AMES 207 Prof. M. Krstic

Control of Distributed Parameter Systems

Homework 1 solutions

May 19, 1999

1. Exercise 2.1.: Consider the matrix $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. From Example 2.1.3 we know that this matrix generates a C_0-semigroup $T(t)$ given by

$$T(t) = e^{At} = \sum_{n=0}^{\infty} \frac{(At)^n}{n!} = \sum_{n=0}^{\infty} \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix} \frac{t^n}{n!}.$$

Here

$$\sum_{n=0}^{\infty} \frac{nt^n}{n!} = \sum_{n=1}^{\infty} \frac{nt^n}{n!} = t \sum_{n=1}^{\infty} \frac{t^{(n-1)}}{(n-1)!} = t \sum_{n=0}^{\infty} \frac{t^n}{n!} = te^t \,,$$

hence $T(t) = \begin{pmatrix} e^t & te^t \\ 0 & e^t \end{pmatrix}$ which has (using Frobenius matrix norm) norm $\|T(t)\| = \sqrt{2+t^2}e^t$ (or something similar using other, equivalent matrix norms). Now we calculate the growth bound of the semigroup:

$$\omega_0 \equiv \lim_{t\to\infty} \frac{1}{t} \log \|T(t)\| = \lim_{t\to\infty} \frac{t + \log\sqrt{2+t^2}}{t} = 1 \,,$$

where we used L'Hospital's rule in the last step. Since the function $\sqrt{2+t^2}$is not bounded, we immediately see that in this case the inequality $\|T(t)\| \leq Me^{\omega_0 t} \quad \forall t \geq 0$ does not hold for any constant M. It is also easy to see how Theorem 2.1.6.e holds in this case. For any $\epsilon > 0$ there exists $M > 0$ such that $\|T(t)\| \leq Me^{(1+\epsilon)t} \quad \forall t \geq 0$:

$$\|T(t)\| = \sqrt{2+t^2}e^t \leq Me^{(1+\epsilon)t} \quad \forall t \geq 0$$

simply take any

$$M \geq \max_{t\geq 0} \frac{\sqrt{2+t^2}}{e^{\epsilon t}} \,.$$

2. Example 2.2.5:

(a) First of all the formula $A^{-1} = \begin{pmatrix} -\alpha A_0^{-1} & -A_0^{-1} \\ I & 0 \end{pmatrix}$ makes sense, because A_0^{-1} is bounded. Also, $Ran\,(A) = Ran\left(A_0^{-1}\right) \bigoplus Z_1 = D\,(A_0) \bigoplus D\left(A_0^{1/2}\right) = D(A)$. Finally,

$$\begin{aligned} A^{-1}A &= \begin{pmatrix} -\alpha A_0^{-1} & -A_0^{-1} \\ I & 0 \end{pmatrix} \begin{pmatrix} 0 & I \\ -A_0 & -\alpha I \end{pmatrix} \\ &= \begin{pmatrix} (-A_0)\left(-A_0^{-1}\right) & -\alpha A_0^{-1} - \alpha\left(-A_0^{-1}\right) \\ 0 & I \end{pmatrix} = I \end{aligned}$$

and

$$\begin{aligned} AA^{-1} &= \begin{pmatrix} 0 & I \\ -A_0 & -\alpha I \end{pmatrix} \begin{pmatrix} -\alpha A_0^{-1} & -A_0^{-1} \\ I & 0 \end{pmatrix} \\ &= \begin{pmatrix} I & 0 \\ (-A_0)\left(-\alpha A_0^{-1}\right) - \alpha I & (-A_0)\left(-A_0^{-1}\right) \end{pmatrix} = I\,. \end{aligned}$$

(b) Note that According to the formula, A is NOT selfadjoint, but $D\,(A^*) = D\,(A)$.

