Integral Operator Feedback for Local Stabilization of a Nonlinear Thermal Convection Loop

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***Abstract*— A state feedback boundary control law that stabilizes fluid flow in a 2D thermal convection loop is presented. The fluid is enclosed between two cylinders, heated from above and cooled from below, which makes it motion unstable for a large enough Rayleigh number. The actuation is at the boundary through rotation (direct velocity actuation) and heat flux (heating or cooling) of the outer boundary. The design is a new approach for this kind of a coupled PDE problem, based on a combination of singular perturbation theory and the backstepping method for infinite dimensional linear systems. Stability is proved by Lyapunov method. Though only a linearized version of the plant is considered in the design, an extensive closed loop simulation study of the nonlinear PDE model shows that the result holds for reasonably large initial conditions.**

I. Introduction

A feedback boundary control law is designed for a thermal fluid confined in a closed convection loop, which is created by heating the lower half of the loop and cooling the upper half. Imposing a temperature gradient induces density differences, which creates a motion that is opposed by viscosity and thermal diffusivity. For a large enough Rayleigh number, which is a function of physical constants of the system, geometry and temperature difference between the top and the bottom, the plant develops an instability that the control law is able to stop.

Other controllers have been designed for this problem, including an LQG controller by Burns et al [2] who formulated the problem, and a nonlinear backstepping design for a discretized version of the plant [1]. The present design is *simpler* than the former, not needing a solution of Ricatti equations, only a linear hyperbolic equation; and more *rigorous* than the latter, which does not hold in the limit when the discrete grid approaches the continuous domain.

Our controller is designed for the linearized plant using a combination of singular perturbation theory and the backstepping method for infinite dimensional linear systems. Singular perturbation theory is a mature area [4] with a wealth of control applications, while backstepping for infinite dimensional linear systems has just been recently developed [7].

Combining both methods it is possible to design a boundary state feedback control law which stabilizes the closed loop; this is proved for a large enough Prandtl number, which is the ratio between kinematic viscosity and thermal difussivity. In this problem, the inverse of the Prandtl number plays the role of the singular perturbation parameter.

We start the paper stating the mathematical model of the convection loop (Section II) and transforming it into a suitable form for application of singular perturbation methods. In Section III we introduce the main assumption of this paper which allows for the application of singular perturbation theory. The quasi-steady state and the reduced model are then found for this problem, and the state feedback controller for velocity is set. Section IV is divided in several subsections, and deals with the reduced system using backstepping to stabilize the PDE. A coordinate transformation (infinite dimensional, represented by a linear Volterra operator) is introduced to transform the original PDE into a stable linear PDE (a heat equation, to be exact). Finding the kernel of the transformation is the main design task in this procedure; a linear hyperbolic PDE which it verifies is derived, and also an equivalent integral equation. Either of them can be used to numerically or symbolically find the kernel. The temperature feedback control law is then presented in terms of this kernel and the state. Finally, the inverse transformation is derived in terms of the direct backstepping transformation. In Section V we present the main result of the paper, a detailed proof of stability based on both singular perturbation and infinite dimensional backstepping theory. The theoretical result is finally suported by a simulation study, presented in Section VI, in which computations of the evolution of the closed loop plant and control effort is shown. In these simulations the Rayleigh number is large enough for the plant to go open loop unstable, but the controller is able to overcome the instability.

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II. Problem Statement

For the convection loop we employ the model derived in [1]. The geometry of the problem is shown in Fig. 1, and consists of fluid confined between two concentric cylinders standing in a vertical plane. The main assumption of this model is that the gap between the cylinders is small compared to the radius of the cylinders, i.e. $R_2 - R_1 \ll R1 < R2$. Then, introducing the Boussinesq approximation, other standard assumptions for the velocity in this 2D configuration, and integrating the momentum equation along circles of fixed radius r, the following plant equations are derived

$$\begin{aligned} v_t &= \frac{\gamma}{2\pi}\int_0^{2\pi} T(t,s,\phi)\cos\phi d\phi \\ &\quad +\nu\left(-\frac{v}{r^2}+\frac{v_r}{r}+v_{rr}\right), && (1) \\ T_t &= -\frac{v}{r}T_\theta \\ &\quad +\chi\left(\frac{T_{\theta\theta}}{r^2}+\frac{T_r}{r}+T_{rr}\right), && (2) \end{aligned}$$

where v stands for velocity, which only depends on the radius r, T for the temperature, which depends on both r and the angle θ, ν is the kinematic viscosity, χ the thermal diffusivity, and $\gamma = g\beta$, with g representing the acceleration due to gravity and β the coefficient of thermal expansion. The boundary conditions are Dirichlet for velocity, with actuation by rotating the outer boundary, while the temperature has Neumann boundary conditions, namely $T_r(t,R_1,\theta) = T_r(t,R_2,\theta) = K\sin\theta$, with K a constant parameter representing the imposed heating and cooling in the boundaries. We actuate the heat flux in the outer boundary, which is more realistic than direct temperature actuation.

