Bessel functions

1. Bessel function J_n

ODE representation ($y(x) = J_n(x)$ is a solution to this ODE)

$$x^2 y''_{xx} + x y'_x + (x^2 - n^2) y = 0 \tag{1}$$

Series representation

$$J_n(x) = \sum_{m=0}^{\infty} \frac{(-1)^m (x/2)^{n+2m}}{m!(m+n)!} \tag{2}$$

Properties

$$2nJ_n(x) = x(J_{n-1}(x) + J_{n+1}(x)) \tag{3}$$

$$J_n(-x) = (-1)^n J_n(x) \tag{4}$$

Differentiation

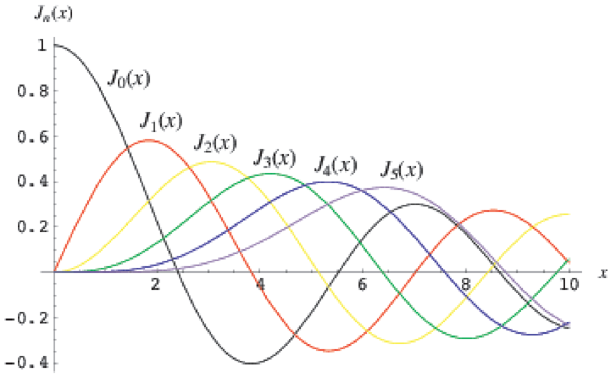
$$\frac{d}{dx} J_n(x) = \frac{1}{2}(J_{n-1}(x) - J_{n+1}(x)) = \frac{n}{x} J_n(x) - J_{n+1}(x) \tag{5}$$

$$\frac{d}{dx}(x^n J_n(x)) = x^n J_{n-1}, \qquad \frac{d}{dx}(x^{-n} J_n(x)) = -x^{-n} J_{n+1} \tag{6}$$

Asymptotic properties

$$J_n(x) \approx \frac{1}{n!}\left(\frac{x}{2}\right)^n, \qquad x \to 0 \tag{7}$$

$$J_n(x) \approx \sqrt{\frac{2}{\pi x}} \cos\left(x - \frac{\pi n}{2} - \frac{\pi}{4}\right), \qquad x \to \infty \tag{8}$$



2. Modified Bessel function I_n

ODE representation ($y(x) = I_n(x)$ is a solution to this ODE)

$$x^2 y''_{xx} + x y'_x - (x^2 + n^2) y = 0 \tag{9}$$

Series representation

$$I_n(x) = \sum_{m=0}^{\infty} \frac{(x/2)^{n+2m}}{m!(m+n)!} \tag{10}$$

Relationship with $J_n(x)$

$$I_n(x) = i^{-n} J_n(ix), \qquad I_n(ix) = i^n J_n(x) \tag{11}$$

Properties

$$2n I_n(x) = x(I_{n-1}(x) - I_{n+1}(x)) \tag{12}$$

$$I_n(-x) = (-1)^n I_n(x) \tag{13}$$

Differentiation

$$\frac{d}{dx} I_n(x) = \frac{1}{2}(I_{n-1}(x) + I_{n+1}(x)) = \frac{n}{x} I_n(x) + I_{n+1}(x) \tag{14}$$

$$\frac{d}{dx}(x^n I_n(x)) = x^n I_{n-1}, \qquad \frac{d}{dx}(x^{-n} I_n(x)) = x^{-n} I_{n+1} \tag{15}$$

Asymptotic properties

$$I_n(x) \approx \frac{1}{n!}\left(\frac{x}{2}\right)^n, \qquad x \to 0 \tag{16}$$

$$I_n(x) \approx \frac{e^x}{\sqrt{2\pi x}}, \qquad x \to \infty \tag{17}$$