1 3D Poiseuille Flow

Over the next two lectures we will be going over the stabilization of the 3-D Poiseuille flow. For those of you who havent had fluids, that means we have a channel of fluid that is infinite in the x and z directions. It is bounded at y = 0 by a wall and also at y = 1. In our scenario, this fluid is incompressible and has a constant density. What I'm going to do is show you the steps to go from the full uncontrolled flow governed by the Navier Stokes equations to a controlled flow that looks like its governed by heat equations. The controllers we're going to use to do this are the following: (Note they are in Fourier Space)

$$(\tilde{V}_{c})_{t} = \frac{1}{Re} \left(\tilde{V}_{yy}(1) - \tilde{V}_{yy}(0) - \alpha^{2} \tilde{V}_{c} \right) -\beta \int_{0}^{1} \tilde{V}(t,\eta)(2\eta - 1) \cosh(\alpha(1-\eta))d\eta ,$$
(1)

$$= \frac{1}{Re} \left(2\pi i \left(k_{x} \left(\tilde{u}_{y}(t,0) - \tilde{u}_{y}(t,1) \right) + k_{z} \left(\tilde{W}_{y}(t,0) - \tilde{W}_{y}(t,1) \right) \right) - \alpha^{2} \tilde{V}_{c} \right) -\beta \int_{0}^{1} \tilde{V}(t,\eta)(2\eta - 1) \cosh(\alpha(1-\eta))d\eta ,$$
(2)

$$\tilde{U}_c = \frac{4\pi^2}{\alpha^2} \left(k_x \tilde{Y}(t,1) + k_z \tilde{\omega}(t,1) \right)$$
(3)

$$= \int_{0}^{1} \frac{4\pi^2}{\alpha^2} \left(k_x K(1,\eta) + k_z \Gamma(1,\eta) \right) \left(k_x \tilde{u}(t,\eta) + k_z \tilde{W}(t,\eta) \right) d\eta \quad (4)$$

$$\tilde{W}_c = \frac{4\pi^2}{\alpha^2} \left(k_z \tilde{Y}(y, 1) - k_x \tilde{\omega}(t, 1) \right)$$
(5)

$$= \int_0^1 \frac{4\pi^2}{\alpha^2} \left(k_z K(1,\eta) - k_x \Gamma(1,\eta) \right) \left(k_x \tilde{u}(t,\eta) + k_z \tilde{W}(t,\eta) \right) d\eta.$$
(6)

and now for the steps to derive these controllers.

2 Notation

We're going to start with notation just to make sure everyone is on the same page. index notation, vector calculus (grad, div, laplacian), material derivative,

2-D Fourier Transform and diff property.

$$\nabla = [\partial_x \quad \partial_z \quad \partial_y] = \frac{\partial}{\partial x_i} \tag{7}$$

$$\nabla^2 = \Delta = \partial_{xx} + \partial_{zz} + \partial_{yy} = \frac{\partial^2}{\partial x_j \partial x_j} \tag{9}$$

$$\frac{D}{Dt} = \partial_t + \mathbf{u} \cdot \nabla = \frac{\partial}{\partial t} + u_i \frac{\partial}{\partial x_i}$$
(11)

Here are some useful examples:

$$\nabla p = \frac{\partial p}{\partial x_i} = \begin{bmatrix} p_x & p_z & p_y \end{bmatrix}$$
(12)

$$\nabla \mathbf{u} = \frac{u_i}{\partial x_i} = u_x + w_z + v_y \tag{13}$$

$$\nabla^{2} \mathbf{u} = \frac{\partial^{2} u_{i}}{\partial x_{j} \partial x_{j}} = \begin{bmatrix} u_{xx} + u_{zz} + u_{yy} \\ w_{xx} + w_{zz} + w_{yy} \\ v_{xx} + v_{zz} + v_{yy} \end{bmatrix}$$
(14)

$$\mathbf{u}.\nabla\mathbf{u} = u_j \frac{\partial u_i}{\partial x_j} = \begin{bmatrix} uu_x + wu_z + vu_y \\ uw_x + ww_z + vw_y \\ uv_x + wv_z + vv_y \end{bmatrix}$$
(15)

We will be using the 2-D Fourier Transform, the forward transform looks like this:

$$\tilde{F}(t,k_x,k_z,y) = \int_{-\infty}^{\infty} \int_{-\infty}^{i} nfty F(t,x,z,y) \mathrm{e}^{-2\pi i (k_x x + k_z z)} dx dz \quad (16)$$

Recall the useful property of the FT under differentiation

$$F_x = 2\pi i k_x \tilde{F} \tag{17}$$

$$F_z = 2\pi i k_z \tilde{F} \tag{18}$$

Remember this only works for differentiation in the x and z directions as there are the directions in which we are taking the FT.

