

Adaptive Boundary Control for Unstable Parabolic PDEs—Part III: Output Feedback Examples with Swapping Identifiers

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This paper is Part III of a series of three companion papers which can be all downloaded for the purpose of review from

<http://flyingv.ucsd.edu/part1.pdf> (cited as reference [12])
<http://flyingv.ucsd.edu/part2.pdf> (cited as reference [20])
<http://flyingv.ucsd.edu/part3.pdf> (present paper)

Abstract— We develop output feedback adaptive controllers for two benchmark parabolic PDEs motivated by a model of thermal instability in solid propellant rockets. Both benchmark plants are unstable, have infinite relative degree, and are controlled from the boundary. One plant has an unknown parameter in the PDE and the other in the boundary condition. In both cases the unknown parameter multiplies the measured output of the system, which is obtained with a boundary sensor located on the “opposite side” of the domain from the actuator. In comparison with the Lyapunov output-feedback adaptive controllers in [12], the controllers presented here employ much simpler update laws and do not require a priori knowledge about the unknown parameters. We show how our two benchmarks examples can be combined and illustrate the adaptive stabilization design by simulation.

Index Terms— adaptive control, boundary control, distributed parameter systems

I. INTRODUCTION

In a companion paper [20] we introduced a novel approach to *adaptive* control of PDEs where a parametrized family of boundary controllers can be combined with “swapping gradient” identifiers to yield global stability of the resulting nonlinear PDE system. Only the *state*-feedback problem was considered in [20]. For a different, narrower, class of systems, the *output*-feedback problem is solvable by this method, which is illustrated on two benchmark examples in this paper.

We consider two parametrically uncertain, *unstable* parabolic PDE plants controlled from the *boundary*. While these benchmark plants are simple in appearance, there does not exist an adaptive control design in the literature that is applicable to them due to the fact that they have infinite relative degree. Infinite relative degree arises in applications where actuators and sensors are on the “opposite sides” of the PDE domains. The two benchmark problems in this paper are motivated by a model of thermal instability in

solid propellant rockets [3]. Our control laws are adaptive versions of the explicit boundary control laws developed in [18], [19]. Our adaptive observers are infinite dimensional extensions of Kreisselmeier observers [13]. Our identifiers are designed using the swapping approach [13], prevalent in adaptive control of finite dimensional systems of relative degree higher than one. These identifiers remove the need for parameter projection and low adaptation gain present in Lyapunov output-feedback designs in [12].

Early works on adaptive control of infinite-dimensional systems were for plants stabilizable by non-identifier based high gain feedback [15], under a relative degree one assumption. State-feedback model reference adaptive control (MRAC) was extended to PDEs in [7], [2], [21], [16], [1] but not for the case of boundary control. Efforts in [5], [22] made use of positive realness assumptions where relative degree one is implicit, except in some examples where this restriction is cleverly overcome. Stochastic adaptive LQR with least-squares parameter estimation and state feedback was pursued in [6]. Adaptive control of nonlinear PDEs was studied in [14], [10], [11]. Adaptive controllers for nonlinear systems on lattices were designed in [9]. An experimentally validated adaptive boundary controller for a flexible beam was presented in [4].

Although for the sake of clarity we consider two separate benchmark problems, it is possible to design an adaptive controller for a combined problem (Section VI). Another reason for a separate consideration is a slightly weaker result for the benchmark plant with the unknown parameter in the boundary condition, due to an inherent difficulty observed in [1], [14].

Throughout the paper we assume well posedness of the closed-loop systems in the interest of space and due to the parabolic character of these systems which ensures their benign behavior, as supported by numerical results that we show in this paper. An example on how one derives the Sobolev estimates of higher order (H_4), the key step in a proof of well posedness, is given in [12].

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a) *Notation.*: The spatial $L_2(0,1)$ norm is denoted by $\|\cdot\|$. The temporal norms are denoted by $\mathcal{L}_\infty$ and $\mathcal{L}_2$ for $t \geq 0$. We denote by l_1 a generic function in $\mathcal{L}_\infty \cap \mathcal{L}_2$.

II. BENCHMARK PLANT WITH UNKNOWN PARAMETER IN THE DOMAIN

Consider the following plant

$$u_t(x, t) = u_{xx}(x, t) + gu(0, t), \quad (1)$$

$$u_x(0, t) = 0, \quad (2)$$

$$u(1, t) = U(t), \quad (3)$$

where $U(t)$ is a control signal. This system is inspired by a model of thermal instability in solid propellant rockets [3]. For $U(t) = 0$ this system is unstable if and only if $g > 2$. The plant can be written in the frequency domain as a transfer function from input $u(1)$ to output $u(0)$:

$$u(0, s) = \frac{s}{(s - g) \cosh \sqrt{s} + g} u(1, s). \quad (4)$$

We can see that it has no zeros (at $s = 0$ the transfer function is $2/(2 - g)$) and has infinitely many poles, one of which is unstable and approximately equal to g as $g \rightarrow +\infty$. So this is an infinite relative degree system.

