Backstepping Boundary Controller and Observer Designs for the Slender Timoshenko Beam

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Abstract— **In this paper we present the first extension of the backstepping methods developed for control of parabolic PDEs (modeling thermal, fluid, and chemical reaction dynamics, including Navier-Stokes equations and turbulence) to second-order PDE systems (often referred loosely as hyperbolic) which model flexible structures and acoustic.**

We introduce controller and observer designs capable of adding damping to a model of beam dynamics using actuation only at the beam base and using sensing only at the beam tip. Interestingly, the backstepping method does not apply to the simplest Euler-Bernoulli model but does apply to more realistic models, including the Timoshenko beam model under the assumption that the beam is "slender." For our method to be applicable it is necessary that the beam model includes a small amount of Kelvin-Voigt damping. Such damping models internal material friction (rather than viscous interaction with the environment) and is present in every realistic material. We don't use the KV damping as a source of dissipation but as a means of controllability of the beam. With only a small amount of KV damping present in the uncontrolled system, we are able to introduce a substantial amount of damping of classical type (velocity-based). The closed-loop system with our boundary feedback included can be transformed into a form where both the added damping and an addition of "stiffness" are evident. As we show, this simultaneous change in damping and stiffness results in an overall shift of the eigenvalues to the left in the complex plane and in the improvement of the damping ratio of all the eigenvalues.

To ease the reader into main concepts, we first present the same method for a wave equation with a small amount of KV damping and then pursue the development for a shear beam and Timoshenko beam model. The same result can be developed for the Rayleigh beam model which is structurally the same as the shear beam model but with different parameters.

THE FINAL VERSION OF THE PAPER WILL INCLUDE SIMULATION RESULTS.

I. Introduction

Flexible beams constitute an important benchmark problem in many application areas ranging from aerospace to civil structures. In some of the exciting modern fields like atomic force microscopy the cantilever beam is more than just a prototype problem and constitutes an important application topic in its own right. Extensive literature exists on control of beam models. In this paper we concentrate on the most realistic of the 1D distributed parameter models, the Timoshenko model, and focus only on the prior literature on control of this model.

Probably the first result on control of the Timoshenko beam is due to Kim and Renardy [6] who proved stabilization under a classical "boundary damper" feedback which relates spatial and temporal derivatives at the beam tip. Morgul [10] proposed a more advanced dynamic feedback design which eliminates the need to measure the tip velocity but retains the requirement to actuate at the tip. Zhang, Dawson, de Queiroz, and Vedagarbha [21], [4] consider a Timoshenko beam with mass/inertial dynamics at the free end and design a Lyapunov-based adaptive version of the boundary damping feedback in [6], which they also demonstrate experimentally. Shi, Hou, and Feng [11] also consider the Timoshenko beam with mass at the tip and prove uniform stabilization with boundary damping feedback laws applied at the tip and applied at both the tip and the base at the same time. Taylor and Yau [17] establish controllability properties of a beam with spatially varying parameters using force actuation at the tip and torque at the base. Macchelli and Melchiorri consider the Timoshenko beam in the framework of distributed port Hamiltonian systems, unify several existing approaches, and develop a new controller based on energy-shaping/Casimir function concepts, with actuation at the tip.

All of the previous approaches, which rely on collocated actuation and sensing at the tip, exploit elegantly the passivity property between the corresponding input and output in the beam model. Such feedbacks can be implemented via passive dampers or as active controllers through more elaborate electromagnetic means of actuation at the tip. Our objective is different—to design controllers implementable through anti-collocated architecture, with actuation only at the base and sensing only at the tip.

For parabolic PDEs such problem formulations have recently led to backstepping controllers [12], [14] and observers [13] which result in closed-form formulae for the controller/observer gains, which are explicit both in the spatial coordinates and in the physical parameters. The explicit parametrization has allowed the development of the first adaptive boundary controllers for unstable PDEs [7], [8], [15], [16]. The primary applications of the backstepping methodology so far have been turbulent fluid systems [1], [2], [20], [18], [19].

In this paper we venture for the first time into the realm of vibrating systems modeled by second order "hyperbolic"[1] PDE systems. We introduce controller and observer designs capable of adding damping to Timoshenko and shear beam models using actuation only at the beam base and using sensing only at the beam tip. Interestingly, the backstepping method does not apply to the simplest Euler-Bernoulli model but does apply to the more realistic models that we consider here, under the assumption that the beam is "slender." For our method to be applicable it is necessary that the beam model includes a small amount of Kelvin-Voigt damping. Such damping models internal material friction (rather than viscous interaction with the environment) and is present in every realistic material. We don't use the KV damping as a source of dissipation but as a means of controllability of the beam. With only a small amount of KV damping present in the uncontrolled system, we are able to introduce a substantial amount of damping of classical type (velocity-based). The closed-loop system with our boundary feedback included can be transformed into a form where

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[1]The word is meant in a loose sense because in the presence of Kelvin-Voigt damping these system share many properties of parabolic PDE systems, though their main characteristic—oscillations with poor damping—remains.

both the added damping and an addition of "stiffness" are evident. As we show, this simultaneous change in damping and stiffness results in an overall shift of the eigenvalues to the left in the complex plane and in the improvement of the damping ratio of all the eigenvalues.

To ease the reader into main concepts, we first present the same method for a wave equation with a small amount of KV damping and then pursue the development for a shear beam and Timoshenko beam model. The same result can be developed for the Rayleigh beam model which is structurally the same as the shear beam model but with different parameters.

a) Notation: The spatial $L_2(0,1)$ norm is denoted by $\|\cdot\|$. The symbols $I_1(\cdot), I_2(\cdot), J_1(\cdot)$, etc., denote the corresponding Bessel functions.

II. An Introductory Example: Wave Equation

To motivate our developments for the Timoshenko beam model, we first present our ideas through an example of boundary controller and observer design for a wave equation. While the Timoshenko beam model is sometimes viewed as a PDE with fourth order derivatives in time and space, it is actually best approached as a set of two coupled wave equations with a very specific form of coupling. Thus our introductory wave equation example will be helpful in preparing the terrain for the further presentation for the Timoshenko model.

