## 1 Review

We did several things last lecture.

- 1. We went over the nonlinear Navier Stokes equations
- 2. We linearized the NS equations around the equilibrium profile
- 3. We took the Fourier Transform of the system in the x and z directions
- 4. We solved for the pressure

p

5. We derived  $V_c$  to put p in backstepping form

## 2 Y and $\omega$ and backstepping

From last time we have

$$= \beta \int_{0}^{y} V(t,\eta)(2\eta-1)\sinh(\alpha(y-\eta))d\eta + \frac{1}{Re} V_{yy}(0) \frac{\cosh(\alpha y) - \cosh(\alpha(1-y))}{\sinh(\alpha)}$$
(1)

and

$$u_t = \frac{1}{Re} (-\alpha^2 u + u_{yy}) + \frac{\beta}{2} y(y-1)u + 4(2y-1)V - 2\pi i k_x p$$
(2)

$$W_t = \frac{1}{Re}(-\alpha^2 W + W_{yy}) + \frac{\beta}{2}y(y-1)W - 2\pi ik_z p$$
(3)

$$V_t = \frac{1}{Re}(-\alpha^2 V + V_{yy}) + \frac{\beta}{2}y(y-1)V - p_y$$
(4)

To continue our derivation of controllers, we're going to do a standard transformation in fluid stability analysis and look at the normal velocity and the vorticity with one caveat. We're going to look at  $V_y$  instead of V.

$$Y = -V_y = 2\pi i (k_x u + k_z W) \tag{5}$$

$$\omega = 2\pi i (k_z u - k_x W) \,. \tag{6}$$

We establish the second equality by recalling the incompressible condition

$$2\pi i k_x u + 2\pi i k_z W + V_y = 0 \tag{7}$$

Stabilizing these two systems stabilizes the entire linearized system. We see this by looking at the inverse transformations.

$$u = \frac{1}{2\pi i} \frac{k_x Y + k_z \omega}{k_x^2 + k_z^2}$$
(8)

$$W = \frac{1}{2\pi i} \frac{k_z Y - k_x \omega}{k_x^2 + k_z^2} \tag{9}$$

$$V(y) = -\int_0^y Y(\eta) d\eta.$$
 (10)

So now we want to see what the dynamics of Y and  $\omega$  look like. Using (2) and (3) we see

$$Y_t = \frac{1}{Re} (-\alpha^2 Y + Y_{yy}) + \frac{\beta}{2} y(y-1)Y + 8\pi i k_x (2y-1)V + \alpha^2 p, \quad (11)$$

$$\omega_t = \frac{1}{Re}(-\alpha^2\omega + \omega_{yy}) + \frac{\beta}{2}y(y-1)\omega + 8\pi ik_z(2y-1)V$$
(12)

By plugging in our solution to the pressure we arrive at

$$Y_{t} = \frac{1}{Re} (-\alpha^{2}Y + Y_{yy}) + \frac{\beta}{2}y(y-1)Y + 8\pi ik_{x}(2y-1)V + \alpha^{2} \Big\{ \beta \int_{0}^{y} V(t,\eta)(2\eta-1)\sinh(\alpha(y-\eta))d\eta + \frac{1}{Re} V_{yy}(0) \frac{\cosh(\alpha y) - \cosh(\alpha(1-y))}{\sinh(\alpha)} \Big\},$$
(13)

$$\omega_t = \frac{1}{Re} (-\alpha^2 \omega + \omega_{yy}) + \frac{\beta}{2} y(y-1)\omega + 8\pi i k_z (2y-1)V$$
(14)