The following transformation has been proposed in [18] for the case of known g :

$$w(x, t) = u(x, t) + \int_0^x \sqrt{g} \sinh \sqrt{g}(x - \xi) u(\xi, t) d\xi, \quad (5)$$

which maps (1)–(2) into an exponentially stable system

$$w_t(x, t) = w_{xx}(x, t), \quad (6)$$

$$w_x(0, t) = 0, \quad (7)$$

$$w(1, t) = 0. \quad (8)$$

A stabilizing control law is given by (5) evaluated at $x = 1$:

$$u(1, t) = - \int_0^1 \sqrt{g} \sinh \sqrt{g}(1 - \xi) u(\xi, t) d\xi. \quad (9)$$

Suppose now that we want to stabilize this system when g is unknown. Our main result for this problem is summarized in the following theorem.

Theorem 1: Consider the system (1)–(2) with the control

$$u(1, t) = \int_0^1 k(1, \xi, \hat{g})(\hat{g}v(\xi, t) + \eta(\xi, t)) d\xi, \quad (10)$$

$$k(x, \xi, \hat{g}) = \begin{cases} -\sqrt{\hat{g}} \sinh \sqrt{\hat{g}}(x - \xi), & \hat{g} \geq 0, \\ \sqrt{-\hat{g}} \sin \sqrt{-\hat{g}}(x - \xi), & \hat{g} < 0, \end{cases} \quad (11)$$

where an update law for $\hat{g}$ is

$$\dot{\hat{g}} = \gamma \frac{(u(0, t) - \hat{g}v(0, t) - \eta(0, t))v(0, t)}{1 + v^2(0, t)}, \quad (12)$$

and the filters $v(x, t)$, $\eta(x, t)$ are defined as

$$v_t(x, t) = v_{xx}(x, t) + u(0, t), \quad (13)$$

$$v_x(0, t) = 0, \quad (14)$$

$$v(1, t) = 0, \quad (15)$$

$$\eta_t(x, t) = \eta_{xx}(x, t), \quad (16)$$

$$\eta_x(0, t) = 0, \quad (17)$$

$$\eta(1, t) = u(1, t). \quad (18)$$

If the closed loop system (1)–(2), (10)–(18) has a classical solution $(u, \hat{g}, v, \eta)$, then for any $\hat{g}(0)$ and any initial conditions $u_0, v_0, \eta_0 \in L_2(0,1)$, the signals $\hat{g}, u, v, \eta$ are bounded and u is regulated to zero for all $x \in [0, 1]$:

$$\lim_{t \rightarrow \infty} \max_{x \in [0,1]} |u(x, t)| = 0. \quad (19)$$

□

Although it may not be immediately obvious from (11), the control law (10)–(11) is a smooth function of $\hat{g}$. Note that a-priori knowledge of a bound on $\hat{g}$ is not required in the swapping scheme (10)–(18) (as opposed to Lyapunov adaptive design in [12]).

III. PROOF OF THEOREM 1

A. Target system

Introducing the error $e = u - gv - \eta$ we get an exponentially stable system

$$e_t(x, t) = e_{xx}(x, t), \quad (20)$$

$$e_x(0, t) = 0, \quad (21)$$

$$e(1, t) = 0. \quad (22)$$

The estimate $\hat{e} = u - \hat{g}v - \eta$ satisfies the following PDE

$$\hat{e}_t(x, t) = \hat{e}_{xx}(x, t) + \tilde{g}u(0, t) - \dot{\hat{g}}v(x, t), \quad (23)$$

$$\hat{e}_x(0, t) = 0, \quad (24)$$

$$\hat{e}(1, t) = 0. \quad (25)$$

The signal $\hat{e}$ can be expressed through e as $\hat{e} = e + \tilde{g}v$.

The transformation

$$\begin{aligned} \hat{w}(x, t) &= \hat{g}v(x, t) + \eta(x, t) \\ &\quad - \int_0^x k(x, \xi, \hat{g})(\hat{g}v(\xi, t) + \eta(\xi, t)) d\xi \end{aligned} \quad (26)$$

with $k(x, \xi, \hat{g})$ given by (11) maps (13)–(18) into the following system (Lemma A.1):

$$\begin{aligned} \hat{w}_t(x, t) &= \hat{w}_{xx}(x, t) + \beta(x)\hat{e}(0, t) + \dot{\hat{g}}v \\ &\quad + \dot{\hat{g}} \int_0^x \alpha(x - \xi)(\hat{g}v(\xi, t) + \hat{w}(\xi, t)) d\xi, \end{aligned} \quad (27)$$

$$\hat{w}_x(0, t) = 0, \quad (28)$$

$$\hat{w}(1, t) = 0, \quad (29)$$

where

$$\alpha(x) = -\frac{1}{\hat{g}}k(x, 0, \hat{g}), \quad (30)$$

$$\beta(x) = k_\xi(x, 0, \hat{g}) = \begin{cases} \hat{g} \cosh \sqrt{\hat{g}}x, & \hat{g} \geq 0, \\ \hat{g} \cos \sqrt{-\hat{g}}x, & \hat{g} < 0, \end{cases} \quad (31)$$

B. Adaptive law

We take the following equation as a parametric model

$$e(0, t) = u(0, t) - gv(0, t) - \eta(0, t). \quad (32)$$

The estimation error is

$$\hat{e}(0, t) = u(0, t) - \hat{g}v(0, t) - \eta(0, t). \quad (33)$$

We use the gradient update law

$$\dot{\hat{g}} = \gamma \frac{\hat{e}(0, t)v(0, t)}{1 + v^2(0, t)}. \quad (34)$$