We are going to consider a wave equation on an interval $x \in [0,1]$ given by

$$\epsilon u_{tt} = (1 + d\partial_t)\, u_{xx} \tag{1}$$
$$u_x(0) = 0 \tag{2}$$
$$u(0) = \text{measured} \tag{3}$$
$$u(1) = \text{controlled}\,, \tag{4}$$

where ϵ and d are positive constants. The value $1/\epsilon$ represents "stiffness" of the string. We do not restrict ϵ to be either small or large. The equation (2) represents the boundary condition at the end $x = 0$, with the dependence on time suppressed to reduce notational burden. The model (1)–(4) might model the dynamics of a string controlled (or clamped) at the end $x = 1$ and with boundary measurements applied at the free end $x = 0$. Our orientation of the x-axis might seem confusing to some readers. Typically the clamped end would be at $x = 0$ whereas the free end would be at $x = 1$. In contrast, our free end is at $x = 0$ and the actuator is at $x = 1$, for consistency with out previous papers on backstepping boundary controllers and observers for PDEs [12], [13].

The operation ∂_t represents partial differentiation with respect to time. The term $d\partial_t$ models the "Kelvin-Voigt" damping which represents the internal material damping (not the damping that arises due to viscous interaction of the string with the surrounding medium). We do not assume that the coefficient d of the Kelvin-Voigt damping is large. We only assume that a small amount of d, occurring in any realistic material, is present in the model. This small amount of Kelvin-Voigt damping will allow us to much more substantially dampen the eigenvalues of the system through the backstepping design. In fact, we do not rely on the Kelvin-Voigt term as "damping," i.e., as a source of energy dissipation. We use it as a means of controllability of the wave equation system, in the same way we use the diffusion operator in the backstepping design for parabolic systems [12].

A. The target system

At the center of our design is the observation that with the backstepping approach we can construct a state transformation and boundary feedback such that, in the transformed variable, the closed-loop system assumes the form

$$\epsilon w_{tt} = (1 + d\partial_t)\,(w_{xx} - cw) \tag{5}$$
$$w_x(0) = 0 \tag{6}$$
$$w(1) = 0\,, \tag{7}$$

where $c \geq 0$ is a design gain. We shall show later how we design such a transformation and a boundary controller but we state first a result that shows the benefits of introducing the term $-cw$ in the w-system.

Proposition 2.1: All the eigenvalues of the system (5)–(7) are in the open left-half-plane, have the damping ratios of at least

$$\frac{\pi d}{4\sqrt{\epsilon}}\sqrt{1 + \frac{4}{\pi^2}c} \tag{8}$$

and all of their real parts are no larger than

$$-\min\left\{\frac{1}{d}, \frac{\pi^2 d}{8\epsilon}\left(1 + \frac{4}{\pi^2}c\right)\right\}\,. \tag{9}$$

At most

$$\frac{4\sqrt{\epsilon}}{\pi d}\sqrt{1 - \frac{d^2}{4\epsilon}c} - 1 \tag{10}$$

of the eigenvalues are complex, whereas the rest are real.[2]

It is clear from this result that, when d is small, the damping ratio of all the eigevalues can be greatly improved and they can be all moved to the left in the complex plane by increasing c. This effect disappears in case of the complete absence of Kelvin-Voigt damping, but it is always present to some extent when $d > 0$. The above proposition is proved by first showing the nth pair of eigenvalues σ_n satisfies the quadratic equation

$$\epsilon\sigma_n^2 + d\left[c + \left(\frac{\pi}{2} + n\pi\right)^2\right]\sigma_n + \left[c + \left(\frac{\pi}{2} + n\pi\right)^2\right] = 0\,, \tag{11}$$

where $n = 0, 1, 2, \ldots$ From this it can be seen that there are two sets of eigenvalues. For lower n's the eignevalues reside on the circle

$$\left(\Re\{\sigma_n\} + \frac{1}{d}\right)^2 + (\Im\{\sigma_n\})^2 = \frac{1}{d^2}\,. \tag{12}$$

For higher n's the eigenvalues are real, with one branch accumulating towards $-1/d$ as $n \to \infty$ and the other branch converging towards $-\infty$ on the real axis. Clearly the least damped of the eigenvalues is the one for $n = 0$, which yields (8). Its negative real part is the second argument in the min function in (9). For small-to-moderate values of c the $n = 0$ eigenvalue will be to the right of the accumulation point $\sigma = -1/d$. However, for large c's, the rightmost eigenvalue might be σ_∞, which corresponds to the first argument in the min function in (9). The expression (10) is obtained by analyzing the discriminant of the quadratic equation (11).

B. Controller design for wave equation

For the model (1)–(4) we introduce an *invertible* spatially-causal/lower-triangular/Volterra state transformation

$$w(x) = u(x) - \int_0^x k(x,y)u(y)\,dy\,, \tag{13}$$

[2]The expression (10) should be understood in the sense of the nearest even integer above the real number given in the expression. It should be also clear that the number (10) is meaningful (real) only when $c \geq 4\epsilon/d^2$. For very high c all the eigenvalues become real and reside on the semi-infinite interval on the real axis $(-\infty, -1/d)$.

and the boundary feedback law

$$u(1)=\int_0^1 k(1,y)u(y)\,dy\,. \qquad (14)$$

It can be shown that (13) and (14) convert the system (1)–(2) into (5)–(7) provided the kernel/gain function $k(x,y)$ satisfies the hyperbolic PDE

$$\begin{aligned} k_{xx} &= k_{yy}+ck && (15)\\ k_y(x,0) &= 0 && (16)\\ k(x,x) &= -\frac{c}{2}x && (17)\end{aligned}$$

on the triangular domain $\{0\le y\le x\le 1\}$. An explicit solution to this PDE was derived in [12]:

$$k(x,y)=-cx\frac{I_1\left(\sqrt{c\,(x^2-y^2)}\right)}{\sqrt{c\,(x^2-y^2)}}\,, \qquad (18)$$

where I_1 is the modified Bessel function of the first kind/first order.

We conclude that the controller defined by (14), (18) turns the lightly damped wave equation (1)–(2) into the well damped target system (5)–(7).

Theorem 1: The controller (14), (18) increases the damping ratio of the nth conjugate complex eigenvalue pair of the wave equation (1)–(2) by a factor of $\sqrt{1+\frac{4}{\pi^2(1+2n^2)}c}$ and moves it leftward in the complex plane by a factor of $1+\frac{4}{\pi^2(1+2n^2)}c$.

