\sigma}} = 1
\]

Therefore the plant is unstable for \( q > 1 \) (since for \( q = 0 \) it is stable).