$$\begin{aligned} \langle Az, \widehat{z} \rangle_Z &= \left\langle \begin{pmatrix} 0 & I \\ -A_0 & -\alpha I \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}, \begin{pmatrix} \widehat{z}_1 \\ \widehat{z}_2 \end{pmatrix} \right\rangle_Z \\ &= \left\langle \begin{pmatrix} z_2 \\ -A_0 z_1 - \alpha z_2 \end{pmatrix}, \begin{pmatrix} \widehat{z}_1 \\ \widehat{z}_2 \end{pmatrix} \right\rangle_Z \\ &= \left\langle A_0^{1/2} z_2, A_0^{1/2} \widehat{z}_1 \right\rangle_X - \langle A_0 z_1, \widehat{z}_2 \rangle_X - \langle \alpha z_2, \widehat{z}_2 \rangle_X \end{aligned}$$

and similarly

$$\begin{aligned} \langle z, A^* \widehat{z} \rangle_Z &= \left\langle \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}, \begin{pmatrix} -\widehat{z}_2 \\ A_0 \widehat{z}_1 - \alpha \widehat{z}_2 \end{pmatrix} \right\rangle \\ &= \left\langle A_0^{1/2} z_1, -A_0^{1/2} \widehat{z}_2 \right\rangle_X + \langle z_2, A_0 \widehat{z}_1 - \alpha \widehat{z}_2 \rangle_X \\ &= \langle -A_0 z_1, \widehat{z}_2 \rangle_X - \langle \alpha z_2, \widehat{z}_2 \rangle_X + \left\langle A_0^{1/2} z_2, A_0^{1/2} \widehat{z}_1 \right\rangle_X . \end{aligned}$$

These calculations confirm the formula for A^*.

(c) In order to check $Re\,\langle w, A^* w \rangle_Z = Re\,\langle Aw, w \rangle_Z = -\alpha \left\| w_2 \right\|_X^2$, note that $Re\,\langle w, Aw \rangle_Z = Re\overline{\langle Aw, w \rangle_Z} = Re\overline{\langle w, A^* w \rangle_Z} = Re\,\langle w, A^* w \rangle_Z$ only for $w \in D(A) \cap D\,(A^*)$, but $D\,(A) = D\,(A^*)$, although A is not selfadjoint. Hence it is enough to consider

$$
\begin{aligned}
Re \langle w, Aw \rangle_Z &= \left\langle \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}, \begin{pmatrix} w_2 \\ -A_0 w_1 - \alpha w_2 \end{pmatrix} \right\rangle_Z \\
&= \left\langle A_0^{1/2} w_1, A_0^{1/2} w_2 \right\rangle_X + \langle w_2, -A_0 w_1 - \alpha w_2 \rangle_X \\
&= \left\langle A_0^{1/2} w_1, A_0^{1/2} w_2 \right\rangle_X - \left\langle A_0^{1/2} w_2, A_0^{1/2} w_1 \right\rangle_X - \alpha \langle w_2, w_2 \rangle_X \\
&= -\alpha \|w_2\|_X^2 .
\end{aligned}
$$

3. Example 2.3.7:

$$
Ah = \frac{d^2 h}{dx^2}, \qquad D(A) = \left\{ h \in S_2^2 : \frac{dh}{dx}(0) = 0 = \frac{dh}{dx}(1) \right\} .
$$

According to A.3.48, A is a closed, densely defined linear operator on $L^2(0,1)$. Example A.3.67 A is also selfadjoint, hence, it is enough to consider

$$
Re \langle Az, z \rangle = \int_0^1 z_{xx} z \, dx = z_x z|_0^1 - \|z_x\|^2 = -\|z_x\|^2 \le 0 = 0 \|z\|^2 .
$$

Hence, by Corollary 2.2.3, A is the infinitesimal generator of a C_0-semigroup $T(t)$ with the property $\|T(t)\| \le e^{0t} = 1$.