From this result it follows that, in the presence of a small amount of Kelvin-Voigt (interior material) damping, we can impart a substantial amount of external-like ("viscous") damping. This is accompanied by an increase in stiffness, which is evident by writing the system (5) in the expanded form

$$\epsilon w_{tt}+d(c-\partial_{xx})w_t+(c-\partial_{xx})w=0 \qquad (19)$$

and viewing $(c-\partial_{xx})/\epsilon$ as the system's stiffness operator. Whether the increase in stiffness (i.e., the increase in the natural frequency) is desirable or not, the overall effect is always the improvement in the damping ratio and the leftward movement of the complex eigenvalues, along with the conversion of some of the complex eigenvalues into real eigenvalues.

With a very high value of c we could make all of the eigenvalues real. While possible, this would not necessarily be a good idea, neither for servo response, nor for disturbance attenuation, and certainly not from the point of view of control effort. Making all of the infinitely many eigenvalues real might make the system response sluggish. Moreover, a very high value of c would increase the density of eigenvalues near $-\frac{1}{d}+i0$, which might result in poor transient response if the eigenvectors become nearly parallel. Thus, the flexibility to improve the damping using the backstepping transformation and controller should be used judiciously, with lower values of c if d is already relatively high.

C. Observer design for wave equation

Based on the duality between the backstepping controller and observer designs introduced in [13], we propose an observer of the form

$$\begin{aligned} \epsilon\hat{u}_{tt} &= (1+d\partial_t)\left[\hat{u}_{xx}+\frac{\tilde{c}(1-x)}{x(2-x)}I_2\left(\sqrt{\tilde{c}x(2-x)}\right)\right.\\ &\qquad \left.\times\left(u(0)-\hat{u}(0)\right)\right] && (20)\\ \hat{u}_x(0) &= -\frac{\tilde{c}}{2}\left(u(0)-\hat{u}(0)\right) && (21)\\ \hat{u}(1) &= u(1)\,, && (22)\end{aligned}$$

where $\tilde{c}$ is a nonnegative design parameter. The function $\frac{\tilde{c}(1-x)}{x(2-x)}I_2\left(\sqrt{\tilde{c}x(2-x)}\right)$ in the PDE (20) and the constant $-\frac{\tilde{c}}{2}$ in the boundary condition (21) are the observer gains which multiply the output estimation error $u(0)-\hat{u}(0)$, which is the error between the measured boundary variable $u(0)$ and its estimate $\hat{u}(0)$. The purpose of the injection of the output estimation error and the associate observer gains is to impart a desired level of damping upon the dynamics of the observer error $u(x,t)-\hat{u}(x,t)$. That this is achieved can be shown using the *invertible* transformation $u-\hat{u}\mapsto\tilde{w}$ defined by

$$u(x)-\hat{u}(x)=\tilde{w}(x)-\int_0^x p(x,y)\tilde{w}(y)dy \qquad (23)$$

and

$$p(x,y)=-\tilde{c}(1-x)\frac{I_1\left(\sqrt{\tilde{c}\left((1-y)^2-(1-x)^2\right)}\right)}{\sqrt{\tilde{c}\left((1-y)^2-(1-x)^2\right)}}, \qquad (24)$$

which transforms the difference $u-\hat{u}$ between the equations (1)–(2) and (20)–(22) into the well damped equation

$$\begin{aligned} \epsilon\tilde{w}_{tt} &= (1+d\partial_t)\left(\tilde{w}_{xx}-\tilde{c}\tilde{w}\right) && (25)\\ \tilde{w}_x(0) &= 0 && (26)\\ \tilde{w}(1) &= 0\,. && (27)\end{aligned}$$

Similar conclusions regarding damping follow as in Sections II-A and II-B. Even if no control is applied to the system (1)–(2), which is very lightly damped through only a small amount of Kelvin-Voigt damping d, good convergence of $\hat{u}(x,t)$ to $u(x,t)$ is ensured due to the good damping of the observer error system (25)–(27).

Moreover, applying the certainty equivalence version of the controller (14), (18), i.e., using

$$\hat{u}(1)=u(1)=-\int_0^1 c\frac{I_1\left(\sqrt{c\,(1-y^2)}\right)}{\sqrt{c\,(1-y^2)}}\hat{u}(y)dy \qquad (28)$$

in place of (22), we obtain an output feedback compensator consisting of the observer (20)–(22) and the controller (28). The input to this compensator is the measured variable $u(0,t)$ and the output is the actuated variable $u(1,t)$. With the two transformations (13) and (23) the closed-loop system can be written as a cascade connection of the $\tilde{w}$-system whose damping is improved with $\tilde{c}$ and the w-system whose damping is improved with c. The eigenvalues of the closed-loop system (1)–(2), (20)–(21), (28) are a union of eigenvalues obtained as the solutions of the quadratic equation (11) and the same equation with c replaced by $\tilde{c}$ (this is so because of the block-triangular structure of the problem). As a final comment, we indicate that a reasonable observer-based design would tune the observer response to be a little faster than the state feedback response, in our case $\tilde{c}>c$, which we assume in the following result.

Theorem 2: All the eigenvalues of the closed-loop system (1)–(2), (20)–(21), (28) are in the open left-half-plane, have the damping ratios of at least $\frac{\pi d}{4\sqrt{\epsilon}}\sqrt{1+\frac{4}{\pi^2}c}$ and all of their real parts are no larger than $-\min\left\{\frac{1}{d},\frac{\pi^2 d}{8\epsilon}\left(1+\frac{4}{\pi^2}c\right)\right\}$.

III. Timoshenko and Shear Beam Models

We consider a beam modeled by the two coupled wave equations given by

$$\begin{aligned} \epsilon u_{tt} &= (1+d\partial_t)\left(u_{xx}-\alpha_x\right) && (29)\\ \mu\epsilon\alpha_{tt} &= (1+d\partial_t)\left(\epsilon\alpha_{xx}+a\left(u_x-\alpha\right)\right), && (30)\end{aligned}$$

where $u(x,t)$ denotes the displacement, $\alpha(x,t)$ denotes the angle of rotation, and the positive constants ϵ,μ,d,a denote the appropriate physical parameters defined in [5], [22]. We consider a beam which is free at the end $x=0$, i.e.,

$$\begin{aligned} u_x(0) &= \alpha(0) && (31)\\ \alpha_x(0) &= 0 && (32)\end{aligned}$$

and which is controlled at the end $x=1$ through the boundary conditions $u(1,t)$ and $\alpha(1,t)$.

