Chapter 2

2.1 Set $z = 1 - x$ in the proofs.

2.2

$$V(t) = \frac{1}{2}\int_0^1 w(x,t)^2\,dx\,.$$

$$\begin{aligned}
\dot{V} &= \int_0^1 w(x)w_t(x)\,dx \\
&= \int_0^1 w(x)w_{xx}(x)\,dx \\
&= w(x)w_x(x)|_0^1 - \int_0^1 w_x(x)^2\,dx \\
&= -\frac{1}{2}w(1)^2 - \int_0^1 w_x(x)^2\,dx \\
&\le -\frac{1}{2}V \quad \text{(by Poincare inequality)}
\end{aligned}$$

Therefore $V(t) \le V(0)e^{-t/2}$ or

$$\|w(t)\| \le e^{-\frac{t}{4}}\|w_0\|\,.$$

2.3

$$V(t) = \frac{1}{2}\int_0^1 w(x,t)^2\,dx\,.$$

$$\begin{aligned}
\dot{V} &= \int_0^1 w(x)w_t(x)\,dx \\
&= \int_0^1 w(x)w_{xx}(x)\,dx - \int_0^1 w(x)^2 w_x(x)\,dx \\
&= -\frac{1}{6}(w(1)^2 + w(1)^4) - \int_0^1 w_x(x)^2\,dx - \int_0^1 w(x)^2 w_x(x)\,dx \\
&= -\frac{1}{6}(w(1)^2 + w(1)^4 + 2w(1)^3) - \int_0^1 w_x(x)^2\,dx \quad \text{(since } (w^3)' = 3w^2 w_x) \\
&= -\frac{1}{6}(w(1) + w(1)^2)^2 - \int_0^1 w_x(x)^2\,dx \quad \text{(completing the square)} \\
&\le -\frac{1}{2}V \quad \text{(by Poincare inequality)}
\end{aligned}$$

Therefore $V(t) \le V(0)e^{-t/2}$ or

$$\|w(t)\| \le e^{-\frac{t}{4}}\|w_0\|\,.$$