Defining $\tau = T - Kr\sin\theta$ we shift the equilibrium to the origin. Then, we introduce nondimensional coordinates and variables, $r' = r/d$, $t' = t\chi/d^2$, $v' = vd/\chi$, $\tau' = \tau/\Delta T$, $R_a = (1/C)\gamma\Delta d^3/2\nu\chi$, $P = \nu/\chi$, where $d = R2 - R1$, $\Delta T = -(4/\pi)K(R_1 + R_2/2)$, C is a constant to be defined, and R_a and P are respectively the Rayleigh and Prandtl numbers. The nondimensional plant equations are, dropping primes, as follows:

$$\begin{aligned} v_t &= \frac{1}{\pi}PR_aC\int_0^{2\pi} \tau(t,s,\phi)\cos\phi d\phi \\ &\quad +P\left(-\frac{v}{r^2}+\frac{v_r}{r}+v_{rr}\right), && (3) \\ \tau_t &= \frac{d\pi}{2(R_1+R_2)}v\cos\theta - \frac{v}{r}\tau_\theta + \frac{\tau_{\theta\theta}}{r^2} \\ &\quad +\frac{\tau_r}{r}+\tau_{rr}. && (4) \end{aligned}$$

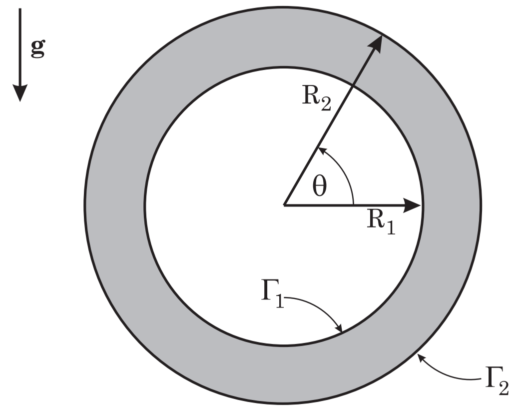

Fig. 1. Convection Loop

The boundary conditions are $v(t,R_1) = 0$, $v(t,R_2) = V(t)$, $\tau_r(t,R_1,\theta) = 0$, and $\tau_r(t,R_2,\theta) = U(t,\theta)$, where V and U is, respectively, the nondimensional velocity and temperature control, and periodic boundary conditions in angle for τ.

Following the lines of the stability study of these equations in [1], the value of C is set so the system is stable for Rayleigh numbers less than unity and unstable otherwise.

Defining $\epsilon = P^{-1}$, $A_1 = R_aC/\pi$, $A_2 = d\pi/2(R_1 + R_2)$, dropping time dependence, and neglecting the nonlinear term, the linearized plant equations are the following:

$$\begin{aligned} \epsilon v_t &= A_1\int_0^{2\pi}\tau(r,\phi)\cos\phi d\phi - \frac{v}{r^2} \\ &\quad +\frac{v_r}{r}+v_{rr}, && (5) \\ \tau_t &= A_2 v\cos\theta + \frac{\tau_{\theta\theta}}{r^2}+\frac{\tau_r}{r}+\tau_{rr}, && (6) \end{aligned}$$

with the same boundary conditions.

We will stabilize this linearized plant around its equilibrium at zero, therefore stabilizing —at least locally— the full nonlinear plant.

III. Reduced Model

For dealing with this plant assume that the parameter ϵ is small enough so we can use singular perturbation theory.

For obtaining the value for the quasi-steady-state, we set $\epsilon = 0$ and solve (5):

$$0 = A_1\int_0^{2\pi}\tau\cos\phi d\phi - \frac{v}{r^2}+\frac{v_r}{r}+v_{rr}. \qquad (7)$$

The general solution for (7) is [6]:

$$\begin{aligned} v &= C_1 r + C_2 \frac{1}{r} - \frac{A_1}{2} \\ &\quad \times \int_{R_1}^{r} \int_0^{2\pi} \frac{r^2 - s^2}{r} \cos\phi \tau(s,\phi) ds d\phi. \quad (8) \end{aligned}$$