Lemma 2: The adaptive law (34) guarantees the following properties:

$$\frac{\hat{e}(0, t)}{\sqrt{1 + v^2(0, t)}} \in \mathcal{L}_2 \cap \mathcal{L}_\infty, \quad \tilde{g} \in \mathcal{L}_\infty, \quad \dot{\hat{g}} \in \mathcal{L}_2 \cap \mathcal{L}_\infty. \quad (35)$$

Proof: Using a Lyapunov function

$$V = \frac{1}{2} \int_0^1 e^2 dx + \frac{1}{2\gamma} \tilde{g}^2 \quad (36)$$

we get

$$\begin{aligned} \dot{V} &= - \int_0^1 e_x^2 dx - \frac{\tilde{g}\hat{e}(0)v(0)}{1 + v^2(0)} \\ &\leq - \int_0^1 e_x^2 dx - \frac{\hat{e}^2(0)}{1 + v^2(0)} + \frac{e(0)\hat{e}(0)}{1 + v^2(0)} \\ &\leq -\|e_x\|^2 - \frac{\hat{e}^2(0)}{1 + v^2(0)} + \frac{\|e_x\|\|\hat{e}(0)\|}{\sqrt{1 + v^2(0)}} \\ &\leq -\frac{1}{2}\|e_x\|^2 - \frac{1}{2} \frac{\hat{e}^2(0)}{1 + v^2(0)} \end{aligned} \quad (37)$$

This gives the following properties

$$\frac{\hat{e}(0, t)}{\sqrt{1 + v^2(0, t)}} \in \mathcal{L}_2, \quad \tilde{g} \in \mathcal{L}_\infty. \quad (38)$$

Since

$$\frac{\hat{e}(0, t)}{\sqrt{1 + v^2(0, t)}} = \frac{e(0, t)}{\sqrt{1 + v^2(0, t)}} + \tilde{g} \frac{v(0, t)}{\sqrt{1 + v^2(0, t)}}, \quad (39)$$

$$\dot{\hat{g}} = \gamma \frac{\hat{e}(0, t)}{\sqrt{1 + v^2(0, t)}} \frac{v(0, t)}{\sqrt{1 + v^2(0, t)}}, \quad (40)$$

we get (35). $\blacksquare$

The explicit bound on $\hat{g}$ in terms of initial conditions of all the signals can be obtained from (37):

$$\begin{aligned} \hat{g}^2(t) &\leq 2g^2 + 2 \left(\tilde{g}(0)^2 + \gamma \int_0^1 e^2(x, 0) dx \right) \\ &\leq 2g^2 + 2(g - \hat{g}(0))^2 \\ &\quad + 2\gamma \int_0^1 (u(x, 0) - gv(x, 0) - \eta(x, 0))^2 dx. \end{aligned} \quad (41)$$

We denote the bound on $\hat{g}$ by g_0 . The above properties imply that functions α and β are bounded, let us denote these bounds by α_0 and β_0 .

C. Boundedness

The filter v can be rewritten in the following way

$$v_t(x, t) = v_{xx}(x, t) + \hat{w}(0, t) + \hat{e}(0, t), \quad (42)$$

$$v_x(0, t) = 0, \quad (43)$$

$$v(1, t) = 0. \quad (44)$$

We have two interconnected systems $\hat{w}$, v driven by a signal $\hat{e}(0, t)$ with properties (35). Consider a Lyapunov function

$$V_v = \frac{1}{2} \int_0^1 v^2(x) dx + \frac{1}{2} \int_0^1 v_x^2(x) dx. \quad (45)$$

We include the H_1 norm in the Lyapunov function because the signal $\hat{e}(0)$ is normalized by $1 + v^2(0)$ and $v^2(0)$ can only be bounded by $\|v_x\|^2$. Using Young's, Poincare's, and Agmon's inequalities we have¹

$$\begin{aligned} \dot{V}_v &= - \int_0^1 v_x^2 dx + (\hat{w}(0) + \hat{e}(0)) \int_0^1 v dx \\ &\quad - \int_0^1 v_{xx}^2 dx - (\hat{w}(0) + \hat{e}(0)) \int_0^1 v_{xx} dx \\ &\leq -\|v_x\|^2 + \frac{1}{8}\|v\|^2 + 4 \frac{\hat{e}^2(0)}{1 + v^2(0)} (1 + \|v_x\|^2) \\ &\quad + 4\|\hat{w}_x\|^2 - \|v_{xx}\|^2 + \frac{1}{2}\|v_{xx}\|^2 + \|\hat{w}_x\|^2 \\ &\quad + \frac{\hat{e}^2(0)}{1 + v^2(0)} (1 + \|v_x\|^2) \\ &\leq -\frac{1}{2}\|v_x\|^2 - \frac{1}{2}\|v_{xx}\|^2 + 5\|\hat{w}_x\|^2 + l_1\|v_x\|^2 + l_1, \end{aligned} \quad (46)$$

where l_1 is a generic function of time in $\mathcal{L}_1 \cap \mathcal{L}_\infty$. Using the following Lyapunov function for the $\hat{w}$ -system,