For the case of μ small we get the so called "slender beam." When μ is set to zero, the fourth-order-in-space/fourth-order-in-time Timoshenko equations (29)–(30) reduce to the "shear beam" model:

$$\begin{aligned} \epsilon u_{tt} &= (1+d\partial_t)\left(u_{xx}-\alpha_x^{ss}\right) && (33)\\ 0 &= \alpha_{xx}^{ss}-b^2\alpha^{ss}+b^2u_x, && (34)\end{aligned}$$

where

$$b=\sqrt{\frac{a}{\epsilon}}. \qquad (35)$$

Let $\alpha^{ss}[u](x)$ denote the solution of the second order two point boundary value ODE problem (34) with boundary conditions for $\alpha^{ss}(x)$ given by $\alpha_x^{ss}(0)=0$ and $\alpha^{ss}(1)=\alpha(1)$, the latter of which we use for control.

Setting $\mu=0$ constitutes a singular perturbation. We will pursue the singular perturbation approach in our design, developing controllers and observers for $\mu=0$, which are also valid for small positive μ's.

Since $\alpha^{ss}(x)$ in (34) is not the actual rotation angle $\alpha(x)$ in the full Timoshenko model (29), (30) but only its "quasi-steady state" singularly perturbed approximation, we introduce the error variable

$$\bar\alpha(x)=\alpha(x)-\alpha^{ss}(x). \qquad (36)$$

With the aid of this variable we can write both the Timoshenko and the shear beam model in a compact way as

$$\begin{aligned} \epsilon u_{tt} &= (1+d\partial_t)\left(u_{xx}-\partial_x\alpha^{ss}[u]-\bar\alpha_x\right) && (37)\\ 0 &= \alpha_{xx}^{ss}-b^2\alpha^{ss}+b^2u_x, && (38)\end{aligned}$$

where $\bar\alpha$ is given by

$$\bar\alpha\equiv 0 \qquad (39)$$

for the shear beam case and

$$\begin{aligned} \mu\bar\alpha_{tt} &= (1+d\partial_t)\left(\bar\alpha_{xx}-b^2\bar\alpha\right)-\mu\alpha^{ss}[u_{tt}] && (40)\\ \bar\alpha_x(0) &= 0 && (41)\\ \bar\alpha(1) &= 0 && (42)\end{aligned}$$

for the Timoshenko case.

Since the TPBV problem (34) with boundary conditions $\alpha_x^{ss}(0)=0,\alpha^{ss}(1)=\alpha(1)$ is linear and second order in x it is easy to solve in various ways (for example, by Laplace transform in x), yielding

$$\alpha^{ss}(x)=\cosh(bx)\alpha^{ss}(0)-b\int_0^x\sinh(b(x-y))u_y(y)dy. \qquad (43)$$

With this definition, and applying integration by parts to the integral in (43), we get the following

$$\begin{aligned} \alpha(x) &= \bar\alpha(x)+\cosh(bx)\left(\alpha(0)-\bar\alpha(0)\right)+b\sinh(bx)u(0)\\ &\quad -b^2\int_0^x\cosh(b(x-y))u(y)dy. && (44)\end{aligned}$$

By setting $x=1$ in this equation we can find an expression for $\alpha(0)-\bar\alpha(0)$ given in terms of $\alpha(1)$,

$$\begin{aligned} \alpha(0) &= \bar\alpha(0)+\frac{1}{\cosh(b)}\left[\alpha(1)-b\sinh(b)u(0)\right.\\ &\quad \left.+b^2\int_0^1\cosh(b(1-y))u(y)dy\right], && (45)\end{aligned}$$

providing us with an alternative form of the solution (44):

$$\begin{aligned} \alpha(x) &= \bar\alpha(x)+\frac{\cosh(bx)}{\cosh(b)}\left[\alpha(1)-b\sinh(b)u(0)\right.\\ &\quad \left.+b^2\int_0^1\cosh(b(1-y))u(y)dy\right]+b\sinh(bx)u(0)\\ &\quad -b^2\int_0^x\cosh(b(x-y))u(y)dy. && (46)\end{aligned}$$

Differentiating (44) and (46) with respect to x and substituting them into (29), (31) we get

$$\begin{aligned} \epsilon u_{tt} &= (1+d\partial_t)\left\{u_{xx}+b^2u\right.\\ &\quad +b^3\int_0^x\sinh(b(x-y))u(y)dy\\ &\quad -b^2\cosh(bx)u(0)\\ &\quad -\frac{b\sinh(bx)}{\cosh(b)}\left[\alpha(1)-b\sinh(b)u(0)\right.\\ &\quad \left.+b^2\int_0^1\cosh(b(1-y))u(y)dy\right]\\ &\quad \left.-\bar\alpha(x)\right\} && (47)\\ u_x(0) &= \bar\alpha(0)+\frac{1}{\cosh(b)}\left[\alpha(1)-b\sinh(b)u(0)\right.\\ &\quad \left.+b^2\int_0^1\cosh(b(1-y))u(y)dy\right] && (48)\end{aligned}$$

and

$$\begin{aligned} \epsilon u_{tt} &= (1+d\partial_t)\left\{u_{xx}+b^2u\right.\\ &\quad +b^3\int_0^x\sinh(b(x-y))u(y)dy\\ &\quad -b^2\cosh(bx)u(0)\\ &\quad -b\sinh(bx)\alpha(0)\\ &\quad \left.-\bar\alpha(x)+b\sinh(bx)\bar\alpha(0)\right\} && (49)\\ u_x(0) &= \alpha(0). && (50)\end{aligned}$$

For $\bar\alpha(x)\equiv 0$ we get the singularly perturbed reduced models (the shear beam models), however, with the $\bar\alpha(x)$ and $\bar\alpha(0)$ present, these models are exact and they are the u-components of a $(u,\bar\alpha)$-model that is equivalent to the original Timoshenko

(u,α)-model.[3] The model (47), (48) will be used in the control design [note the presence of control input $\alpha(1)$], whereas the model (49), (50) will be used in the observer design [note the presence of the measured output $\alpha(0)$].