3 Navier Stokes and our equilibrium

So we start with the Navier Stokes equations

$$\frac{D\mathbf{u}}{Dt} = \frac{1}{Re} \nabla^2 \mathbf{u} - \nabla p \tag{19}$$

$$\mathbf{u}_t = \frac{1}{Re} \nabla^2 \mathbf{u} - \mathbf{u} \cdot \nabla \mathbf{u} - \nabla p \tag{20}$$

$$\nabla .\mathbf{u} = 0 \tag{21}$$

$$\mathbf{u}(y=0) = 0 \tag{22}$$

$$\mathbf{u}(y=1) = 1 \tag{23}$$

Remember (19) is the same as (20), only we have written out the material derivative operator. Note that these equations are already non dimensionalized. We assumed density is constant, and that gravity and other forces are neglected. We are including viscosity and then we have a pressure gradient to force the flow. The Reynolds number (Re) is a ratio of the inertial forces to the viscous forces. It is the characteristic velocity times the characteristic length divided by the kinematic viscosity. $Re = \frac{UL}{\nu}$ The second equation you see is the incompressible condition. This means the density of a parcel of fluid is constant. It does not mean the density of the flow is constant. We need the additional assumption that density is constant to drop it altogether.

Our boundary conditions mean two things. One is that the flow cannot penetrate the walls at the top and bottom, so the normal velocity (V) must be zero. The other is that there is no slip at the walls, ie there is viscosity, so the tangential velocities (U and W) must be zero at the boundaries.

We can use (21) to derive an equation for the pressure. We take the divergence of the velocity equations ∇ . \mathbf{u}_t , plug in (21), and we arrive at the following equation for the pressure. We get the boundary conditions by looking at the V_t equation and plugging in 0 and 1.

$$\nabla .(\mathbf{u}_t) = \frac{1}{Re} \nabla .(\nabla^2 \mathbf{u}) - \nabla .(\mathbf{u} . \nabla \mathbf{u}) - \nabla .(\nabla p)$$
(24)

$$\nabla (\mathbf{u})_t = \frac{1}{Re} \nabla^2 (\nabla \cdot \mathbf{u}) - \frac{\partial}{\partial x_i} (u_j \frac{\partial u_i}{\partial x_j}) - \nabla^2 p \qquad (25)$$

$$0 = 0 - \left(\frac{\partial u_j}{\partial x_i}\frac{\partial u_i}{\partial x_j} + u_j\frac{\partial^2 u_i}{\partial x_i\partial x_j}\right) - \Delta p \tag{26}$$

$$\frac{\partial^2 p}{\partial x_j \partial x_j} = -\frac{\partial u_j}{\partial x_i} \frac{\partial u_i}{\partial x_j}$$
(27)

$$\Delta p = -(U_x^2 + W_z^2 + V_y^2 + 2U_z W_x + 2U_y V_y + 2W_y V_z)$$
(28)

$$p_y(0) = \frac{1}{Re} V_{yy}(0)$$
 (29)

$$p_y(1) = \frac{1}{Re} V_{yy}(1)$$
 (30)

We said we were looking at the Poiseuille flow. What is that? well, it means we have a flow which is parallel to the walls and is an equilibrium soln to the NS eqns. In our case, this flow looks like this

$$U^{e} = 4y(1-y)$$
(31)

$$V = W = 0 \tag{32}$$

$$P^e = P_0 - \frac{8}{Re}x\tag{33}$$

When the system remains at this equilibrium, the flow is smooth - laminar. However, when the Reynolds number is high, the flow is unstable and we do not remain at this equilibrium. First we see delaminarization into a transition regime, and for higher Re the flow goes fully turbulent. We want to stabilize the system around the Poiseuille flow so that turbulence is reduced.

4 linearization around the equilibrium

Now we shift the system so that we are looking at just the perturbations and then we linearize this resulting system;

Our perturbation variables are the following

$$u = U - U^e, \qquad p = P - P^e, \tag{34}$$

And now we plug them in. First we shift the system.

$$(u+U^{e})_{t} = \frac{1}{Re}((u+U^{e})_{xx} + (u+U^{e})_{zz} + (u+U^{e})_{yy}) - (u+U^{e})(u+U^{e})_{x} - V(u+U^{e})_{y} - W(u+U^{e})_{z} - (p+P^{e})_{x}, \qquad (35)$$

$$u_{t} = \frac{1}{Re}(u_{xx} + u_{zz} + u_{yy} - U^{e}_{yy}) - (u+U^{e})u_{x} - V(u_{y} + U^{e}_{y}) - Wu_{z} - (p_{x} - P^{3}_{x}), \qquad (36)$$