$$V_{\hat{w}} = \frac{1}{2} \int_0^1 \hat{w}^2(x) dx \quad (47)$$

we get

$$\begin{aligned} \dot{V}_{\hat{w}} &= - \int_0^1 \hat{w}_x^2 dx + \hat{e}(0) \int_0^1 \beta \hat{w} dx + \dot{\hat{g}} \int_0^1 \hat{w} v dx \\ &\quad + \dot{\hat{g}} \int_0^1 \hat{w}(x) \int_0^x \alpha(x - y)(\hat{g}v(y) + \hat{w}(y)) dy dx \\ &\leq -\|\hat{w}_x\|^2 + \frac{c_1}{2}\|\hat{w}\|^2 + \frac{\beta_0^2}{2c_1} \frac{\hat{e}^2(0)}{1 + v^2(0)} (1 + \|v_x\|^2) \\ &\quad + \frac{|\dot{\hat{g}}|^2(1 + \alpha_0 g_0)^2}{2c_1} \|v\|^2 + c_1\|\hat{w}\|^2 + \frac{|\dot{\hat{g}}|^2 \alpha_0^2}{2c_1} \|\hat{w}\|^2 \\ &\leq -(1 - 6c_1)\|\hat{w}_x\|^2 + l_1\|\hat{w}\|^2 + l_1\|v_x\|^2 + l_1. \end{aligned} \quad (48)$$

Choosing $c_1 = 1/24$ and using a Lyapunov function $V = V_{\hat{w}} + (1/20)V_v$, we get

$$\begin{aligned} \dot{V} &\leq -\frac{1}{2}\|\hat{w}_x\|^2 - \frac{1}{40}\|v_x\|^2 - \frac{1}{40}\|v_{xx}\|^2 \\ &\quad + l_1\|\hat{w}\|^2 + l_1\|v_x\|^2 + l_1 \\ &\leq -\frac{1}{4}V + l_1V + l_1 \end{aligned} \quad (49)$$

and by Lemma A.2 we obtain $\|\hat{w}\|, \|v\|, \|v_x\| \in \mathcal{L}_2 \cap \mathcal{L}_\infty$. Using these properties we get

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|\hat{w}_x\|^2 &\leq -\|\hat{w}_{xx}\|^2 + \beta_0|\hat{e}(0)|\|\hat{w}_{xx}\| \\ &\quad + |\dot{\hat{g}}|\|\hat{w}_{xx}\|((1 + \alpha_0 g_0)\|v\| + \alpha_0\|\hat{w}\|) \\ &\leq -\frac{1}{8}\|\hat{w}_x\|^2 + l_1, \end{aligned} \quad (50)$$

so that $\|\hat{w}_x\| \in \mathcal{L}_2 \cap \mathcal{L}_\infty$.

¹We drop the dependence on time in the proofs to reduce notational burden.

D. Regulation

Using the fact that $\|v_x\|$, $\|\hat{w}_x\|$ are bounded we get

$$\left| \frac{d}{dt} (\|v\|^2 + \|\hat{w}\|^2) \right| \leq l_1 \|\hat{w}_x\|^2 + l_1 \|v_x\|^2 + l_1 < \infty. \quad (51)$$

By Barbalat's lemma $\|\hat{w}\| \rightarrow 0$, $\|v\| \rightarrow 0$. From (A.2) we have $\|\eta\| \rightarrow 0$ and $\|\eta_x\|$ is bounded. Since $u = e + qv + \eta$, we get $\|u\| \rightarrow 0$ and $\|u_x\|$ is bounded. Finally, using Agmon's inequality we get

$$\lim_{t \rightarrow \infty} \max_{x \in [0,1]} |u(x,t)| \leq \lim_{t \rightarrow \infty} (2\|u\| \|u_x\|)^{1/2} = 0. \quad (52)$$

IV. BENCHMARK PLANT WITH UNKNOWN PARAMETER IN THE BOUNDARY CONDITION

Consider the following plant

$$u_t(x,t) = u_{xx}(x,t), \quad (53)$$

$$u_x(0,t) = -qu(0,t), \quad (54)$$

$$u(1,t) = U(t), \quad (55)$$

where $U(t)$ is the control signal. This is an example of a system with a parametric uncertainty in the boundary condition, a hard-to-stabilize case even with full state feedback with in-domain actuation [1]. With $U(1) = 0$ this PDE is unstable if and only if $q > 1$. The plant can be written in the frequency domain as a transfer function from input $u(1)$ to output $u(0)$:

$$u(0,s) = \frac{\sqrt{s}}{\sqrt{s} \cosh \sqrt{s} - q \sinh \sqrt{s}} u(1,s). \quad (56)$$

Since this transfer function has infinitely many poles and no zeros (at $s = 0$ the transfer function is $1/(1-q)$), this is an infinite relative degree system. One of the poles is unstable and is approximately equal to q^2 as $q \rightarrow +\infty$.