IV. Control Design

We start by assuming that $\bar{\alpha}(x) \equiv 0$ in (47), (48) and seek a kernel function $k(x,y)$ in the change of variable (13) and in the boundary feedback law (14) to convert (47), (48) into (5)–(7). Unfortunately, because of the term $\int_0^1 \cosh(b(1-y))u(y)dy$, which is an integral over the entire interval $[0,1]$, the model (47) is not in the strict-feedback form and therefore backstepping does not apply. However we can also notice that the control input $\alpha(1)$ can cancel the non-strict-feedback integral and cast the problem into the form needed for a backstepping design via the input $u(1)$. Hence, we select

$$\alpha(1) = b\sinh(b)u(0) - b^2\int_0^1 \cosh(b(1-y))u(y)dy\,. \quad (51)$$

It can then be shown that a kernel $k(x,y)$ that satisfies the PDE

$$\begin{aligned} k_{xx} &= k_{yy} + \left(c+b^2\right)k - b^3\sinh(b(x-y)) \\ &\quad + b^3\int_y^x k(x,\xi)\sinh(b(\xi-y))d\xi \quad (52) \\ k(x,x) &= -\frac{c+b^2}{2}x \quad (53) \\ k_y(x,0) &= b^2\left(-\cosh(bx) + \int_0^x k(x,y)\cosh(by)dy\right) \quad (54) \end{aligned}$$

converts (47), (48) into (5)–(7). It was shown in [12] that this type of a PDE has a unique smooth solution.

When the full state $u(y)$ is not available for measurement, instead of (14) we use the feedback law

$$u(1) = \int_0^1 k(1,y)\hat{u}(y)\,dy\,, \quad (55)$$

with $\hat{u}(y)$ generated by an observer presented in Section V. Likewise, we replace (51) by

$$\alpha(1) = b\sinh(b)u(0) - b^2\int_0^1 \cosh(b(1-y))\hat{u}(y)dy\,. \quad (56)$$

V. Observer Design

Next we focus our attention on the plant model (49), (50) with $\bar{\alpha}(x)$ set identically to zero. This system is also not in the strict-feedback format required in the observer design procedure in [13]. However, the term $\alpha(0)$, which violates the structure, turns out to be measurable, which means that we can cancel it in our observer.

According to the observer design theory in [13], our PDE observer should consist of a copy of the plant (49), (50) and output injection. The observer is given by

$$\begin{aligned} \epsilon\hat{u}_{tt} &= (1+d\partial_t)\left\{\hat{u}_{xx} + b^2\hat{u} + b^3\int_0^x \sinh(b(x-y))\hat{u}(y)dy - b^2\cosh(bx)u(0) - b\sinh(bx)\alpha(0) + p_y(x,0)\left(u(0)-\hat{u}(0)\right)\right\} \quad (57) \\ \hat{u}_x(0) &= \alpha(0) + p(0,0)\left(u(0)-\hat{u}(0)\right) \quad (58) \\ \hat{u}(1) &= u(1)\,, \quad (59) \end{aligned}$$

where the quantities $p_y(x,0)$ in the last line of (57) and $p(0,0)$ in (58) are determined by solving the PDE

$$\begin{aligned} p_{yy} &= p_{xx} + \left(\tilde{c}+b^2\right)p - b^3\sinh(b(x-y)) \\ &\quad + b^3\int_y^x p(\xi,y)\sinh(b(x-\xi))d\xi \quad (60) \\ p(x,x) &= \frac{\tilde{c}+b^2}{2}(x-1) \quad (61) \\ p(1,y) &= 0\,. \quad (62) \end{aligned}$$

It should be noted that the observer (57)–(59) can be used whether the input $u(1)$ in (59) is substituted from the controller (55), set to zero, or set to some other (perhaps open-loop, purely time dependent) control policy. Denoting the observer error as

$$\tilde{u} = u - \hat{u} \quad (63)$$

and substituting (57)–(59) from (49), (50) we obtain the observer error dynamics

$$\begin{aligned} \epsilon\tilde{u}_{tt} &= (1+d\partial_t)\left\{\tilde{u}_{xx} + b^2\tilde{u} + b^3\int_0^x \sinh(b(x-y))\tilde{u}(y)dy + p_y(x,0)\tilde{u}(0)\right\} \quad (64) \\ \tilde{u}_x(0) &= -p(0,0)\tilde{u}(0) \quad (65) \\ \tilde{u}(1) &= 0\,. \quad (66) \end{aligned}$$

It can be shown that the transformation

$$\tilde{u}(x) = \tilde{w}(x) - \int_0^x p(x,y)\tilde{w}(y)dy \quad (67)$$

converts the error system (64)–(66) into

$$\begin{aligned} \epsilon\tilde{w}_{tt} &= (1+d\partial_t)\left(\tilde{w}_{xx} - \tilde{c}\tilde{w} + \widetilde{\Pi}\right) \quad (68) \\ \tilde{w}_x(0) &= 0 \quad (69) \\ \tilde{w}(1) &= 0\,, \quad (70) \end{aligned}$$

where

$$\begin{aligned} \widetilde{\Pi}(x) &= -\bar{\alpha}(x) - \int_0^x r(x,y)\bar{\alpha}_y(y)dy \\ &\quad + b\left(\sinh(bx) + \int_0^x r(x,y)\sinh(by)dy\right)\bar{\alpha}(0) \quad (71) \end{aligned}$$

and the smooth function $r(x,y)$ represents the kernel of the inverse transformation of (67) given by

$$\tilde{w}(x) = \tilde{u}(x) + \int_0^x r(x,y)\tilde{u}(y)dy\,. \quad (72)$$

The observer error system (68)–(70) is the same as (25)–(27) when $\bar{\alpha}(x) \equiv 0$, that is, it is well damped for the singularly perturbed/reduced Timoshenko model or for the shear beam model.

[3] The PDE governing the $\bar{\alpha}$ dynamics will be presented later in the paper. In singular perturbation terminology, the $\bar{\alpha}$ equation represents the "boundary layer" model.

When the control (55) is is used, we have to jointly analyze the plant (47), (48), (55) and the observer (57)–(59). However, analyzing the overall system in the variables $(u, \hat{u})$ is not the only possible choice. Other choices are $(u, \tilde{u})$ and $(\tilde{u}, \hat{u})$. The last choice turns out to be particularly convenient, so we will analyze the systems (64)–(66) and (57)–(59), (55).

