MAE 287 FINAL EXAM (Fall 2008)

Problem 1.

Consider the system

$$\begin{aligned} u_{tt} &= u_{xx} \\ u_x(0) &= -q u_t(0) \\ u_x(1) &= U\,, \end{aligned}$$

where q is a positive constant. All eigenvalues of the open-loop system ($U = 0$) lie in the right half plane, therefore the system is "anti-stable" (stable in reverse time).

Design the stabilizing controller for this system. Use the transformation

$$w(x) = u(x) - k_1 \int_0^x u_y(y)\,dy - k_2 \int_0^x u_t(y)\,dy$$

and the target system

$$\begin{aligned} w_{tt} &= w_{xx} \\ w_x(0) &= c_1 w_t(0) \\ w_x(1) &= -c_0 w(1)\,, \end{aligned}$$

where k_1, k_2 are constants and $c_0, c_1 > 0$.

(a) Find k_1, k_2.

(b) Write down the explicit controller.

Problem 2.

Assuming the measurements of $u(1)$ and $u_t(1)$, design the observer for the plant from Problem 1. Use the following observer

$$\begin{aligned} \hat{u}_{tt} &= \hat{u}_{xx} + p_1[u(1) - \hat{u}(1)] + p_2[u_t(1) - \hat{u}_t(1)] \\ \hat{u}_x(0) &= -q\hat{u}_t(0) + p_3[u(1) - \hat{u}(1)] + p_4[u_t(1) - \hat{u}_t(1)] \\ \hat{u}_x(1) &= U + p_5[u(1) - \hat{u}(1)] + p_6[u_t(1) - \hat{u}_t(1)]\,, \end{aligned}$$

where p_i, $i = 1, \ldots, 6$ are constants.
With the transformation

$$\tilde{u}(x,t) = \tilde{w}(x,t) + \beta \int_x^1 \tilde{w}_t(y,t)\,dy$$

map the error system for $\tilde{u} = u - \hat{u}$ into the target system

$$\begin{aligned} \tilde{w}_{tt} &= \tilde{w}_{xx} \\ \tilde{w}_x(0) &= c_2 \tilde{w}_t(0) \\ \tilde{w}_x(1) &= -c_0 \tilde{w}(1)\,. \end{aligned}$$

Find the constants β and p_i, $i = 1, \ldots, 6$, and write the observer explicitly.

Problem 3.

Consider the cascade of a heat equation and an LTI finite-dimensional system given by

$$\dot{X}(t) = AX(t) + Bu(0,t) \tag{1}$$
$$u_t(x,t) = u_{xx}(x,t) \tag{2}$$
$$u_x(0,t) = 0 \tag{3}$$
$$u(D,t) = U(t)\,, \tag{4}$$

where $X \in \mathbb{R}^n$ is the ODE state, U is the scalar input to the entire system, and $u(x,t)$ is the state of the PDE dynamics of a diffusive actuator. Show that the transformation

$$w(x,t) = u(x,t) - \int_0^x m(x-y)u(y,t)dy - KM(x)X(t)\,,$$

where

$$\begin{aligned} m(s) &= \int_0^s KM(\xi)Bd\xi\,, \\ M(\xi) &= \begin{bmatrix} I & 0 \end{bmatrix} \mathrm{e}^{\begin{bmatrix} 0 & A \\ I & 0 \end{bmatrix}\xi} \begin{bmatrix} I \\ 0 \end{bmatrix}, \end{aligned}$$

and the controller

$$U(t) = K \begin{bmatrix} I & 0 \end{bmatrix} \left\{ \mathrm{e}^{\begin{bmatrix} 0 & A \\ I & 0 \end{bmatrix} D} \begin{bmatrix} I \\ 0 \end{bmatrix} X(t) + \int_0^D \left(\int_0^{D-y} \mathrm{e}^{\begin{bmatrix} 0 & A \\ I & 0 \end{bmatrix}\xi} d\xi \right) \begin{bmatrix} I \\ 0 \end{bmatrix} Bu(y,t)dy \right\}$$

convert the system (1)–(4) into the exponentially stable cascade PDE-ODE system

$$\begin{aligned} \dot{X}(t) &= (A+BK)X(t) + Bw(0,t) \\ w_t(x,t) &= w_{xx}(x,t) \\ w_x(0,t) &= 0 \\ w(D,t) &= 0\,. \end{aligned}$$