For the case of known q the transformation

$$w(x,t) = u(x,t) - \int_0^x k(x,\xi) u(\xi,t) d\xi \quad (57)$$

was used in [18] to map (53)–(54) into the target system

$$w_t(x,t) = w_{xx}(x,t) - cw(x,t), \quad (58)$$

$$w_x(0,t) = -qw(0,t), \quad (59)$$

$$w(1,t) = 0, \quad (60)$$

which is exponentially stable for $c \geq \max\{q|q|, 0\}$. However, this stability condition cannot be used when q is unknown. Instead, let us use (57) to map (53)–(54) into a different target system,

$$w_t(x,t) = w_{xx}(x,t), \quad (61)$$

$$w_x(0,t) = 0, \quad (62)$$

$$w(1,t) = 0. \quad (63)$$

It can be shown that the kernel $k(x,\xi)$ must satisfy the following conditions:

$$k_{xx} - k_{\xi\xi} = 0, \quad (64)$$

$$k_\xi(x,0) = -qk(x,0), \quad (65)$$

$$k(x,x) = -q. \quad (66)$$

The solution to this PDE is

$$k(x,\xi) = -qe^{q(x-\xi)}. \quad (67)$$

Suppose now that we want to stabilize the plant (53)–(55) when q is unknown. We have the following result.

Theorem 3: Consider the system (53)–(54) with the control

$$u(1,t) = - \int_0^1 \hat{q} e^{\hat{q}(1-\xi)} (\hat{q}v(\xi,t) + \eta(\xi,t)) d\xi, \quad (68)$$

where the update law for $\hat{q}$ is

$$\dot{\hat{q}} = \gamma \frac{(u(0,t) - \hat{q}v(0,t) - \eta(0,t))v(0,t)}{1 + v^2(0,t)}, \quad (69)$$

and the filters $v(x,t)$, $\eta(x,t)$ are defined as

$$v_t(x,t) = v_{xx}(x,t), \quad (70)$$

$$v_x(0,t) = -u(0,t), \quad (71)$$

$$v(1,t) = 0, \quad (72)$$

$$\eta_t(x,t) = \eta_{xx}(x,t), \quad (73)$$

$$\eta_x(0,t) = 0, \quad (74)$$

$$\eta(1,t) = u(1,t). \quad (75)$$

If the closed loop system (53)–(54), (68)–(75) has a classical solution $(u, \hat{q}, v, \eta)$, then for any $\hat{q}(0)$ and any initial conditions $u_0, v_0, \eta_0 \in L_2(0,1)$, the signals $\hat{q}(t)$, $\|u\|$, $\|v\|$, $\|\eta\|$ are bounded and $\|u\|$ is regulated to zero:

$$\lim_{t \rightarrow \infty} \|u\| = 0. \quad (76)$$

In addition, $u(x,t)$ is square integrable in t for all $x \in [0,1]$.

Although the plants considered in Sections II (g -system) and IV (q -system) look quite similar, the adaptive stabilization problem for the latter is substantially harder due to uncertainty in the boundary condition. The proof becomes harder and the end result is a little weaker — L_2 boundedness and regulation instead of pointwise boundedness and regulation.

V. PROOF OF THEOREM 3

A. Target system

Introducing the error $e = u - qv - \eta$ we get an exponentially stable system

$$e_t(x,t) = e_{xx}(x,t), \quad (77)$$

$$e_x(0,t) = 0, \quad (78)$$

$$e(1,t) = 0. \quad (79)$$

The transformation

$$\begin{aligned} \hat{w}(x,t) &= \hat{q}v(x,t) + \eta(x,t) \\ &+ \int_0^x \hat{q} e^{\hat{q}(x-\xi)} (\hat{q}v(\xi,t) + \eta(\xi,t)) d\xi \end{aligned} \quad (80)$$

maps (53)–(54), (68) into the following system (Lemma A.1):

$$\begin{aligned} \hat{w}_t(x,t) &= \hat{w}_{xx}(x,t) + \dot{\hat{q}}^2 e^{\hat{q}x} \hat{e}(0,t) + \dot{\hat{q}}v \\ &+ \dot{\hat{q}} \int_0^x e^{\hat{q}(x-\xi)} (\hat{q}v(\xi,t) + \hat{w}(\xi,t)) d\xi, \end{aligned} \quad (81)$$

$$\hat{w}_x(0,t) = -\hat{q}\hat{e}(0,t), \quad (82)$$

$$\hat{w}(1,t) = 0. \quad (83)$$

B. Adaptive law properties

This step is almost the same as in Section III for the g -system. We take the following equation as a parametric model

$$e(0, t) = u(0, t) - qv(0, t) - \eta(0, t). \quad (84)$$

The estimation error is

$$\hat{e}(0, t) = u(0, t) - \hat{q}v(0, t) - \eta(0, t). \quad (85)$$

Using the gradient update law

$$\dot{\hat{q}} = \gamma \frac{\hat{e}(0, t)v(0, t)}{1 + v^2(0, t)} \quad (86)$$

we get the following properties (as in Lemma 2)

$$\frac{\hat{e}(0, t)}{\sqrt{1 + v^2(0, t)}} \in \mathcal{L}_2 \cap \mathcal{L}_\infty, \quad \tilde{q} \in \mathcal{L}_\infty, \quad \dot{\hat{q}} \in \mathcal{L}_2 \cap \mathcal{L}_\infty. \quad (87)$$

We denote the bound on $\hat{q}$ by q_0 .