We start by noticing that the observer system (57)–(59) contains the term $\alpha(0)$. We substitute the controller (56) into the expression (45) for $\alpha(0)$ and get

$$\alpha(0) = \bar{\alpha}(0) + \frac{b^2}{\cosh(b)} \int_0^1 \cosh(b(1-y))\tilde{u}(y)dy\,. \tag{73}$$

Plugging this result into (57)–(59) we get

$$\begin{aligned}
\epsilon\hat{u}_{tt} &= (1+d\partial_t)\left\{\hat{u}_{xx} + b^2\hat{u} \right.\\
&\quad + b^3\int_0^x \sinh(b(x-y))\hat{u}(y)dy\\
&\quad - b^2\cosh(bx)\hat{u}(0)\\
&\quad + \left(p_y(x,0) - b^2\cosh(bx)\right)\tilde{u}(0)\\
&\quad - \frac{b^2\sinh(bx)}{\cosh(b)}\int_0^1 \cosh(b(1-y))\tilde{u}(y)dy\\
&\quad \left. - b\sinh(bx)\bar{\alpha}(0)\right\} &(74)\\
\hat{u}_x(0) &= \bar{\alpha}(0) + p(0,0)\tilde{u}(0)\\
&\quad + \frac{b^2}{\cosh(b)}\int_0^1 \cosh(b(1-y))\tilde{u}(y)dy &(75)\\
\hat{u}(1) &= \int_0^1 k(1,y)\hat{u}(y)\,dy\,, &(76)
\end{aligned}$$

We have already converted the $\tilde{u}$-system (64)–(66) into the $\tilde{w}$-system (68)–(70). We do the same next with the observer system (74)–(76). Applying the transformation

$$\hat{w}(x) = \hat{u}(x) - \int_0^x k(x,y)\hat{u}(y)dy \tag{77}$$

with $k(x, y)$ defined by (52)–(54), we get

$$\begin{aligned}
\epsilon\hat{w}_{tt} &= (1+d\partial_t)\left(\hat{w}_{xx} - \hat{c}\tilde{w} + \widehat{\Gamma} + \widehat{\Pi}\right) &(78)\\
\hat{w}_x(0) &= \widehat{\Gamma}_0 + \widehat{\Pi}_0 &(79)\\
\hat{w}(1) &= 0\,, &(80)
\end{aligned}$$

where

$$\begin{aligned}
\widehat{\Gamma}(x) &= Q_0(x)\tilde{w}(0) + Q_1(x)\int_0^1 Q_p(1,y)\tilde{w}(y)dy &(81)\\
\widehat{\Pi}(x) &= Q_2(x)\bar{\alpha}(0) &(82)\\
\widehat{\Gamma}_0 &= p(0,0)\tilde{w}(0)\\
&\quad + \frac{b^2}{\cosh(b)}\int_0^1 Q_p(1,y)\tilde{w}(y)dy &(83)\\
\widehat{\Pi}_0 &= \bar{\alpha}(0) &(84)\\
Q_0(x) &= p_y(x,0) - b^2\cosh(bx)\\
&\quad - \int_0^x k(x,\eta)\left(p_y(\eta,0) - b^2\cosh(b\eta)\right)d\eta &(85)\\
Q_1(x) &= \frac{b^2}{\cosh(b)}Q_2(x) &(86)\\
Q_2(x) &= -b\left(\sinh(bx) - \int_0^x k(x,\eta)\sinh(b\eta)d\eta\right) &(87)\\
Q_p(x,y) &= b^2\left(\cosh(b(x-y))\right.\\
&\quad \left. - \int_y^x \cosh(b(x-\xi))p(\xi,y)d\xi\right)\,. &(88)
\end{aligned}$$

We have thus written the $(\tilde{u}, \hat{u})$-system in the $(\tilde{w}, \hat{w})$ variables, given by the equations (68)–(70) and (78)–(80). The $\hat{w}$-system is driven by both $\tilde{w}$ and $\bar{\alpha}$ (through the perturbations $\widehat{\Gamma}, \widehat{\Gamma}_0, \widehat{\Pi}, \widehat{\Pi}_0$), whereas the $\tilde{w}$-system is driven by only $\bar{\alpha}$ (through the perturbation $\widetilde{\Pi}$).

VI. Stability Analysis

To study the stability of the reduced model (68)–(70) and (78)–(80) with $\bar{\alpha} = 0$ (which sets $\widehat{\Pi} \equiv \widehat{\Pi}_0 \equiv \widetilde{\Pi} \equiv 0$), we introduce the Lyapunov functions

$$\begin{aligned}
\widetilde{V} &= \frac{1}{2}\left[\left(1+\tilde{\delta}d\right)\left(\|\tilde{w}_x\|^2 + \tilde{c}\|\tilde{w}\|^2\right)\right.\\
&\quad \left. + \epsilon\|\tilde{w}_t\|^2 + 2\tilde{\delta}\epsilon\langle\tilde{w},\tilde{w}_t\rangle\right] &(89)\\
\widehat{V} &= \frac{1}{2}\left[\left(1+\hat{\delta}d\right)\left(\|\hat{w}_x\|^2 + c\|\hat{w}\|^2\right)\right.\\
&\quad \left. + \epsilon\|\hat{w}_t\|^2 + 2\hat{\delta}\epsilon\langle\hat{w},\hat{w}_t\rangle\right]\,. &(90)
\end{aligned}$$

Using Poincare's inequality, it is easy to see that for sufficiently small positive $\hat{\delta}, \tilde{\delta}$ there exist positive constants $\tilde{m}_1, \tilde{m}_2, \hat{m}_1, \hat{m}_2$ such that

$$\begin{aligned}
\tilde{m}_1\widetilde{U} \le \widetilde{V} \le \tilde{m}_2\widetilde{U} &\qquad (91)\\
\hat{m}_1\widehat{U} \le \widehat{V} \le \hat{m}_2\widehat{U}, &\qquad (92)
\end{aligned}$$

where

$$\begin{aligned}
\widetilde{U} &= \|\tilde{w}_x\|^2 + \|\tilde{w}_t\|^2 &(93)\\
\widehat{U} &= \|\hat{w}_x\|^2 + \|\hat{w}_t\|^2\,. &(94)
\end{aligned}$$

Furthermore, a long calculation shows that (for $\bar{\alpha} = 0$)

$$\begin{aligned}
\dot{\widetilde{V}} &= -\tilde{\delta}\left(\tilde{c}\|\tilde{w}\|^2 + \|\tilde{w}_x\|^2\right)\\
&\quad - \left(\tilde{c}d - \tilde{\delta}\epsilon\right)\|\tilde{w}_t\|^2 - d\|\tilde{w}_{xt}\|^2 &(95)
\end{aligned}$$

and

$$\begin{aligned}
\dot{\widehat{V}} &= -\hat{\delta}\left(c\|\hat{w}\|^2 + \|\hat{w}_x\|^2\right)\\
&\quad - \left(cd - \hat{\delta}\epsilon\right)\|\hat{w}_t\|^2 - d\|\hat{w}_{xt}\|^2\\
&\quad + \Xi\,, &(96)
\end{aligned}$$

where

$$\begin{aligned}
\Xi &= -\left(\widehat{\Gamma}_0 + d\dot{\widehat{\Gamma}}_0\right)\left(\hat{w}_t(0) + \hat{\delta}\hat{w}(0)\right)\\
&\quad + \left\langle \hat{w}_t + \hat{\delta}\hat{w}, \widehat{\Gamma} + d\widehat{\Gamma}_t\right\rangle\,. &(97)
\end{aligned}$$