The values of C_1 and C_2 depend on the boundary conditions, and therefore on the velocity actuation. The quasi-steady-state, substituted into (6), gives the reduced system, which will be stabilized via the backstepping method. For this procedure to be applicable we need the quasy-steady-state to have a strict integral feedback form, i.e., $v(t,r)$ should not depend on any value of τ after r. Based on this consideration we set the velocity actuation:

$$V = -\frac{A_1}{2} \int_{R_1}^{R_2} \int_0^{2\pi} \frac{R_2^2 - s^2}{R_2} \cos\phi \tau(s,\phi) ds d\phi, \quad (9)$$

and then the final expression for the quasi-steady-state is

$$v = -\frac{A_1}{2} \int_{R_1}^{r} \int_0^{2\pi} \frac{r^2 - s^2}{r} \cos\phi \tau(t,s,\phi) ds d\phi, \quad (10)$$

which plugged into equation (6) renders the following reduced system:

$$\begin{aligned} \tau_t &= -A_{12} \int_{R_1}^{r} \int_0^{2\pi} \frac{r^2 - s^2}{r} \cos\phi \cos\theta \tau(s,\phi) ds d\phi \\ &\quad + \frac{\tau_{\theta\theta}}{r^2} + \frac{\tau_r}{r} + \tau_{rr}, \quad (11) \end{aligned}$$

where $A_{12} = A_1 A_2 / 2$. Note that the reduced system has an integral term which is in the desired strict feedback form.

IV. Backstepping Controller for Temperature

For stabilization of the reduced system we apply the backstepping technique for parabolic PDEs [7], which allows for compensation of integral terms like the one that appears in (11).

A. Target system

The target system is going to be:

$$w_t = \frac{w_{\theta\theta}}{r^2} + \frac{w_r}{r} + w_{rr}, \quad (12)$$

with periodic boundary conditions in θ and the following boundary conditions in r: $w_r(R_1) = 0$, $w_r(R_2) = q w(R_2)$, where q is a negative real number. Note that this system is exponentially stable, which follows from a standard argument taking as a Lyapunov functional the L^2 norm of w.

B. Backstepping transformation

For transforming (11) into (12) we are going to use the following change of variables:

$$w(r,\theta) = \tau(r,\theta) - \int_{R_1}^{r} \int_0^{2\pi} k(r,\theta,s,\phi) \tau(s,\phi) ds d\phi. \quad (13)$$

For calculating the kernel, we introduce (13) into (12) and then we apply integration by parts to arrive at an ultra-hyperbolic PDE which must be verified by the kernel,

$$\begin{aligned} k_{rr} &= -\frac{k_{\theta\theta}}{r^2} - \frac{k_r}{r} + \frac{k_{\phi\phi}}{s^2} - \frac{k_s}{s} + k_{ss} + \frac{k}{s^2} \\ &\quad + A_{12} \left(\int_s^r \int_0^{2\pi} k(r,\theta,\rho,\psi) \frac{\rho^2 - s^2}{\rho} \right. \\ &\quad \left. \times \cos\psi d\rho d\psi - \frac{r^2 - s^2}{r} \cos\theta \right) \cos\phi, (14) \end{aligned}$$

with periodic boundary conditions in both ϕ and ψ, and the following boundary conditions in the radial variables:

$$\begin{aligned} k_s(r,\theta,R_1,\phi) &= \frac{k(r,\theta,R_1,\phi)}{R_1}, \quad (15) \\ k(r,\theta,r,\phi) &= 0. \quad (16) \end{aligned}$$

By inspection of (14) and looking for a solution, we insert the following particular shape of the kernel:

$$k(r,\theta,s,\phi) = \cos\theta \cos\phi \bar{k}(r,s), \quad (17)$$

which verifies the periodic boundary conditions, and substituted in (14) renders:

$$\begin{aligned} \bar{k}_{rr} &= \frac{\bar{k}}{r^2} - \frac{\bar{k}_r}{r} - \frac{\bar{k}_s}{s} + \bar{k}_{ss} - A_{12} \left(\frac{r^2 - s^2}{r} \right. \\ &\quad \left. -\pi \int_s^r \bar{k}(r,\rho) \frac{\rho^2 - s^2}{\rho} d\rho \right), \quad (18) \end{aligned}$$

completely eliminating the angular dependence. Also, introducing $\bar{k} = \sqrt{\frac{s}{r}} \hat{k}(r,s)$ in the last equation we get:

$$\begin{aligned} \hat{k}_{rr} - \hat{k}_{ss} &= \frac{3}{4} \left(\frac{1}{r^2} - \frac{1}{s^2} \right) \hat{k} - A_{12} \left(\frac{r^2 - s^2}{\sqrt{rs}} \right. \\ &\quad \left. -\pi \int_s^r \hat{k}(r,\rho) \frac{\rho^2 - s^2}{\sqrt{\rho s}} d\rho \right), \quad (19) \end{aligned}$$

a hyperbolic partial integro-differential equation, in the region $\mathcal{T} = \{(r,s) : R_1 \le r \le R_2, R_1 \le s \le r\}$ with boundary conditions:

$$\begin{aligned} \hat{k}_s(r,R_1) &= \frac{\hat{k}(r,R_1)}{2R_1}, \quad (20) \\ \hat{k}(r,r) &= 0. \quad (21) \end{aligned}$$