C. Boundedness

First we rewrite v -filter as

$$v_t(x, t) = v_{xx}(x, t), \quad (88)$$

$$v_x(0, t) = -\hat{w}(0, t) - \hat{e}(0, t), \quad (89)$$

$$v(1, t) = 0, \quad (90)$$

We have two interconnected systems for $\hat{w}$ and v driven by the signal $\hat{e}(0, t)$ with properties (87). Consider a Lyapunov function

$$V = \frac{1}{2} \int_0^1 \hat{w}^2(x) dx + \frac{1}{2} \int_0^1 v^2(x) dx. \quad (91)$$

We have

$$\begin{aligned} \dot{V} &= -\hat{w}(0)\hat{w}_x(0) - \int_0^1 \hat{w}_x^2 dx + \dot{\hat{q}} \int_0^1 \hat{w}(x)v(x) dx \\ &\quad + \dot{\hat{q}} \int_0^1 \hat{w}(x) \int_0^x e^{\hat{q}(x-\xi)} (\hat{q}v(\xi) + \hat{w}(\xi)) d\xi dx \\ &\quad + \hat{e}(0) \int_0^1 \hat{q}^2 e^{\hat{q}x} \hat{w}(x) dx - v(0)v_x(0) - \int_0^1 v_x^2 dx \\ &\leq -\|\hat{w}_x\|^2 + |\hat{e}(0)|(q_0|\hat{w}(0)| + q_0^2 e^{q_0} \|\hat{w}\|) + c_1 \|\hat{w}\|^2 \\ &\quad + \frac{(1 + q_0 e^{q_0})^2 |\dot{\hat{q}}|^2}{2c_1} \|v\|^2 + \frac{e^{2q_0} |\dot{\hat{q}}|^2}{2c_1} \|\hat{w}\|^2 \\ &\quad - \|v_x\|^2 + \frac{1}{2} \|v_x\|^2 + \frac{1}{2} \|\hat{w}_x\|^2 + |v(0)||\hat{e}(0)|. \quad (92) \end{aligned}$$

Estimates of particular terms:

$$\begin{aligned} q_0 |\hat{e}(0)| |\hat{w}(0)| &\leq q_0 |\hat{w}(0)| \frac{\hat{e}(0)}{\sqrt{1 + v^2(0)}} (1 + |v(0)|) \\ &\leq c_2 \|\hat{w}_x\|^2 + \frac{q_0^2}{4c_2} \frac{\hat{e}^2(0)}{1 + v^2(0)} \\ &\quad + 2q_0 \sqrt{\|\hat{w}\| \|\hat{w}_x\| \|v\| \|v_x\|} \frac{|\hat{e}(0)|}{\sqrt{1 + v^2(0)}} \\ &\leq c_2 \|\hat{w}_x\|^2 + l_1 \\ &\quad + \frac{q_0 |\hat{e}(0)|}{\sqrt{1 + v^2(0)}} (\|\hat{w}\| \|\hat{w}_x\| + \|v\| \|v_x\|) \\ &\leq c_2 \|\hat{w}_x\|^2 + l_1 + c_3 \|\hat{w}_x\|^2 + c_4 \|v_x\|^2 \\ &\quad + q_0^2 \frac{\hat{e}^2(0)}{1 + v^2(0)} \left(\frac{\|v\|^2}{4c_3} + \frac{\|\hat{w}\|^2}{4c_4} \right) \\ &\leq c_2 \|\hat{w}_x\|^2 + c_3 \|\hat{w}_x\|^2 + c_4 \|v_x\|^2 \\ &\quad + l_1 \|v\|^2 + l_1 \|\hat{w}\|^2 + l_1, \quad (93) \end{aligned}$$

$$\begin{aligned} q_0^2 e^{q_0} |\hat{e}(0)| \|\hat{w}\| &\leq q_0^2 e^{q_0} \|\hat{w}\| \frac{\hat{e}(0)}{\sqrt{1 + v^2(0)}} (1 + |v(0)|) \\ &\leq c_5 \|\hat{w}\|^2 + \frac{q_0^4 e^{2q_0}}{4c_5} \frac{\hat{e}^2(0)}{1 + v^2(0)} + c_6 \|v_x\|^2 \\ &\quad + \frac{q_0^4 e^{2q_0}}{4c_6} \frac{\hat{e}^2(0)}{1 + v^2(0)} \|\hat{w}\|^2 \\ &\leq c_5 \|\hat{w}\|^2 + c_6 \|v_x\|^2 + l_1 \|\hat{w}\|^2 + l_1, \quad (94) \end{aligned}$$

$$\begin{aligned} |v(0)| |\hat{e}(0)| &\leq \frac{|v(0)| |\hat{e}(0)|}{1 + v^2(0)} (1 + 2\|v\| \|v_x\|) \\ &\leq \frac{c_7}{2} \|v_x\|^2 + \frac{1}{2c_7} \frac{\hat{e}^2(0)}{1 + v^2(0)} + \frac{c_7}{2} \|v_x\|^2 \\ &\quad + \frac{2}{c_7} \left(\frac{|v(0)| |\hat{e}(0)|}{1 + v^2(0)} \right)^2 \|v\|^2 \\ &\leq c_7 \|v_x\|^2 + l_1 \|v\|^2 + l_1. \quad (95) \end{aligned}$$