Remembering that  $V_y = -Y$  and  $V = -\int_0^y Y(\eta) d\eta$  we now see

$$Y_{t} = \frac{1}{Re} (-\alpha^{2}Y + Y_{yy}) + \frac{\beta}{2} y(y-1)Y - 8\pi i k_{x}(2y-1) \int_{0}^{y} Y(\eta) d\eta$$
$$+ \alpha^{2} \left\{ -\beta \int_{0}^{y} \left( \int_{0}^{\eta} Y(\sigma) d\sigma \right) (2\eta-1) \sinh(\alpha(y-\eta)) d\eta$$
$$- \frac{1}{Re} Y_{y}(0) \frac{\cosh(\alpha y) - \cosh(\alpha(1-y))}{\sinh(\alpha)} \right\},$$
(15)

$$\omega_t = \frac{1}{Re} (-\alpha^2 \omega + \omega_{yy}) + \frac{\beta}{2} y(y-1)\omega - 8\pi i k_z (2y-1) \int_0^y Y(\eta) d\eta \quad (16)$$

If we define the following:

$$\epsilon = \frac{1}{Re}, \qquad (17)$$

$$\phi(y) = \frac{\beta}{2}y(y-1) = 8\pi i k_x y(y-1), \qquad (18)$$

$$f(y,\eta) = 8i \left\{ \pi k_x (2y-1) - 4\pi \frac{k_x}{\alpha} \sinh(\alpha(y-\eta)) - 2\pi k_x (2n-1) \cosh(\alpha(y-n)) \right\}$$
(19)

$$-2\pi\kappa_x(2\eta-1)\cosh(\alpha(y-\eta))\},$$
(19)
$$\cosh(\alpha(1-\eta)) - \cosh(\alpha\eta)$$

$$g(y) = \epsilon \alpha^2 \frac{\cosh(\alpha(1-y)) - \cosh(\alpha y)}{\sinh(\alpha)}, \qquad (20)$$

$$h(y) = -8\pi k_z i(2y - 1)$$
(21)

then our Y and  $\omega$  systems satisfy the following

$$Y_t = \epsilon(-\alpha^2 Y + Y_{yy}) + \phi(y)Y + g(y)Y_y(t,0) + \int_0^y f(y,\eta)Y(t,\eta)d\eta(22)$$

$$\omega_t = \epsilon(-\alpha^2 \omega + \omega_{yy}) + \phi(y)\omega + h(y) \int_0^y Y(\eta) d\eta$$
(23)

$$Y(t,0) = 0, \quad Y(t,1) = k_x u(t,1) + k_x W(t,1)$$
(24)

$$\omega(t,0) = 0, \quad \omega(t,1) = k_z u(t,1) - k_x W(t,1)$$
(25)

But wait - we can see that we already know how to find a controller for the Y system as it is autonomous, but we dont know yet how to find a controller for the  $\omega$  system as it is not autonomous. We have a cascade system, so what do we do? Well, it turns out, since the viscous coefficient here is the same, we can use the following transformations to both decouple and stabilize the system.

$$\Psi = Y - \int_0^y K(k_x, k_z, y, \eta) Y(t, k_x, k_z, \eta) d\eta, \qquad (26)$$

$$\Omega = \omega - \int_0^y \Gamma(k_x, k_z, y, \eta) Y(t, k_x, k_z, \eta) d\eta, \qquad (27)$$

(28)

The first transformation is the standard one we've seen, and it stabilizes the two systems. The second decouples Y and  $\omega$ . As  $\omega$  is stable without the Y forcing, all we are doing is decoupleing it from Y. (Note that  $\phi$  does not cause instability because it is a purely imaginary valued function). These transformations take us to the following target systems

$$\Psi_t = \epsilon(-\alpha^2 \Psi + \Psi_{yy}) + \phi(y)\Psi, \qquad (29)$$

$$\Omega_t = \epsilon(-\alpha^2 \Omega + \Omega_{yy}) + \phi(y)\Omega$$
(30)

$$\Psi(t,0) = \Psi(t,1) = 0, \qquad (31)$$

$$\Omega(t,0) = \Omega(t,1) = 0,$$
(32)

both of these are autonomous and stable systems. Again, they have an extra term that might be unfamiliar  $(\phi(y)\Psi$  and  $\phi(y)\Omega)$ , but this term does not cause instability as the coefficient is purely imaginary. We see this in a homework problem.