The kernel in this form can be calculated numerically, using a simple finite difference scheme, or rewritten into an integral equation (useful for proving well-posedness and smoothness). This last step can be done introducing the following variables $\xi = r+s$, $\eta = r-s$, and denoting

$$G(\xi,\eta) = \hat{k}(r,s) = \hat{k}\left(\frac{\xi+\eta}{2}, \frac{\xi-\eta}{2}\right) \quad (22)$$

transforming the problem into the following PIDE:

$$\begin{aligned} G_{\xi\eta} &= 3\left(\frac{\xi\eta}{(\xi^2-\eta^2)^2}\right)G - A_{12}\left(\frac{\xi\eta}{2\sqrt{\xi^2-\eta^2}}\right. \\ &\quad -\pi\int_0^{2\eta} G(\xi+\frac{\rho}{2},\eta-\frac{\rho}{2}) \\ &\quad \left.\times\frac{(\rho+\xi-\eta)^2-(\xi-\eta)^2}{2\sqrt{(\rho+\xi-\eta)(\xi-\eta)}}d\rho\right). \end{aligned} \quad (23)$$

This equation can be transformed into a pure integral equation, doing several integrations and employing the boundary conditions, arriving at

$$\begin{aligned} G &= -A_{12}\left(\int_{2R_1+\eta}^{\xi}\int_0^{\eta}\frac{\gamma\sigma}{2\sqrt{\sigma^2-\gamma^2}}d\gamma d\sigma\right. \\ &\quad \left.+\int_0^{\eta}\int_0^{\sigma}e^{\frac{\eta-\sigma}{R_1}}\frac{2(2R_1+\sigma)\gamma}{\sqrt{(2R_1+\sigma)^2-\gamma^2}}d\gamma d\sigma\right) \\ &\quad +\int_{2R_1+\eta}^{\xi}\int_0^{\eta}\left(3\left(\frac{\gamma\sigma}{(\sigma^2-\gamma^2)^2}\right)G(\gamma,\sigma)\right. \\ &\quad +\int_0^{2\gamma}A_{12}\pi G(\sigma+\frac{\rho}{2},\gamma-\frac{\rho}{2}) \\ &\quad \left.\times\frac{(\rho+\sigma-\gamma)^2-(\sigma-\gamma)^2}{2\sqrt{(\rho+\sigma-\gamma)(\sigma-\gamma)}}d\rho\right)d\gamma d\sigma \\ &\quad +\int_0^{\eta}\int_0^{\sigma}e^{\frac{\eta-\sigma}{R_1}}\left(6G(\gamma,\sigma)\frac{(2R_1+\sigma)\gamma}{((2R_1+\sigma)^2-\gamma^2)^2}\right. \\ &\quad +\int_0^{2\gamma}\frac{(\rho+2R_1+\sigma-\gamma)^2-(2R_1+\sigma-\gamma)^2}{\sqrt{(\rho+2R_1+\sigma-\gamma)(2R_1+\sigma-\gamma)}} \\ &\quad \left.\times A_{12}\pi G(2R_1+\sigma+\frac{\rho}{2},\gamma-\frac{\rho}{2})d\rho\right)d\gamma d\sigma. \end{aligned} \quad (24)$$

Using (24) and the same argument as in [7] the following result holds:

Theorem 1: The equation (19) with boundary conditions (20)-(21) has a unique $\mathcal{C}^2(\mathcal{T})$ solution.
Therefore a smooth solution exists for equation (14) with boundary conditions (15)-(16).

C. Control law

Once the kernel is found, it is easy to derive the control law. Substituting the backstepping transformation into the outer boundary condition for the target system,

$$\begin{aligned} &\tau_r(R_2,\theta) - \int_{R_1}^{R_2}\int_0^{2\pi}k_r(R_2,\theta,s,\phi)\tau(s,\phi)dsd\phi \\ &-\int_0^{2\pi}k(R_2,\theta,R_2,\phi)\tau(R_2,\phi)dsd\phi \\ &= q\tau(R_2,\theta) \\ &-q\int_{R_1}^{R_2}\int_0^{2\pi}k(R_2,\theta,s,\phi)\tau(s,\phi)dsd\phi, \end{aligned} \quad (25)$$

and then the control law for the derivative of the temperature at the outer boundary becomes

$$\begin{aligned} U(t,\theta) &= q\tau(R_2,\theta) + \int_{R_1}^{R_2}\int_0^{2\pi}(k_r(R_2,\theta,s,\phi) \\ &\quad -qk(R_2,\theta,s,\phi))\,\tau(s,\phi)dsd\phi. \end{aligned} \quad (26)$$

Note that q is a design parameter that does not enter the kernel equations at any point; it is set externally and enhances stability.

