Chapter 3

3.1 Let $u(x,t) = X(x)T(t)$, then

$$\frac{\dot{T}(t)}{T(t)} = \frac{X''(x) + \lambda X(x)}{X(x)} = \sigma\,,$$

so that $T(t) = T(0)\mathrm{e}^{\sigma t}$ and $X(x)$ satisfies the following ODE:

$$X''(x) + (\lambda - \sigma)X(x) = 0$$

with boundary conditions $X'(0) = 0$ and $X(1) = 0$. General solution of this ODE is

$$X(x) = A\sin\sqrt{\lambda - \sigma}x + B\cos\sqrt{\lambda - \sigma}x\,.$$

Boundary conditions give $A = 0$ and

$$\cos\sqrt{\lambda - \sigma} = 0 \quad \Rightarrow \quad \lambda - \sigma = \left(\frac{\pi}{2} + \pi n\right)^2,\ n = 0, 1, 2, \ldots$$

Therefore we have ($C_n = T(0)B$)

$$u(x,t) = \sum_{n=0}^{\infty} C_n \mathrm{e}^{\left[\lambda - \left(\frac{\pi}{2} + \pi n\right)^2\right]t} \cos\left[\left(\frac{\pi}{2} + \pi n\right)x\right]$$

The constants C_n are determined from the initial condition:

$$u_0(x)\cos\left[\left(\frac{\pi}{2} + \pi m\right)x\right] = \sum_{n=0}^{\infty} C_n \cos\left[\left(\frac{\pi}{2} + \pi n\right)x\right]\cos\left[\left(\frac{\pi}{2} + \pi m\right)x\right]$$

$$\int_0^1 u_0(x)\cos\left[\left(\frac{\pi}{2} + \pi m\right)x\right]\,dx = \frac{1}{2}C_m\,.$$

Finally,

$$u(x,t) = 2\sum_{n=0}^{\infty} \mathrm{e}^{\left[\lambda - \left(\frac{\pi}{2} + \pi n\right)^2\right]t} \cos\left[\left(\frac{\pi}{2} + \pi n\right)x\right]\int_0^1 u_0(x)\cos\left[\left(\frac{\pi}{2} + \pi n\right)x\right]\,dx$$

The PDE is unstable when

$$\lambda - \left(\frac{\pi}{2} + \pi n\right)^2 > 0$$

for at least one n, which gives $\lambda > \pi^2/4$.

3.2 Let $u(x,t) = X(x)T(t)$, then

$$\frac{\dot{T}(t)}{T(t)} = \frac{X''(x)}{X(x)} = \sigma\,,$$

so that $T(t) = T(0)e^{\sigma t}$ and $X(x)$ satisfies the following ODE:

$$X''(x) - \sigma X(x) = 0$$

with boundary conditions $X'(0) = -qX(0)$ and $X(1) = 0$. General solution of this ODE is

$$X(x) = A \sinh \sqrt{\sigma} x + B \cosh \sqrt{\sigma} x \,.$$

From the boundary condition at $x = 0$ we get $\sqrt{\sigma} A = -qB$. Then $X(1) = 0$ gives

$$-q \sinh \sqrt{\sigma} + \sqrt{\sigma} \cosh \sqrt{\sigma} = 0 \,.$$

This equation cannot be solved in closed form, but we still can find the critical value of q by letting $\sigma \to 0$:

$$\begin{aligned} q &= \frac{\sqrt{\sigma} \cosh \sqrt{\sigma}}{\sinh \sqrt{\sigma}} \\ q_0 &= \lim_{\sigma \to 0} \frac{\sqrt{\sigma} \cosh \sqrt{\sigma}}{\sinh \sqrt{\sigma}} = \lim_{\sigma \to 0} \frac{\sqrt{\sigma} \cosh \sqrt{\sigma}}{\sqrt{\sigma}} = 1 \end{aligned}$$

Therefore the plant is unstable for $q > 1$ (since for $q = 0$ it is stable).