Homework 7

Using the Lyapunov function

$$\Lambda(t) \;=\; \frac{1}{2} \int_0^1 |\Omega|^2(t, y) dy \tag{1}$$

where Ω is a complex valued function and $|\cdot|$ denotes the modulus ($\Omega\bar{\Omega}$), show that the system

$$\Omega_t \;=\; \epsilon(-\alpha^2\Omega + \Omega_{yy}) + \phi(y)\Omega \tag{2}$$
$$\Omega(t, 0) \;=\; \Omega(t, 1) = 0 \tag{3}$$

is exponentially stable when $\phi(y)$ is a purely imaginary valued function.

Problem 2.

Consider the plant

$$Y_t \;=\; \epsilon(-\alpha^2 Y + Y_{yy}) \tag{4}$$
$$\omega_t \;=\; \epsilon(-\alpha^2\omega + \omega_{yy}) + h(y) \int_0^y Y(t, \eta) d\eta \tag{5}$$
$$Y(t, 0) \;=\; \omega(t, 0) = 0\,. \tag{6}$$

Show that the transformation

$$\Omega = \omega - \int_0^y \Gamma(y, \eta) Y(t, \eta) d\eta \tag{7}$$

decouples Y and ω by transforming the plant (4)–(6) into the target system

$$Y_t \;=\; \epsilon(-\alpha^2 Y + Y_{yy}) \tag{8}$$
$$\Omega_t \;=\; \epsilon(-\alpha^2\Omega + \Omega_{yy}) \tag{9}$$
$$Y(t, 0) \;=\; \Omega(t, 0) = 0 \tag{10}$$

when the following pde for Γ is satisfied:

$$\epsilon\Gamma_{yy} \;=\; \epsilon\Gamma_{\eta\eta} - h(y) \tag{11}$$
$$\Gamma(y, 0) \;=\; 0 \tag{12}$$
$$\Gamma(y, y) \;=\; 0\,. \tag{13}$$

Problem 3.

Show that the transformation

$$\Psi \;=\; Y - \int_0^y K(y,\eta)Y(t,\eta)d\eta \tag{14}$$

transforms the plant

$$Y_t \;=\; \epsilon(-\alpha^2 Y + Y_{yy}) + \phi(y)Y + g(y)Y_y(t,0) \tag{15}$$
$$Y(t,0) \;=\; 0 \tag{16}$$

into the target system

$$\Psi_t \;=\; \epsilon(-\alpha^2 \Psi + \Psi_{yy}) + \phi(y)\Psi \tag{17}$$
$$\Psi(t,0) \;=\; 0 \tag{18}$$

when the following pde for K is satisfied:

$$\epsilon K_{yy} \;=\; \epsilon K_{\eta\eta} + (\phi(\eta) - \phi(y))K \tag{19}$$
$$\epsilon K(y,0) \;=\; \int_0^y K(y,\eta)g(\eta)d\eta - g(y) \tag{20}$$
$$\epsilon K(y,y) \;=\; -g(0) . \tag{21}$$

Problem 4.

The double backstepping transformation

$$\Psi \;=\; Y - \int_0^y K(y,\eta)Y(t,\eta)d\eta, \tag{22}$$
$$\Omega \;=\; \omega - \int_0^y \Gamma(y,\eta)Y(t,\eta)d\eta, \tag{23}$$

with inverse

$$Y \;=\; \Psi + \int_0^y L(y,\eta)\Psi(t,\eta)d\eta, \tag{24}$$
$$\omega \;=\; \Omega + \int_0^y \Theta(y,\eta)\Psi(t,\eta)d\eta \tag{25}$$

transforms the plant

$$Y_t \;=\; \epsilon(-\alpha^2 Y + Y_{yy}) + \phi(y)Y + g(y)Y_y(0) + \int_0^y f(y,\eta)Y(\eta)d\eta, \tag{26}$$
$$\omega_t \;=\; \epsilon(-\alpha^2 \omega + \omega_{yy}) + \phi(y)\omega + h(y)\int_0^y Y(\eta)d\eta, \tag{27}$$
$$Y(t,0) \;=\; \omega(t,0) = 0 \tag{28}$$

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into the target system

$$\Psi_t = \epsilon(-\alpha^2\Psi + \Psi_{yy}) + \phi(y)\Psi, \tag{29}$$

$$\Omega_t = \epsilon(-\alpha^2\Omega + \Omega_{yy}) + \phi(y)\Omega \tag{30}$$

$$\Psi(t,0) = \Omega(t,0) = 0 \tag{31}$$

Show that

$$\Theta = \Gamma + \int_\eta^y \Gamma(y,\sigma)L(\sigma,\eta)d\sigma \tag{32}$$

by first plugging (24) into (23), secondly plugging the modified (23) into (25), and thirdly using integration rules we have used in previous homeworks.