Using the Poincare, Agmon, and Cauchy-Schwartz inequalities, it can be shown that

$$\begin{aligned}
\left\|\widehat{\Gamma}\right\|, \left|\widehat{\Gamma}_0\right| &\le m\|\tilde{w}_x\| &(98)\\
\left\|\widehat{\Gamma}_t\right\|, \left|\dot{\widehat{\Gamma}}_0\right| &\le m\|\tilde{w}_{xt}\| &(99)
\end{aligned}$$

for sufficiently large m, and further that

$$|\Xi| \le \bar{m}\left(\|\tilde{w}_x\|^2 + \|\tilde{w}_{xt}\|^2 + \|\hat{w}_x\|^2 + \|\hat{w}_{xt}\|^2\right) \tag{100}$$

for sufficiently large $\bar{m}$. Taking a Lyapunov function of the form

$$V = \widehat{V} + \Lambda\widetilde{V}\,, \tag{101}$$

using (89)–(100) one can show that there exists a sufficiently large positive Λ such that

$$\dot{V} \le -\lambda V \tag{102}$$

for some (small) $\lambda > 0$. From this result, along with (91), (92), it follows that

$$\widehat{U}(t) + \widetilde{U}(t) \le M e^{-t/M} \left(\widehat{U}(0) + \widetilde{U}(0) \right) \tag{103}$$

for sufficiently large $M > 0$. From the invertibility of the transformations (67) and (77) [and from the smoothness of their kernels $p(x, y)$ and $k(x, y)$], it follows that

$$\begin{aligned} &\|u_x(t)\|^2 + \|u_t(t)\|^2 + \|\hat{u}_x(t)\|^2 + \|\hat{u}_t(t)\|^2 \le \bar{M} e^{-t/\bar{M}} \\ &\times \left(\|u_x(0)\|^2 + \|u_t(0)\|^2 + \|\hat{u}_x(0)\|^2 + \|\hat{u}_t(0)\|^2 \right) . \end{aligned} \tag{104}$$

Theorem 3: Consider the system consisting of the plant (47), (48), the controller (55), (56), and the observer (57)–(59), (45), with $\bar{\alpha} \equiv 0$. The trivial solution $u_t(x, t) \equiv u_x(x, t) \equiv \hat{u}_t(x, t) \equiv \hat{u}_x(x, t) \equiv 0$ is exponentially stable in the L_2 sense, as specified by the estimate (104).

Of course, a stronger property than mere exponential stability is achieved. All the eigenvalues of the closed loop system, which is block-triangular as clearly displayed in the representation (68)–(70), (78)–(80) with $\widehat{\Pi} \equiv \widehat{\Pi}_0 \equiv \widetilde{\Pi} \equiv 0$, have damping ratios of at least $\frac{\pi d}{4\sqrt{\epsilon}} \sqrt{1 + \frac{4}{\pi^2} \bar{c}}$ and all of their real parts are no larger than $-\min\left\{ \frac{1}{d}, \frac{\pi^2 d}{8\epsilon} \left(1 + \frac{4}{\pi^2} \bar{c}\right) \right\}$, where $\bar{c} = \min\{c, \tilde{c}\}$. As mentioned before, a good engineering choice would be $\tilde{c} > c$, which would make the observer operate on a faster time scale than the certainty-equivalent state feedback controller.

All the conclusions so far in this section are for the shear beam model. To study the closed-loop Timoshenko model we include $\widehat{\Pi}, \widehat{\Pi}_0, \widetilde{\Pi}$ in (68)–(70), (78)–(80) and write (40)–(42), (44), (73), (67) as

$$\mu \bar{\alpha}_{tt} = (1 + d\partial_t) \left(\bar{\alpha}_{xx} - b^2 \bar{\alpha} \right) - \mu \bar{\Gamma} \tag{105}$$
$$\bar{\alpha}_x(0) = 0 \tag{106}$$
$$\bar{\alpha}(1) = 0 , \tag{107}$$

where

$$\begin{aligned} \bar{\Gamma}(x) &= b \sinh(bx) \left(\tilde{w}_{tt}(0) + \hat{w}_{tt}(0) \right) \\ &+ \frac{\cosh(bx)}{\cosh(b)} \int_0^1 Q_p(1, y) \tilde{w}_{tt}(y) dy \\ &- \int_0^x Q_p(x, y) \tilde{w}_{tt}(y) dy \\ &- \int_0^x Q_l(x, y) \hat{w}_{tt}(y) dy \end{aligned} \tag{108}$$

$$\begin{aligned} Q_l(x, y) &= b^2 \left(\cosh(b(x - y)) \right. \\ &\left. + \int_y^x \cosh(b(x - \xi)) l(\xi, y) d\xi \right) \end{aligned} \tag{109}$$

and the smooth function $l(x, y)$ represents the kernel of the inverse transformation of (77) given by

$$\hat{u}(x) = \hat{w}(x) + \int_0^x l(x, y) \hat{w}(y) dy . \tag{110}$$

Thus the "boundary layer model" in standard singular perturbation analysis is

$$\mu \bar{\alpha}_{\tau\tau} = (1 + d\partial_t) \left(\bar{\alpha}_{xx} - b^2 \bar{\alpha} \right) \tag{111}$$
$$\bar{\alpha}_x(0) = 0 \tag{112}$$
$$\bar{\alpha}(1) = 0 , \tag{113}$$

where τ is the time variable of the fast subsystem. Hence, both the reduced model (68)–(70), (78)–(80) with $\widehat{\Pi} \equiv \widehat{\Pi}_0 \equiv \widetilde{\Pi} \equiv 0$ and the boundary layer model (111)–(113) are exponentially stable, satisfying the conditions in the singular perturbation theory for overall stability of the un-approximated system. Indeed, due to the fact that μ multiplies the perturbation $\bar{\Gamma}$ in (105), the overall $(\tilde{w}, \hat{w}, \bar{\alpha})$-system (68)–(70), (78)–(80), (105)–(107), with the $\bar{\alpha}$-perturbations $\widehat{\Pi}, \widehat{\Pi}_0, \widetilde{\Pi}$ included in (68)–(70), (78)–(80), is exponentially stable for sufficiently small μ, however, the Lyapunov proof involves higher order norms and we don't pursue it here.