D. Inverse transformation

Having found the backstepping change of variables, we also look for the inverse of it. Postulating it as

$$\tau(r,\theta) = w(r,\theta) - \int_{R_1}^{r}\int_0^{2\pi}l(r,\theta,s,\phi)w(s,\phi)dsd\phi, \quad (27)$$

then, introducing the expression for w in terms of τ an integral equation is found for this inverse kernel; introducing, as it was done for the direct transformation,

$$l(r,\theta,s,\phi) = \cos\theta\cos\phi\bar{l}(r,s), \quad (28)$$

the equation for the inverse transformation is

$$\bar{l}(r,s) = -\bar{k}(r,s) + \pi\int_s^r \bar{l}(r,\rho)\bar{k}(\rho,s)d\rho. \quad (29)$$

Using this integral equation a similar result to Theorem 1 holds for the inverse kernel.

V. Singular Pertubation Analysis for the Entire System

Now that we have derived a control law for the reduced system, we can drop the assumption that $\epsilon = 0$ and instead consider it a small but nonzero parameter, and analyze the stability of the closed loop system. Now the quasi-steady-state solution is no longer the exact solution of the v PDE, but still plays an important role. Calling this previously calculated fast solution v_{ss},

$$v_{ss} = -\frac{A_1}{2}\int_{R_1}^{r}\int_0^{2\pi}\frac{r^2-s^2}{r}\cos\phi\tau(s,\phi)dsd\phi, \quad (30)$$

an error variable z that measures the deviation of the velocity from the fast solution can be introduced:

$$z(t,r) = v(t,r) - v_{ss}(t,r). \tag{31}$$

We start by deriving the PDE that is verified by z:

$$\begin{aligned} \epsilon z_t &= -\frac{z}{r^2} + \frac{z_r}{r} + z_{rr} + \epsilon \frac{A_1}{2} \int_{R_1}^{r} \int_0^{2\pi} \frac{r^2 - s^2}{r} \\ &\quad \times \cos\phi \tau_t(s,\phi) ds d\phi, \end{aligned} \tag{32}$$

where we have used the fact that v_{ss} verifies equation (7). This PDE without the last term is usually referred to as the Boundary Layer model; note that it is exponentially stable. The last term of (32) can be expressed in terms of τ introducing its differential equation and applying integration by parts and the τ boundary conditions, and then in terms w by using the inverse kernel.

The overall plant written in (z,w) variables has the form

$$\begin{aligned} \epsilon z_t &= -\frac{z}{r^2} + \frac{z_r}{r} + z_{rr} + \epsilon \left(\int_{R_1}^{r} Q_{zz}(r,s) z(s) ds \right. \\ &\quad + \int_{R_1}^{r} \int_0^{2\pi} Q_{zw}^1(r,s,\phi) w(s,\phi) ds d\phi \\ &\quad + \int_0^{2\pi} Q_{zw}^2(r,\phi) w(r,\phi) d\phi \\ &\quad \left. + \int_0^{2\pi} Q_{zw_0}(r,\phi) w(R_1,\phi) d\phi \right), \end{aligned} \tag{33}$$

$$\begin{aligned} w_t &= \frac{w_{\theta\theta}}{r^2} + \frac{w_r}{r} + w_{rr} + Q_{wz}^2(r,\theta) z(r) \\ &\quad + \int_{R_1}^{r} Q_{wz}^1(r,s,\theta) z(s) ds, \end{aligned} \tag{34}$$

together with boundary conditions $z(R_1) = z(R_2) = 0$, $w_r(R_1,\theta) = 0$, $w_r(R_2,\theta) = q w(R_2,\theta)$, and periodic angular boundary conditions for w. For simplicity, we have denoted the following kernels:

$$Q_{zz} = A_{12}\pi \frac{r^2 - s^2}{r}, \tag{35}$$

$$\begin{aligned} Q_{zw}^1 &= -\frac{A_1}{2} \cos\phi \left(2\pi \bar{l}(r,s) \right. \\ &\quad \left. + A_{12} \cos\phi \frac{r^4 - s^4 - 4r^2 s^2 \ln\frac{r}{s}}{4r} \right), \end{aligned} \tag{36}$$

$$Q_{zw}^2 = A_1 \cos\phi, \tag{37}$$

$$Q_{wz}^1 = -A_2 \pi \cos\theta \bar{k}(r,s), \tag{38}$$

$$Q_{wz}^2 = A_2 \cos\theta, \tag{39}$$

$$Q_{zw_0} = \frac{A_1}{2} \frac{r^2 + R_1^2}{r R_1} \cos\phi. \tag{40}$$

For the stability proof we are going to use the following energy Lyapunov functionals:

$$E_w(t) = \frac{1}{2} \int_0^{2\pi} \int_{R_1}^{R_2} w^2(t,s,\phi) s ds d\phi, \tag{41}$$

$$E_z(t) = \frac{1}{2} \int_{R_1}^{R_2} z^2(t,s) s ds. \tag{42}$$

The time derivative of E_w can be bounded in the following way:

