Homework 3

Problem 1.

For the plant

$$\begin{aligned} u_t &= u_{xx} + \lambda u && (1) \\ u_x(0) &= 0 && (2) \end{aligned}$$

design the Neumann stabilizing controller ($u_x(1)$ actuated).

Hint: use the target system

$$\begin{aligned} w_t &= w_{xx} && (3) \\ w_x(0) &= 0 && (4) \\ w_x(1) &= -\frac{1}{2} w(1)\,. && (5) \end{aligned}$$

As you showed in Homework 1, this system is asymptotically stable. Note also that you do not need to find $k(x, y)$, it has already been found in class. You only need to use the condition (5) to derive the controller.

Problem 2.

Find the PDE for the kernel $l(x, y)$ of the inverse transformation

$$u(x) = w(x) + \int_0^x l(x, y) w(y)\, dy\,, \tag{6}$$

which relates the systems u and w from Problem 1. By comparison with the PDE for $k(x, y)$ derived in class, show that

$$l(x, y) = -\lambda x \frac{J_1\left(\sqrt{\lambda(x^2 - y^2)}\right)}{\sqrt{\lambda(x^2 - y^2)}}\,. \tag{7}$$

Problem 3.

Design the Dirichlet boundary controller for the heat equation

$$\begin{aligned} u_t &= u_{xx} && (8) \\ u_x(0) &= -q u(0) && (9) \end{aligned}$$

Follow these steps:
1) Use the transformation

$$w(x) = u(x) - \int_0^x k(x,y)u(y)\,dy \tag{10}$$

to map the plant into the target system

$$\begin{aligned} w_t &= w_{xx} && (11)\\ w_x(0) &= 0 && (12)\\ w(1) &= 0\,. && (13) \end{aligned}$$

Show that $k(x,y)$ satisfies the following PDE:

$$\begin{aligned} k_{xx}(x,y) &= k_{yy}(x,y) && (14)\\ k_y(x,0) &= -qk(x,0) && (15)\\ k(x,x) &= -q\,. && (16) \end{aligned}$$

2) The general solution of the PDE (14) has the form $k(x,y) = \phi(x-y) + \psi(x+y)$, where ϕ and ψ are arbitrary functions. Using (16) it can be shown that $\psi \equiv 0$ (*you are not required to prove the statements in the last two sentences, but will get extra credit if you do it*).

Find ϕ from the conditions (15) and (16). Write the solution for $k(x,y)$.

3) Write down the controller.

Problem 4.

Show that the solution of the closed-loop system from the Problem 3 is ($\sigma_n = \pi(2n+1)/2$)

$$\begin{aligned} u(x,t) &= 2\sum_{n=0}^{\infty} e^{-\sigma_n^2 t}\left(\sigma_n \cos(\sigma_n x) - q\sin(\sigma_n x)\right)\\ &\quad \times \int_0^1 \frac{\sigma_n\cos(\sigma_n\xi) - q\sin(\sigma_n\xi) + (-1)^n q e^{q(1-\xi)}}{\sigma_n^2 + q^2} u_0(\xi)\,d\xi\,. \end{aligned} \tag{17}$$

To do this, first write the solution of the system (11)–(13) (you already found this solution in Problem 1 from Homework 2, just take it for $\lambda = 0$). Then use the transformation (10) with the $k(x,y)$ that you found in Problem 3 to express the initial condition $w_0(x)$ in terms of $u_0(x)$ (you will need to change the order of integration in one of the terms to do this). Finally, write the solution for $u(x,t)$ using the inverse transformation

$$u(x) = w(x) - q\int_0^x w(y)\,dy \tag{18}$$

(i.e., $l(x,y) = -q$ in this problem; if you have a lot of extra time, feel free to prove it).

Note that it is not possible to write a closed form solution for the open loop plant, but it is possible to do so for the closed loop system!