$$u_{t} = \frac{1}{Re}(u_{xx} + u_{zz} + u_{yy}) - (u+U^{e})u_{x} - V(u_{y} + U^{e}_{y}) - Wu_{z} - p_{x}, \qquad (37)$$

$$W_{t} = \frac{1}{Re}(W_{xx} + W_{zz} + W_{yy})$$

$$W_{t} = \frac{1}{Re} (W_{xx} + W_{zz} + W_{yy}) - (u + U^{e})W_{x} - VW_{y} - WW_{z} - (p + P^{e})_{z}, \quad (38)$$

$$V_t = \frac{1}{Re} (V_{xx} + V_{zz} + V_{yy}) -(u + U^e) V_x - V V_y - W V_z - (p + P^e)_y, \qquad (39)$$

$$(u + U_e)_x + W_z + V_y = 0 (40)$$

$$u_x + W_z + V_y = 0 \tag{41}$$

$$(p + P^{e})_{xx} + (p + P^{e})_{zz} + (p + P^{e})_{yy} = -2V_{y}^{2} - 2(u + U^{e})_{x}W_{z} - 2(u + U^{e})_{y}V_{x} - 2(u + U^{e})_{z}W_{x} - 2V_{z}W_{y},$$

$$p_{xx} + p_{zz} + p_{yy} =$$

$$-2V_y^2 - 2u_x W_z - 2(u_y + U_y^e)V_x - 2u_z W_x - 2V_z W_y, \qquad (43)$$

And now we linearize it

$$u_t = \frac{1}{Re}(u_{xx} + u_{zz} + u_{yy}) - U^e u_x - U^e_y V - p_x, \quad (44)$$

$$W_t = \frac{1}{Re} (W_{xx} + W_{zz} + W_{yy}) - U^e W_x - p_z , \qquad (45)$$

$$V_t = \frac{1}{Re} (V_{xx} + V_{zz} + V_{yy}) - U^e V_x - p_y , \qquad (46)$$

$$p_{xx} + p_{zz} + p_{yy} = -2U_y^e V_x (47)$$

$$u_x + W_z + V_y = 0 \tag{48}$$

5 stability of linear system

Now this linearized system is still unstable. To see this, we look at two coupled equations. To find the first equation, we take the Laplacian of the V_t equation

(46) and use the pressure equation (47) to make it autonomous. The second equation consists of the vorticity in the y direction $(u_z - W_x)$. The resulting equations look like this:

$$\nabla^2 V_t = \frac{1}{Re} \nabla^4 V - U \nabla^2 V_x + U_{yy} V_x \tag{49}$$

$$\omega_t = \frac{1}{Re} \nabla^2 \omega - U^e \omega_x - U^e_y V_z \tag{50}$$

These are the equations from which the Orr-Sommerfeld (1907,1908) and Squire(1933) equations are derived. For those of you who have studied flow stability - take the normal modes of the form

$$V = \tilde{v}(y)e^{i(\alpha x + \beta z - \sigma t)}$$
(51)

$$\omega = \tilde{\omega}(y) e^{i(\alpha x + \beta z - \sigma t)} \tag{52}$$

and you arrive at the OS and Squire equations.

$$\left[(-i\sigma + i\alpha U)(D^2 - (\alpha^2 + \beta^2)) - i\alpha U'' - \frac{1}{Re} (D^2 - (\alpha^2 + \beta^2))^2 \right] \tilde{v} = 0$$
(53)

$$\left[\left(-i\sigma_{i}\alpha U\right)-\frac{1}{Re}\left(D^{2}-\left(\alpha^{2}+\beta^{2}\right)\right)\right]\tilde{\omega} = -i\beta U'\tilde{v} \quad (54)$$

Notice that the Squire equation $(50\rightarrow54)$ is forced by the OS equation $(49\rightarrow53)$. Without that forcing, the Squire equation is stable. This leads to the theorum, that for any unstable 3-D flow, there exists a 2-D flow at a *lower* Reynolds number which is also unstable.

Now we are going to use (50) to find our controllers, but instead of using (49) we are going to do something similar to what we did in the beam example, as we dont like those higher order derivatives. What we shall do instead, is use V_y and p similar to how we used α and u in the beam example.