You know how to derive the pdes for K and Gamma, but Ill write them down for you.

$$\epsilon K_{yy} = \epsilon K_{\eta\eta} - f(y,\eta) + (\phi(\eta) - \phi(y))K + \int_{\eta}^{y} K(y,\xi)f(\xi,\eta)d\xi, \qquad (33)$$

$$\epsilon K(y,0) = \int_0^y K(y,\eta)g(\eta)d\eta - g(y), \qquad (34)$$

$$\epsilon K(y,y) = -g(0). \tag{35}$$

$$\epsilon \Gamma_{yy} = \epsilon \Gamma_{\eta\eta} - h(y) + (\phi(\eta) - \phi(y))\Gamma + \int_{\eta}^{y} \Gamma(y,\sigma) f(\sigma,\eta) d\sigma$$
(36)

$$\epsilon \Gamma(y, y) = 0 \tag{37}$$

$$\epsilon \Gamma(y,0) = \int_0^y \Gamma(y,\eta) g(\eta) d\eta$$
(38)

Now we have controllers of the Y and omega systems that look like this.

$$Y(1) = \int_{0}^{1} K(1,\eta) Y(\eta) d\eta$$
 (39)

$$\omega(1) = \int_0^1 \Gamma(1,\eta) Y(\eta) d\eta \tag{40}$$

Again, the  $\omega$  controller only needs to use Y as it decouples  $\omega$  from Y. Now what do we do with these controllers? well, we use the inverse transforms we wrote down earlier

$$u = \frac{1}{2\pi i} \frac{k_x Y + k_z \omega}{k_x^2 + k_z^2} \tag{41}$$

$$W = \frac{1}{2\pi i} \frac{k_z Y - k_x \omega}{k_x^2 + k_z^2} \tag{42}$$

(43)

to find controllers for **u** and **W**.

$$U_c = \frac{-2\pi i}{\alpha^2} \left( k_x Y(t,1) + k_z \omega(t,1) \right)$$
(44)

$$W_c = \frac{-2\pi i}{\alpha^2} \Big( k_z Y(y,1) - k_x \omega(t,1) \Big)$$
(45)

(46)

and then use the forward tranformation

$$Y = 2\pi i (k_x u + k_z W) \tag{47}$$

to get  $U_c$  and  $W_c$  in terms of u and W.

$$U_{c} = \int_{0}^{1} \frac{4\pi^{2}}{\alpha^{2}} \left( k_{x}K(1,\eta) + k_{z}\Gamma(1,\eta) \right) \left( k_{x}u(t,\eta) + k_{z}W(t,\eta) \right) d\eta \quad (48)$$
$$W_{c} = \int_{0}^{1} \frac{4\pi^{2}}{\alpha^{2}} \left( k_{z}K(1,\eta) - k_{x}\Gamma(1,\eta) \right) \left( k_{x}u(t,\eta) + k_{z}W(t,\eta) \right) d\eta. \quad (49)$$

So to take this back to physical space we first want to check that the energy is not infinite. One way to do this is to remember Parseval's theorem - the energy in wavespace is the same in physical space - and then to cut the wavenumbers that we actuate off at a certain number. So we can cut off high wavenumbers because the paramaterized system is stable when  $k_x$  and  $k_z$  are high. Now  $U_c$  in physical space becomes

$$U_{c} = \int_{0}^{1} \int_{-M}^{M} \int_{-M}^{M} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{4\pi^{2}}{k_{x}^{2} + k_{z}^{2}} \times \left(k_{x}K(k_{x},k_{z},1,\eta) + k_{z}\Gamma(k_{x},k_{z},1,\eta)\right) \left(k_{x}u(t,\tilde{x},\tilde{z},\eta) + k_{z}W(t,\tilde{x},\tilde{z},\eta)\right) \times e^{2\pi i \left(k_{x}(x-\tilde{x}) + k_{z}(z-\tilde{z})\right)} d\tilde{x}d\tilde{z}dk_{x}dk_{z}d\eta$$

$$(50)$$

where M is our cutoff. Physically, actuating small wavenumbers means actuators can be spaced further apart and the changes in space are relatively slow. Actuating high wavenumbers means that actuators spaced close together will change quickly in space.