In the last inequality we used the fact that $\dot{\hat{q}}^2$ is an l_1 function. We have

$$\begin{aligned} \dot{V} &\leq -\left(\frac{1}{2} - 4c_1 - c_2 - c_3 - 4c_5 \right) \|\hat{w}_x\|^2 + l_1 \|\hat{w}\|^2 \\ &\quad - \left(\frac{1}{2} - c_4 - c_6 - c_7 \right) \|v_x\|^2 + l_1 \|v\|^2 + l_1. \quad (96) \end{aligned}$$

Choosing $4c_1 = c_2 = c_3 = 4c_5 = 1/16$, $c_4 = c_6 = c_7 = 1/12$, we get

$$\dot{V} \leq -\frac{1}{8} V + l_1 V + l_1 \quad (97)$$

and by Lemma A.2 we obtain $\|\hat{w}\|, \|v\| \in \mathcal{L}_2 \cap \mathcal{L}_\infty$.

D. Regulation

It is easy to see from (97) that $\dot{V}$ is bounded from above. By using an alternative to Barbalat's lemma [14, Lemma 3.1] we get $V \rightarrow 0$, that is $\|\hat{w}\| \rightarrow 0$, $\|v\| \rightarrow 0$. From (A.3) we have $\|\eta\| \rightarrow 0$. Since $u = e + qv + \eta$, we get $\|u\| \rightarrow 0$.

By integrating (96) we get $\|\hat{w}_x\|, \|v_x\| \in \mathcal{L}_2$, and from (A.3) $\|\eta_x\| \in \mathcal{L}_2$ and therefore $\|u_x\| \in \mathcal{L}_2$. Square integrability in time of $u(x, t)$ for all $x \in [0, 1]$ follows from Agmon's inequality.

VI. PLANT WITH TWO UNKNOWN PARAMETERS

For the sake of clarity and due to different adaptive regulation properties that can be achieved, we considered two benchmark problems separately. It is also possible to design an output feedback adaptive controller for the combined system

$$u_t(x, t) = u_{xx}(x, t) + gu(0, t), \quad (98)$$

$$u_x(0, t) = -qu(0, t), \quad (99)$$

This system is unstable if and only if $2q + g > 2$. The non-adaptive control law can be designed based on the controllers for separate problems by using the method described in [18, Sec. VIII-E]. We state here the stabilization result without a proof.

Theorem 4: Consider the plant (98)–(99) with the controller

$$u(1) = \int_0^1 \frac{r_1^2 e^{r_1(1-x)} - r_2^2 e^{r_2(1-x)}}{2\sqrt{\hat{g} + \hat{q}^2/4}} (\hat{g}v + \hat{q}p + \eta) dx, \quad (100)$$

where the update laws for $\hat{g}$ and $\hat{q}$ are

$$\dot{\hat{g}} = \gamma_1 \frac{\hat{e}(0)v(0, t)}{1 + v^2(0, t) + p^2(0, t)}, \quad (101)$$

$$\dot{\hat{q}} = \gamma_2 \frac{\hat{e}(0)p(0, t)}{1 + v^2(0, t) + p^2(0, t)}, \quad (102)$$

the input filter is

$$\eta_t = \eta_{xx} \quad (103)$$

$$\eta_x(0) = 0 \quad (104)$$

$$\eta(1) = u(1) \quad (105)$$

and the output filters are

$$\begin{aligned} v_t &= v_{xx} + u(0) & p_t &= p_{xx} \\ v_x(0) &= 0 & p_x(0) &= -u(0) \\ v(1) &= 0 & p(1) &= 0 \end{aligned} \quad (106)$$

with $\hat{e}(0) = u(0) - \hat{g}v(0) - \hat{q}p(0) - \eta(0)$ and

$$r_{1,2} = \frac{\hat{q}}{2} \mp \sqrt{\hat{g} + \frac{\hat{q}^2}{4}}. \quad (107)$$

If the closed loop system (98)–(107) has a classical solution $(u, \hat{g}, \hat{q}, v, p, \eta)$, then for any $\hat{g}(0), \hat{q}(0)$ and any initial conditions $u_0, v_0, p_0, \eta_0 \in L_2(0, 1)$, the signals $\hat{g}(t), \hat{q}(t), \|u\|, \|v\|, \|p\|, \|\eta\|$ are bounded and $\|u\|$ is regulated to zero:

$$\lim_{t \rightarrow \infty} \|u\| = 0. \quad (108)$$

In addition, $u(x, t)$ is square integrable in t for all $x \in [0, 1]$.

Remark 1: If the expression $\hat{g} + \hat{q}^2/4$ becomes negative, $r_{1,2}$ become complex. However, the control gain in (100) remains real and well defined.

VII. SIMULATIONS

We present now the results of closed-loop simulations for the system (98)–(99). The plant parameters are set to $g = 4$ and $q = 2$, with these values the unstable eigenvalue ≈ 10 . For the update laws we take $\hat{g}(0) = 3, \hat{q}(0) = 1$, and $\gamma_1 = \gamma_2 = 15$. The results are shown in Fig. 1–2. We can see that although the instability occurs at the $x = 0$ boundary, the system is successfully regulated to zero by the control from the opposite boundary.

