Homework 7

Problem 1.

Using the Lyapunov function

$$\Lambda(t) = \frac{1}{2}\int_0^1 |\Omega|^2(t,y)dy \tag{1}$$

where Ω is a complex valued function and $|\cdot|$ denotes the modulus $(\Omega\bar{\Omega})$, show that the system

$$\Omega_t = \epsilon(-\alpha^2\Omega + \Omega_{yy}) + \phi(y)\Omega \tag{2}$$

$$\Omega(t,0) = \Omega(t,1) = 0 \tag{3}$$

is exponentially stable when $\phi(y)$ is a purely imaginary valued function.

Problem 2.

Consider the plant

$$Y_t = \epsilon(-\alpha^2 Y + Y_{yy}) \tag{4}$$

$$\omega_t = \epsilon(-\alpha^2\omega + \omega_{yy}) + h(y)\int_0^y Y(y,\eta)d\eta \tag{5}$$

$$Y(t,0) = \omega(t,0) = 0\,. \tag{6}$$

Show that the transformation

$$\Omega = \omega - \int_0^y \Gamma(y,\eta)Y(t,\eta)d\eta \tag{7}$$

decouples Y and ω by transforming the plant (4)–(6) into the target system

$$Y_t = \epsilon(-\alpha^2 Y + Y_{yy}) \tag{8}$$

$$\Omega_t = \epsilon(-\alpha^2\Omega + \Omega_{yy}) \tag{9}$$

$$Y(t,0) = \Omega(t,0) = 0 \tag{10}$$

when the following pde for Γ is satisfied:

$$\epsilon\Gamma_{yy} = \epsilon\Gamma_{\eta\eta} - h(y) \tag{11}$$

$$\Gamma(y,0) = 0 \tag{12}$$

$$\Gamma(y,y) = 0\,. \tag{13}$$

Problem 3.

Show that the transformation

$$\Psi = Y - \int_0^y K(y,\eta)Y(t,\eta)d\eta \tag{14}$$

transforms the plant

$$\begin{aligned} Y_t &= \epsilon(-\alpha^2 Y + Y_{yy}) + \phi(y)Y + g(y)Y_y(t,0) &(15)\\ Y(t,0) &= 0 &(16)\end{aligned}$$

into the target system

$$\begin{aligned} \Psi_t &= \epsilon(-\alpha^2 \Psi + \Psi_{yy}) + \phi(y)\Psi &(17)\\ \Psi(t,0) &= 0 &(18)\end{aligned}$$

when the following pde for K is satisfied:

$$\begin{aligned} \epsilon K_{yy} &= \epsilon K_{\eta\eta} + (\phi(\eta) - \phi(y))K &(19)\\ \epsilon K(y,0) &= \int_0^y K(y,\eta)g(\eta)d\eta - g(\eta) &(20)\\ K(y,y) &= -g(0)\,. &(21)\end{aligned}$$

Problem 4.

The double backstepping transformation

$$\begin{aligned} \Psi &= Y - \int_0^y K(y,\eta)Y(t,\eta)d\eta, &(22)\\ \Omega &= \omega - \int_0^y \Gamma(y,\eta)Y(t,\eta)d\eta, &(23)\end{aligned}$$

with inverse

$$\begin{aligned} Y &= \Psi + \int_0^y L(y,\eta)\Psi(t,\eta)d\eta, &(24)\\ \omega &= \Omega + \int_0^y \Theta(y,\eta)\Psi(t,\eta)d\eta &(25)\end{aligned}$$

transforms the plant

$$\begin{aligned} Y_t &= \epsilon(-\alpha^2 Y + Y_{yy}) + \phi(y)Y + g(y)Y_y(0) + \int_0^y f(y,\eta)Y(\eta)d\eta, &(26)\\ \omega_t &= \epsilon(-\alpha^2 \omega + \omega_{yy}) + \phi(y)\omega + h(y)\int_0^y Y(\eta)d\eta, &(27)\\ Y(t,0) &= \omega(t,0) = 0 &(28)\end{aligned}$$

into the target system

$$\begin{aligned}
\Psi_t &= \epsilon(-\alpha^2\Psi + \Psi_{yy}) + \phi(y)\Psi, && (29)\\
\Omega_t &= \epsilon(-\alpha^2\Omega + \Omega_{yy}) + \phi(y)\Omega && (30)\\
\Psi(t,0) &= \Omega(t,0) = 0 && (31)
\end{aligned}$$

Show that

$$\Theta = \Gamma + \int_\eta^y \Gamma(y,\sigma)L(\sigma,\eta)d\sigma \qquad (32)$$

by first plugging (24) into (23), secondly plugging (23) into (25), and thirdly using integration rules we have used in previous homeworks.