6 Fourier Transform

We are also not going to take normal modes to find out controllers, but we are going to do something similar. Instead of making assumptions in both space and time, we will only make assumptions in space. We shall take the fourier transform of the system in the x and z directions, as our system is infinite in these directions. This gives us wavenumbers in these directions, k_x and k_z . Why do we do this? Well, we started with 1 system defined by 4 3-D nonlinear pdes. Now, with linearization and the FT, we have an infinite number of systems, each paramaterized by the wave numbers k_x and k_z , and defined by 4 1-D linear pdes. Yeah! - we only have to look at 1-D linear systems. This approach is not just used for flow control, we can use it in other applications, for example, if one had an infinite or periodic beam - one could use the Fourier Transform. After taking the Fourier Transform of the previous linear system (44)–(48), the resulting equations are

$$\alpha^2 = 4\pi^2 (k_x^2 + k_z^2), \qquad \beta = 16\pi k_x i, \qquad (55)$$

$$u_t = \frac{1}{Re} (-\alpha^2 u + u_{yy}) + \frac{\beta}{2} y(y-1)u + 4(2y-1)V - 2\pi i k_x p, \qquad (56)$$

$$W_t = \frac{1}{Re} (-\alpha^2 W + W_{yy}) + \frac{\beta}{2} y(y-1)W - 2\pi i k_z p, \qquad (57)$$

$$V_t = \frac{1}{Re} (-\alpha^2 V + V_{yy}) + \frac{\beta}{2} y(y-1)V - p_y, \qquad (58)$$

$$u(y=0) = V(y=0) = W(y=0) = 0$$
(59)

$$u(y=1) = U_c \tag{60}$$

$$W(y=1) = W_c \tag{61}$$

$$V(y=1) = V_c \tag{62}$$

$$-\alpha^2 p + p_{yy} = \beta(2y - 1)V, \qquad (63)$$

$$p_y(0) = \frac{1}{Re} V_{yy}(0)$$
 (64)

$$p_y(1) = \frac{1}{Re}(-\alpha^2 V_c + V_{yy}(1)) - (V_c)_t$$
(65)

$$k_x u + k_z W = -\frac{V_y}{2\pi i} \tag{66}$$

7 Solution to p and V_c

So we want to find our 2 equations one for V_y and p and then other, (50). However, first, lets solve the p equation. The p equation, is nondynamic similar to the α equation in the beam exercise. We will now solve this equation and substitute it back into the other 3 equations. Looking at our equation for p we see

$$-\alpha^2 p + p_{yy} = \beta(2y - 1)V,$$
 (67)

$$p_y(0) = \frac{1}{Re} V_{yy}(0)$$
 (68)

$$p_y(1) = \frac{1}{Re} (-\alpha^2 V_c + V_{yy}(1)) - (V_c)_t$$
(69)

We have an ODE in p forced by V. The solution to this ODE and p is

$$p = \left\{ \beta \int_{0}^{y} V(t,\eta)(2\eta - 1) \sinh(\alpha(y - \eta)) d\eta -\beta \frac{\cosh(\alpha y)}{\sinh(\alpha)} \int_{0}^{1} V(t,\eta)(2\eta - 1) \cosh(\alpha(1 - \eta)) d\eta -\frac{\cosh(\alpha(1 - y))}{\sinh(\alpha)} p_{y}(t,x,z,0) +\frac{\cosh(\alpha y)}{\sinh(\alpha)} p_{y}(t,x,z,1) \right\},$$
(70)

$$p = \beta \int_{0}^{y} V(t,\eta)(2\eta-1) \sinh(\alpha(y-\eta))d\eta$$

+ $\frac{\cosh(\alpha y)}{\sinh(\alpha)} \left\{ -\beta \int_{0}^{1} V(t,\eta)(2\eta-1) \cosh(\alpha(1-\eta))d\eta$
+ $\frac{1}{Re}(-\alpha^{2}V_{c}+V_{yy}(1)) - (V_{c})_{t} \right\}$
- $\frac{\cosh(\alpha(1-y))}{\sinh(\alpha)} p_{y}(t,x,z,0)$ (71)

This next step is again similar to the beam case, we design one controller to put the system into a strict feedback form so that we can use backstepping, ie we dont like the $\int_0^1 V(2\eta - 1) \cosh(\alpha(1 - \eta)) d\eta$ term, so lets get rid of it. How do we do this? Notice the V_c and $(V_c)_t$ terms in the equation. We will use these to get rid of our unwanted term by setting V_c in the following way.

$$(V_c)_t = \frac{1}{Re} \left(V_{yy}(1) - V_{yy}(0) - \alpha^2 V_c \right) -\beta \int_0^1 V(t,\eta) (2\eta - 1) \cosh(\alpha(1-\eta)) d\eta , \qquad (72)$$

Notice we added a $V_{yy}(0)$ term which we have to account for in our resulting p equation.

$$p = \beta \int_{0}^{y} V(t,\eta)(2\eta - 1)\sinh(\alpha(y - \eta))d\eta$$
$$+ \frac{\cosh(\alpha y)}{\sinh(\alpha)} \frac{1}{Re} V_{yy}(0)$$
$$- \frac{\cosh(\alpha(1 - y))}{\sinh(\alpha)} \frac{1}{Re} V_{yy}(0)$$
(73)

8 Y and ω and backstepping