$$\begin{aligned} \frac{dE_w}{dt} &\leq -\int_0^{2\pi} \int_{R_1}^{R_2} \frac{w_\theta^2}{s^2} s ds d\phi \\ &\quad - \frac{1}{2} \int_0^{2\pi} \int_{R_1}^{R_2} w_r^2 s ds d\phi \\ &\quad + \left(q + \frac{R_2}{4(R_2 - R_1)}\right) \int_0^{2\pi} R_2 w(R_2,\phi)^2 d\phi \\ &\quad - \frac{1}{8(R_2 - R_1)^2} \int_0^{2\pi} \int_{R_1}^{R_2} w^2 s ds d\phi \\ &\quad + \beta_1 \left(\int_0^{2\pi} \int_{R_1}^{R_2} w^2(s,\phi) s ds d\phi \right)^{\frac{1}{2}} \\ &\quad \times \left(\int_{R_1}^{R_2} z^2(s) s ds \right)^{\frac{1}{2}}, \end{aligned} \tag{43}$$

where

$$\begin{aligned} \beta_1 &= \sqrt{2\pi} \left(\|Q_{wz}^2\|_\infty + \sqrt{(R_2^2 - R_1^2) \ln \frac{R_2}{R_1}} \right. \\ &\quad \left. \times \|Q_{wz}^1\|_\infty \right). \end{aligned} \tag{44}$$

The time derivative of E_z has the following bound:

$$\begin{aligned} \frac{dE_z}{dt} &\leq -\left(\frac{1}{\epsilon R_2^2} - \gamma \right) \int_{R_1}^{R_2} z^2 s ds \\ &\quad + \beta_2 \left(\int_0^{2\pi} \int_{R_1}^{R_2} w^2(s,\phi) s ds d\phi \right)^{\frac{1}{2}} \\ &\quad \times \left(\int_{R_1}^{R_2} z^2(s) s ds \right)^{\frac{1}{2}} \\ &\quad + \beta_3 \int_0^{2\pi} w^2(t,R_2,\phi) d\phi \\ &\quad + \frac{1}{2} \int_{R_1}^{R_2} \int_0^{2\pi} w_r^2(t,r,\phi) s ds d\phi, \end{aligned} \tag{45}$$

where

$$\begin{aligned}\beta_2 &= \sqrt{2\pi}\left(\|Q_{zw}^2\|_\infty\right. \\ &\left.+\sqrt{(R_2^2-R_1^2)\ln\frac{R_2}{R_1}}\|Q_{zw}^1\|_\infty\right), \quad (46)\\ \beta_3 &= -\frac{R_2}{2}(q+\frac{R_2}{4(R_2-R_1)}), \quad (47)\\ \gamma &= \gamma_1+2\gamma_2+\frac{\gamma_3}{\beta_3}, \quad (48)\\ \gamma_1 &= \sqrt{(R_2^2-R_1^2)\ln\frac{R_2}{R_1}}\|Q_{zz}\|_\infty, \quad (49)\\ \gamma_2 &= \pi^2(R_2-R_1)^2\|Q_{zw_0}\|_\infty^2 R_2, \quad (50)\\ \gamma_3 &= \frac{\pi^2}{2}(R_2-R_1)\|Q_{zw_0}\|_\infty^2. \quad (51)\end{aligned}$$

In both of the previous calculations repeated use of Cauchy-Schwartz's and Young's inequality has been made, and a version of Poincare's inequality tailored for this system has been employed (see the Appendix). Now, selecting the following Lyapunov function,

$$E(t) = E_w(t) + E_z(t), \quad (52)$$

we find its time derivative to be:

$$\begin{aligned}\frac{dE(t)}{dt} \le & -\int_0^{2\pi}\int_{R_1}^{R_2}\frac{w_\theta^2}{s^2}sdsd\phi \\ &+\left(\frac{R_2}{2}(q+\frac{R_2}{4(R_2-R_1)})\right) \\ &\times\int_0^{2\pi} w(t,R_2,\phi)^2 d\phi \\ &-\frac{1}{8(R_2-R_1)^2}\int_0^{2\pi}\int_{R_1}^{R_2} w^2 sdsd\phi \\ &+(\beta_1+\beta_2)\left(\int_{R_1}^{R_2} z^2(s)sds\right)^{\frac{1}{2}} \\ &\times\left(\int_0^{2\pi}\int_{R_1}^{R_2} w^2(s,\phi)sdsd\phi\right)^{\frac{1}{2}} \\ &-\left(\frac{1}{\epsilon R_2^2}-\gamma\right)\int_{R_1}^{R_2} z^2 sds. \quad (53)\end{aligned}$$

In this equation we have to choose q and ϵ so the final expression is negative definite. We set the first as:

$$q = -1 - \frac{R_2}{4(R_2-R_1)}. \quad (54)$$

For finding a value for ϵ, we identify the quadratic form which appears in (53) and call its matrix A:

$$A = \begin{pmatrix} \frac{1}{8(R_2-R_1)^2} & -\frac{\beta_1+\beta_2}{2} \\ -\frac{\beta_1+\beta_2}{2} & \frac{1}{\epsilon R_2^2}-\gamma \end{pmatrix}. \quad (55)$$