4. Exercise 2.9:

   (a) We use the results of A.4.26.

$$
\begin{aligned}
\frac{\partial z}{\partial t}(x,t) &= \frac{\partial^2 z}{\partial x^2}(x,t), \qquad z(0,t) = z(1,t) = 0, && (0.1) \\
z(x,0) &= z_0(x) && (0.2)
\end{aligned}
$$

   If we chose state space $Z = L^2(0,1)$, operator A with $Ah = \frac{d^2h}{dx^2}$, and domain $D(A) = \{h \in S_2^2 : h(0) = 0 = h(1)\}$, then A is a closed, densely defined unbounded linear operator on Z, and (0.1)-(0.2) can be reformulated as

$$
\dot{z}(t) = Az(t); \qquad z(0) = z_0 .
$$

   (b) Checking the assumptions of Corollary 2.2.3 we see that $A^{-1}$is selfadjoint and compact, which implies A is selfadjoint and closed. Because of the selfadjointness it is enough to check

$$
Re \langle Az, z \rangle = \int_0^1 z_{xx} z \, dx = z_x z|_0^1 - \|z_x\|^2 \le 0 \le 0 \|z\|^2
$$

   (Poincaré's inequality even gives us $-\|z_x\|^2 \le \|z\|^2$) Hence, A is the infinitesimal generator of a C_0-semigroup $T(t)$ with $\|T(t)\| \le e^{0t} = 1$, i.e. the semigroup is contraction semigroup.

(c) From Example A.4.26 we have that the eigenvalues of $-A^{-1}$ are $\left\{\frac{1}{n^2\pi^2}\right\}_{n=1}^{\infty}$ with corresponding eigenvectors $\left\{\sqrt{2}\sin n\pi x\right\}_{n=1}^{\infty}$. This implies that the eigenvalues of A are $\{n^2\pi^2\}_{n=1}^{\infty}$ with the same system of eigenvectors. We see that the closure of the pointspectrum of $A$is totally disconnected, and the set of eigenvectors form not only Riesz basis in $Z = L^2(0,1)$, but an orthonormal basis. Hence, $A$is a Riesz-spectral operator.

(d) Because Ahas the representation

$$\begin{aligned} Az &= \sum_{n=1}^{\infty} \left(-n^2\pi^2\right) \left\langle z, \sqrt{2}\sin\left(n\pi\cdot\right)\right\rangle \sqrt{2}\sin\left(n\pi\cdot\right) , \\ D(A) &= \left\{ z \in L^2(0,1) \, : \, \sum_{n=1}^{\infty} n^4\pi^4 \left|\left\langle z, \sqrt{2}\sin\left(n\pi\cdot\right)\right\rangle\right| < \infty \right\} \end{aligned}$$

and

$$\sup_{n\geq 1} \left(-n^2\pi^2\right) = -\pi^2 < \infty \, ,$$

Theorem 2.3.5 tells us that the semigroup $T(t)$ generated by A has the representation

$$T(t)z = \sum_{n=1}^{\infty} e^{-n^2\pi^2 t} \left\langle z, \sqrt{2}\sin\left(n\pi\cdot\right)\right\rangle \sqrt{2}\sin\left(n\pi\cdot\right) \, .$$

(e) The growth bound of the semigroup obtained above is $\omega_0 \equiv \sup\left(-n^2\pi^2\right) = -\pi^2$.

(f) Negative growth bound implies that the L^2-norm of the solution is decreasing. The solution represents the temperature of the rod, hence the temperature is decreasing. This is because the heat dissipates through the endpoints of the rod.

(g) For $t < 0$ we obtain that

$$\|T(t)\| = \sup_{\|z\|=1} \|T(t)z\| \geq \sup_{n} \left\|T(t)\sqrt{2}\sin\left(n\pi\cdot\right)\right\| = \sup_{n} e^{-n^2\pi^2 t} = \infty \, .$$

Hence $T(t)$is not a bounded operator for $t < 0$.