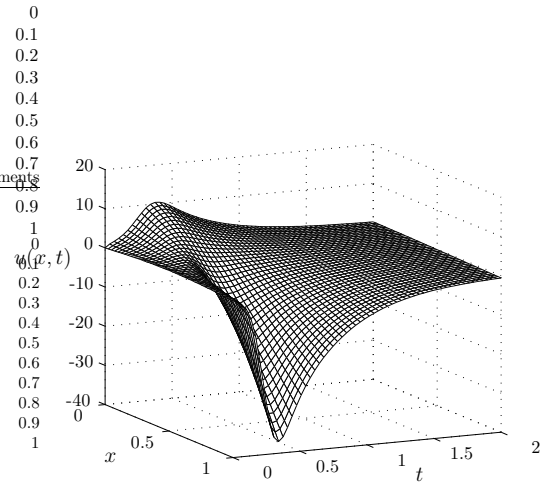


Fig. 1. The state $u(x, t)$ with the adaptive output feedback controller (100).

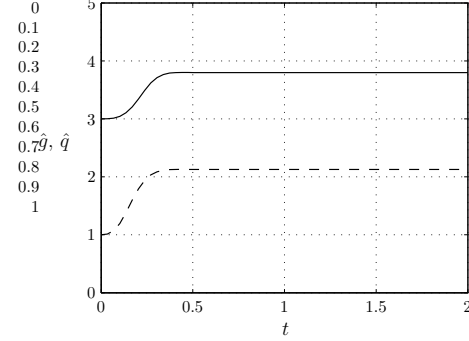


Fig. 2. The parameters $\hat{g}$ (solid) and $\hat{q}$ (dashed). The unknown parameters are set to $g = 4$ and $q = 2$.

VIII. CONCLUSIONS

It would be highly desirable to develop output-feedback versions of the state-feedback designs for reaction-advection-diffusion systems in [12], [20]. While the nonadaptive versions of such results were developed in [19], we have so far not been able to make them adaptive in a way that guarantees global stability. This may contradict the finite-dimensional intuition where output-feedback adaptive designs are available for a very general class of linear systems [8]. However, those designs rely on transfer function representations or particular canonical state space forms—steps that do not easily translate into the PDE framework, particularly if one wants to preserve a finite parametrization.

In general, swapping-based adaptive schemes [20] have a considerably higher dynamic order than Lyapunov-based schemes [12]. However, for the systems studied in this paper we have been able to use the same set of Kreisselmeier filters for both designing an observer and for achieving a static parametrization from which a gradient update law is derived. Thus, the dynamic order for the output feedback designs in the present paper and in [12] is the same. The advantage of the swapping update laws in the present paper is that they are considerably simpler, whereas the advantage of the Lyapunov update laws in [12] is that they are derived from a complete Lyapunov function that incorporates the plant, the filters, and the update law, providing a tighter control over transient performance.

APPENDIX

Lemma A.1: The transformation (26) maps the system (1)–(2), (10) into (27)–(29). The transformation (80) maps the system (53)–(54), (68) into (81)–(83).

Proof: It is easy to check that boundary conditions (28) and (29) are satisfied. Substituting (26) into (1) we get

$$\begin{aligned} \hat{w}_t = & \hat{w}_{xx} + \hat{g}v - \hat{g} \int_0^x \{(k_{\hat{g}}(x, \xi, \hat{g})\hat{g} + k(x, \xi, \hat{g}))v(\xi) \\ & + k_{\hat{g}}(x, \xi, \hat{g})\eta(\xi)\} d\xi + k_{\xi}(x, 0, \hat{g})\hat{e}(0). \end{aligned} \quad (\text{A.1})$$

To express the signal η in terms of v and $\hat{w}$ we use the inverse transformation to (26):

$$\hat{g}v(x, t) + \eta(x, t) = \hat{w}(x, t) - \hat{g} \int_0^x (x - \xi)\hat{w}(\xi, t) d\xi. \quad (\text{A.2})$$

Changing the order of integration and taking the necessary derivatives of $k(x, \xi, \hat{g})$ we come to (27)–(29).

The second part of the lemma is proved in the same way. It is easy to check that (80) satisfies the boundary conditions (82) and (83). Substituting (80) into (53) we get (A.1) but with $\hat{g}$ changed to $\hat{q}$ everywhere. To express the signal η in terms of v and $\hat{w}$ we use the inverse transformation of (80):

$$\hat{q}v(x, t) + \eta(x, t) = \hat{w}(x, t) - \hat{q} \int_0^x \hat{w}(\xi, t) d\xi. \quad (\text{A.3})$$

Changing the order of integration and taking necessary derivatives of $k(x, \xi, \hat{q})$ we come to (81)–(83). ■

Lemma A.2 (Lemma B.6 in [13]): Let v , l_1 , and l_2 be real-valued functions defined on R_+ , and let c be a positive constant. If l_1 and l_2 are nonnegative and in $\mathcal{L}_1$ and satisfy the differential inequality

$$\dot{v} \leq -cv + l_1(t)v + l_2(t), \quad v(0) \geq 0 \quad (\text{A.4})$$

then $v \in \mathcal{L}_{\infty} \cap \mathcal{L}_1$.

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