Our interest is to find the maximum possible value of ϵ so $A > 0$. From Sylvester's criterion we get the condition for A to be positive definite:

$$0 < \left(\frac{1}{\epsilon R_2^2}-\gamma\right) - 2(R_2-R_1)^2(\beta_1+\beta_2)^2. \quad (56)$$

Solving for $1/\epsilon$,

$$\frac{1}{\epsilon} > 2R_2^2(R_2-R_1)^2(\beta_1+\beta_2)^2 + R_2^2\gamma. \quad (57)$$

Substituting γ, we can define an upper bound for ϵ:

$$\begin{aligned}\frac{1}{\epsilon *} &= 2R_2^2(R_2-R_1)^2\ (\beta_1+\beta_2)^2 \\ &+R_2^2\left(\gamma_1+2\gamma_2+2\frac{\gamma_3}{R_2}\right). \quad (58)\end{aligned}$$

Note that this bound is a function which depends exclusively of the geometry and physical parameters of the plant.

This establishes asymptotic stability for the plant in the z, w coordinates, when $\epsilon \in (0, \epsilon^*)$. Stability in the original coordinates follows from the following inequalities:

$$\begin{aligned}\|\tau\|_2^2 \le & \|w\|_2^2\left(1+\|\bar{l}\|_\infty\right. \\ &\left.\times\sqrt{\frac{\pi(R_2-R_1)(R_2^2-R_1^2)}{R_1}}\right)^2 \quad (59)\end{aligned}$$

and

$$\begin{aligned}\|v\|_2^2 \le & 2\|z\|_2^2 + 2\|w\|_2^2\left(\frac{(R_2-R_1)(R_2^2-R_1^2)^2}{R_1^3}\right) \\ &\times\left(1+\|\bar{l}\|_\infty\right. \\ &\left.\times\sqrt{\frac{\pi(R_2-R_1)(R_2^2-R_1^2)}{R_1}}\right)^2 \quad (60)\end{aligned}$$

which are derived taking norm in the respective definitions. We have just proved the following theorem:

Theorem 2: For a sufficiently small ϵ, the system (5)-(6) with inner boundary conditions $v(R_1) = 0$, $\tau_r(R_1,\theta) = 0$, outer boundary conditions $v(R_2) = V(t)$, $\tau(R_2,\theta) = U(t,\theta)$ where V and U are specified by control laws (9) and (26) respectively, has unique classical solutions and is exponentially stable at the origin in the L^2 sense, that is, there exist positive constants M and α, independent of the initial conditions,

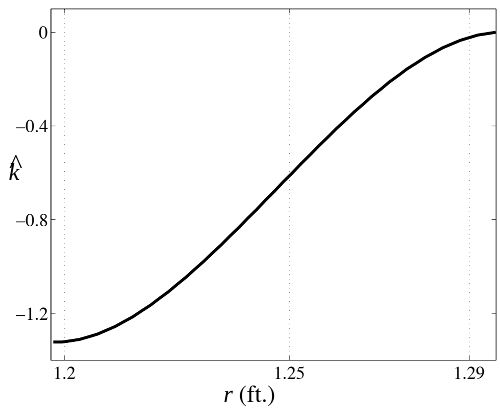

Fig. 2. Kernel for temperature actuaction

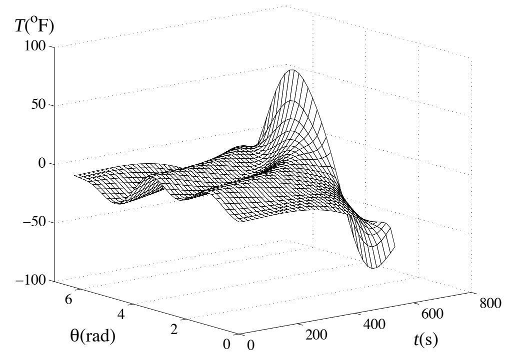

Fig. 3. Open loop evolution of temperature at radius $r = 1.21$.

such that

$$\begin{aligned} &\int_{R_1}^{R_2} \left(v^2(t,s) + \int_0^{2\pi} \tau^2(t,s,\phi) d\phi \right) s ds \\ \leq \quad & M \mathrm{e}^{-\alpha t} \int_{R_1}^{R_2} \Big(v^2(0,s) \\ & + \int_0^{2\pi} \tau^2(0,s,\phi) d\phi \Big) s ds \end{aligned} \tag{61}$$

The proof of existence and uniqueness of classical solutions has been skipped, but follows from standard arguments due to linearity of (5)-(6) and due to the form of the boundary conditions.