Finally, in addition to the eigenvalues of the reduced model being well damped, the eigenvalues of the boundary layer model have damping ratios of at least $\frac{\pi d}{4\sqrt{\mu}} \sqrt{1 + \frac{4}{\pi^2} \frac{a}{\epsilon}}$ and all of their real parts are no larger than $-\min\left\{ \frac{1}{d}, \frac{\pi^2 d}{8\mu} \left(1 + \frac{4}{\pi^2} \frac{a}{\epsilon}\right) \right\}$. So for small μ all the eigenvalues of the Timoshenko model are well damped.

VII. Controller and Observer Gains

The controller gain $k(1, y)$ and the observer gain $p_y(x, 0), p(0, 0)$ can be computed by numerically calculating the solutions to the hyperbolic PIDEs (52)–(54) and (60)–(62). However their solutions can also be computed symbolically as

$$k(x, y) = \lim_{n \to \infty} k_n(x, y) \tag{114}$$

$$\begin{aligned} k_0 &= -\frac{b}{2} \left[-\sinh(b(x - y)) + by \cosh(b(x - y)) \right] \\ &\quad -\frac{c}{2} x \end{aligned} \tag{115}$$

$$\begin{aligned} k_{n+1} &= k_0 \\ &+ (c + b^2) \int_{\frac{x-y}{2}}^{\frac{x+y}{2}} \int_0^{\frac{x-y}{2}} k_n(\sigma + s, \sigma - s) \, ds \, d\sigma \\ &+ 2(c + b^2) \int_0^{\frac{x-y}{2}} \int_0^{\sigma} k_n(\sigma + s, \sigma - s) \, ds \, d\sigma \\ &- b^2 \int_0^{x-y} \int_0^{\sigma} k_n(\sigma, s) \cosh(bs) \, ds \, d\sigma \\ &+ b^3 \int_{\frac{x-y}{2}}^{\frac{x+y}{2}} \int_0^{\frac{x-y}{2}} \int_{\sigma - s}^{\sigma + s} k_n(\sigma + s, \xi) \\ &\times \sinh(b(\xi - \sigma + s)) \, d\xi \, ds \, d\sigma \\ &+ 2b^3 \int_0^{\frac{x-y}{2}} \int_0^{\sigma} \int_{\sigma - s}^{\sigma + s} k_n(\sigma + s, \xi) \\ &\times \sinh(b(\xi - \sigma + s)) \, d\xi \, ds \, d\sigma \end{aligned} \tag{116}$$

and

$$p(x, y) = \lim_{n \to \infty} p_n(x, y) \tag{117}$$

$$\begin{aligned} p_0 &= -\frac{b}{2} \left[\sinh(b(x - y)) + b(1 - x) \cosh(b(x - y)) \right] \\ &\quad -\frac{\tilde{c}}{2} (1 - y) \end{aligned} \tag{118}$$

$$\begin{aligned}
p_{n+1} \;=\; & p_0 \\
& +(\tilde{c}+b^2)\int_{\frac{x-y}{2}}^{\frac{2-x-y}{2}}\int_0^{\frac{x-y}{2}} p_n(\sigma+s,\sigma-s)\,ds\,d\sigma \\
& +2(\tilde{c}+b^2)\int_0^{\frac{x-y}{2}}\int_0^{\sigma} p_n(\sigma+s,\sigma-s)\,ds\,d\sigma \\
& +b^3\int_{\frac{x-y}{2}}^{\frac{2-x-y}{2}}\int_0^{\frac{x-y}{2}}\int_{\sigma-s}^{\sigma+s} p_n(\sigma+s,\xi) \\
& \times\sinh(b(\xi-\sigma+s))\,d\xi\,ds\,d\sigma \\
& +2b^3\int_0^{\frac{x-y}{2}}\int_0^{\sigma}\int_{\sigma-s}^{\sigma+s} p_n(\sigma+s,\xi) \\
& \times\sinh(b(\xi-\sigma+s))\,d\xi\,ds\,d\sigma \qquad (119)
\end{aligned}$$

As for the initial iterates of the controller and observer gains, they are

$$\begin{aligned}
(k(1,y))_0 \;=\; & -\frac{b}{2}\left[-\sinh(b(1-y))+by\cosh(b(1-y))\right] \\
& -\frac{c}{2} \qquad (120)
\end{aligned}$$

$$\begin{aligned}
(p_y(x,0))_0 \;=\; & \frac{b^2}{2}\left[\cosh(bx)+b(1-x)\sinh(bx)\right] \\
& +\frac{\tilde{c}}{2}\,. \qquad (121)
\end{aligned}$$

The observer gain in the boundary condition (58) is known exactly,

$$p(0,0) = -\frac{\tilde{c}+b^2}{2}\,. \qquad (122)$$

VIII. Discussion

The main distinction between the bacstepping compensators designed here and the damping feedbacks in the prior literature, besides the control architecture required, is that the backstepping controllers, which are not relying on a passivity property from the actuator to the sensor, can be employed to pursue more ambitious objectives such as trajectory tracking. Our major future research effort will be in developing motion planning techniques that result in explicit formulae for state and input reference functions to achieve asymptotic tracking of desired beam tip trajectories. Once the motion planning problems are solved, the backstepping compensators can be modified in a straightforward way from serving the purpose of stabilization of the equilibrium state to stabilization of trajectories. Such an explicit parametrization will be possible for trajectories consisting of sinusoids, exponentials, and polynomial functions of time.

We presented our results for a model of a beam with a free end. These results can be extended to the case where the beam tip is subject to a force that is the result of interaction with the environment and is a function of the tip displacement. This extension is of interest in atomic force microscopes.

We focused on the Dirichlet form of actuation and sensing. Similar designs can be produced for the cases off Neumann actuation and/or sensing.[4]

As mentioned before, our method does not work for the Euler-Bernoulli beam model, though it works for more realistic models, an irony which has kept the backstepping approach restricted to parabolic problems only for several years. It is useful to see that when $\epsilon, \mu \to 0$, which is when the Timoshenko model degenerates into the Euler-Bernoulli model, our controller and observer gains grow towards infinity.

[4]Involving $u_x(0), \alpha_x(0), u_x(1), \alpha_x(1)$.

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