Homework 6

Problem 1.

Derive the shear beam model

\[ u_{tt} - \varepsilon u_{xxtt} + u_{xxxx} = 0 \]  

(0)

from the alternative model

\[
\begin{align*}
\varepsilon u_{tt} &= u_{xx} - \alpha_x \\
0 &= \varepsilon \alpha_{xx} - \alpha + u_x.
\end{align*}
\]

(1)

(2)

Problem 2.

Show that

\[
\alpha(x) = \cosh(bx)\alpha(0) - b \int_0^x \sinh(b(x - y))u_y(y) \, dy
\]

is a solution of the equation

\[ \alpha_{xx} - b^2 \alpha + b^2 u_x = 0 \]

(3)

(4)

with the initial condition \( \alpha_x(0) = 0 \).

Problem 3.

Consider the following plant

\[ u_t = u_x + g(x)u(0) + \int_0^x f(x, y)u(y) \, dy. \]

(5)

Show that the transformation

\[ w(x) = u(x) - \int_0^x k(x, y)u(y) \, dy, \]

(6)

where \( k(x, y) \) is given by the PDE

\[
\begin{align*}
k_x + k_y &= \int_y^x k(x, \xi)f(\xi, y) \, d\xi - f(x, y) \\
k(x, 0) &= \int_0^x k(x, y)g(y) \, dy - g(x).
\end{align*}
\]

(7)

(8)
maps (5) into the target system

\[ w_t = w_x. \]  \hfill (9)

Hint: use the formula

\[
\int_0^x \int_0^\xi k(x, \xi) f(\xi, y) u(y) \, dy \, d\xi = \int_0^x \int_y^x k(x, \xi) f(\xi, y) u(y) \, d\xi \, dy
\]  \hfill (10)

Problem 4.

Consider the system

\[
\begin{align*}
\dot{X} &= AX \\
Y(t) &= CX(t - D),
\end{align*}
\]  \hfill (11, 12)

where the output equation can be also represented as

\[
\begin{align*}
\dot{u} &= u_x, \\
\dot{u}(D, t) &= CX(t), \\
Y(t) &= u(0, t).
\end{align*}
\]  \hfill (13, 14, 15)

Introduce the observer

\[
\begin{align*}
\dot{\tilde{X}} &= A\tilde{X} + e^{AD} L (Y(t) - \hat{u}(0, t)) \\
\dot{\tilde{u}} &= \tilde{u}_x + Ce^{Ax}L(Y(t) - \hat{u}(0, t)) \\
\hat{u}(D, t) &= C\tilde{X}(t)
\end{align*}
\]  \hfill (16, 17, 18)

where \( L \) is chosen such that \( A - LC \) is Hurwitz. Show that the transformation

\[
\begin{align*}
\tilde{w}(x) &= \tilde{u}(x) - Ce^{A(x-D)}X,
\end{align*}
\]  \hfill (19)

where \( \tilde{X} = X - \dot{X}, \tilde{u} = u - \dot{u} \), converts the \((\tilde{X}, \tilde{u})\) system into

\[
\begin{align*}
\dot{\tilde{X}} &= \left( A - e^{AD} LC e^{-AD} \right) \tilde{X} - e^{AD} L\tilde{w}(0) \\
\dot{\tilde{w}} &= \tilde{w}_x \\
\tilde{w}(D) &= 0
\end{align*}
\]  \hfill (20, 21, 22)

Note that the \( \tilde{w} \) system is exponentially stable and that the matrix \( A - e^{AD} LC e^{-AD} \) is Hurwitz (you can see this by using a similarity transformation \( e^{AD} \), and using the fact that it commutes with \( A \)).