VI. Simulation Study

We show a prototypical simulation case. For numerical computations, a spectral method combined with the well-known Crank-Nicholson method (see, for example, [3]) has been used, using the following numerical values:$R_1 = 1.1975$ ft., $R_2 = 1.2959$ ft., $P = 8.06$, $Ra = 50$, $C = 7.8962 \times 10^3$, $K = 5$ °F/ft.

In Fig. 2 the shape of the control kernel, $\hat{k}(R_2, s)$ is plotted, showing that information near the inner boundary is given more weight in the control law, which makes sense as the boundary controller is on the opposite side and therefore has to react more agressively to compensate fluctuations of temperature in the interior part of the domain.

Fig. 3 is an open loop simulation of temperature, which grows very positive or very negative, depending on the angle, eventually becoming too large for further computations. In Fig. 4 closed loop simulations of the plant are shown in physical variables (velocity and temperature) showing how they reach the equilibrium state quickly, staying there afterwards. The magnitude of heat flux control is also shown, while the velocity actuation can be seen just looking at the $r = R_2$ section in the velocity plot, which is the outer cylinder rotation imposed by the control law.

VII. Conclusions and Future Work

A combination of singular perturbation theory and backstepping for parabolic PDEs has been successfully employed to stabilize a thermal fluid confined in a convection loop which is open loop unstable. The equations of the plant consists on a 1D evolution equation nonlinearly coupled with a 2D evolution equation, which is complex enough to make very hard any analytical attempt to design a boundary controller and then prove stability of the closed loop. Our controller, based on the singular perturbation assumption of a large Prandtl number —which is true for many fluids— succeeds at this, at least for the linearized plant, employing rotation of the outer boundary and a Neumann type of boundary state feedback controller for the temperature (heat flux actuation) which is a realistic setting. This controller is found using a backstepping design procedure, which is both conceptually and computationally simple, requiring only to solve a hyperbolic linear equation for obtaining the control law. A simulation study has been done to show how the plant is stabilized and the magnitude of the control exerted through the boundary.

Full state feedback can be used in a CFD setting in which the state is known at every point of the domain, but in a real physical experiment this is not possible. Future research includes developing an output feedback controller, which will need to measure the temperature only at one or both of the boundaries.

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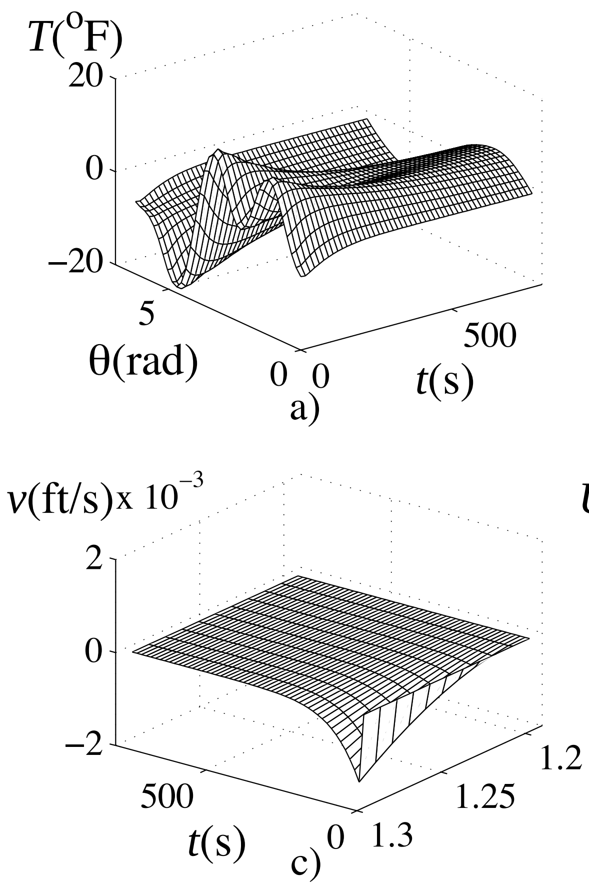

Fig. 4. Closed loop simulation. a) temperature at radius $r = 1.21$ ft. b) temperature at radius $r = 1.25$ ft. c) velocity d) temperature control effort

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Appendix

The following lemma has been used in Section V.

Lemma 1: For any $\tau \in \mathcal{C}^1[R_1, R_2]$ the following inequality holds:

$$\begin{aligned} & \int_0^{2\pi} \int_{R_1}^{R_2} \tau^2(r,\theta) r dr d\theta \\ \leq \quad & 2R_2(R_2 - R_1) \int_0^{2\pi} \tau^2(R_2,\theta) d\theta \\ & + 4(R_2 - R_1)^2 \int_0^{2\pi} \int_{R_1}^{R_2} \tau_r^2(r,\theta) r dr d\theta. \end{aligned} \quad (62)$$

We skip the proof which is standard, see, e.g., [5].