Homework 6

Problem 1.

Derive the shear beam model

$$u_{tt} - \varepsilon u_{xxtt} + u_{xxxx} = 0 \tag{0}$$

from the alternative model

$$\begin{aligned} \varepsilon u_{tt} &= u_{xx} - \alpha_x & (1) \\ 0 &= \varepsilon \alpha_{xx} - \alpha + u_x \,. & (2) \end{aligned}$$

Problem 2.

Show that

$$\alpha(x) = \cosh(bx)\alpha(0) - b \int_0^x \sinh(b(x-y)) u_y(y)\, dy \tag{3}$$

is a solution of the equation

$$\alpha_{xx} - b^2 \alpha + b^2 u_x = 0 \tag{4}$$

with the initial condition $\alpha_x(0) = 0$.

Problem 3.

Consider the following plant

$$u_t = u_x + g(x)u(0) + \int_0^x f(x,y)u(y)\, dy \,. \tag{5}$$

Show that the transformation

$$w(x) = u(x) - \int_0^x k(x,y)u(y)\, dy \,, \tag{6}$$

where $k(x,y)$ is given by the PDE

$$\begin{aligned} k_x + k_y &= \int_y^x k(x,\xi) f(\xi,y)\, d\xi - f(x,y) & (7) \\ k(x,0) &= \int_0^x k(x,y) g(y)\, dy - g(x) \,, & (8) \end{aligned}$$

maps (5) into the target system

$$w_t = w_x \,. \tag{9}$$

Hint: use the formula

$$\int_0^x \int_0^\xi k(x,\xi) f(\xi,y) u(y)\, dy\, d\xi = \int_0^x \int_y^x k(x,\xi) f(\xi,y) u(y)\, d\xi\, dy \tag{10}$$

Problem 4.

Consider the system

$$\begin{aligned} \dot{X} &= AX && (11)\\ Y(t) &= CX(t-D)\,, && (12) \end{aligned}$$

where the output equation can be also represented as

$$\begin{aligned} u_t &= u_x && (13)\\ u(D,t) &= CX(t) && (14)\\ Y(t) &= u(0,t)\,. && (15) \end{aligned}$$

Introduce the observer

$$\begin{aligned} \dot{\hat{X}} &= A\hat{X} + e^{AD} L\left(Y(t) - \hat{u}(0,t)\right) && (16)\\ \hat{u}_t &= \hat{u}_x + Ce^{Ax} L\left(Y(t) - \hat{u}(0,t)\right) && (17)\\ \hat{u}(D,t) &= C\hat{X}(t) && (18) \end{aligned}$$

where L is chosen such that $A - LC$ is Hurwitz. Show that the transformation

$$\tilde{w}(x) = \tilde{u}(x) - Ce^{A(x-D)}\tilde{X}\,, \tag{19}$$

where $\tilde{X} = X - \hat{X}$, $\tilde{u} = u - \hat{u}$, converts the $(\tilde{X}, \tilde{u})$ system into

$$\begin{aligned} \dot{\tilde{X}} &= \left(A - e^{AD} LCe^{-AD}\right)\tilde{X} - e^{AD} L\tilde{w}(0) && (20)\\ \tilde{w}_t &= \tilde{w}_x && (21)\\ \tilde{w}(D) &= 0 && (22) \end{aligned}$$

Note that the $\tilde{w}$ system is exponentially stable and that the matrix $A - e^{AD}LCe^{-AD}$ is Hurwitz (you can see this by using a similarity transformation e^{AD}, and using the fact that it commutes with A).