Backstepping Boundary Control of Navier-Stokes Channel Flow: A 3D Extension

Jennie Cochran, Rafael Vazquez and Miroslav Krstic

***Abstract*— We present an extension from 2D to 3D of a boundary control law which stabilizes the parabolic profile of an infinite channel flow. In this 3D case, we include an additional controller in the spanwise direction. This result guarantees exponential stability in the L_2 sense of the linearized Navier-Stokes equations. We also present explicit controllers for the control of the system with an averaged streamwise velocity and a Taylor series expansion of the kernels for the controllers up to third order.**

I. Introduction

In this paper we present a boundary control law which stabilizes a benchmark 3D linearized Navier-Stokes system. This result extends a previous result [14] of a control law for the 2D version of the system we examine here. By adding a third dimension, we add an additional controller in the spanwise direction and modify the derivation accordingly. Studying this problem in 3D continues to be hard — the extention to 3D does not detract from the numerous complex issues underlying the problem [9].

While we use a backstepping technique to stabilize the system, other methods, such as optimal control methods [6], [11], [10], model reduction techniques [4], separation control [3] have been studied extensively. Currently, the most successful technique [8] involves discretizing the Navier-Stokes equations and employs high-dimensional algebraic Riccati equations which has formidable computational complexity for large Reynolds numbers. A simple and explicit design, using a Lyapunov/passivity technique [1], [5], was also developed. However, though this approach did not use discretization or linearization, and though it was successful in simulations using high Reynolds numbers, the theory behind this design was restricted to low Reynolds numbers. The backstepping technique used here is based on the recently developed technique for parabolic systems [13], which has been successfully used in flow control problems [14], [2], [16]. It does not employ Riccati equations, nor is the theory limited to low Reynolds numbers.

We start the paper, in Section II, by reviewing the 3D Navier-Stokes equations in the geometry we are considering (an infinite channel) and the Poiseuille equilibrium profile we stabilize. Next, we linearize around the equilibrium profile and transform the system to Fourier space. This allows us to analyze each wave number pair separately, as all pairs are uncoupled from each other. We split the wave numbers into two sets, induce control on one set of wave numbers, and leave the other set uncontrolled. After transforming the equations to Fourier space, we solve for the pressure. Eliminating the pressure from the velocity equations allows us to design a normal velocity controller, V_c, using a strict-feedback structure. This also allows us to transform the streamwise and spanwise velocities from a cascade Volterra operator form to two stable heat equation systems using boundary feedback. With this in mind, we design streamwise and spanwise velocity controllers, U_c and W_c, using the backstepping technique. In Section III, we prove the stability of the system with controlled wave numbers and then the stability of the system with the uncontrolled wave numbers. Using these results we prove stability of the entire physical system with the designed controllers. After this proof, Section IV examines the special case when the wavenumber in the streamwise direction equals zero. This case corresponds to studying the system with an averaged streamwise velocity. In this case, we derive explicit solutions of the kernels used in U_c and W_c. After examining the $k_x = 0$ case, we study the controllers around small wavenumbers in Section V. We provide a Taylor series expansion of the kernels used in U_c and W_c up to third order. We finish the paper in Section VI by discussing the results.

Scripps Institute of Oceanography and Department of Mechanical and Aerospace Engineering, University of California at San Diego (jcochran@ucsd.edu,krstic@ucsd.edu)

II. Derivation of Controllers

In this section, we present the mathematical model of the channel flow problem and then derive controllers to stabilize the Poiseuille equilibrium profile. The geometry we consider here is a semi-infinite box $(x, z, y) \in (-\infty, \infty) \times (-\infty, \infty) \times [0, 1]$. The governing equations for the dimensionless velocity field of the incompressible channel flow we consider are the Navier-Stokes equations

$$\begin{aligned} U_t &= \frac{1}{Re}(U_{xx} + U_{zz} + U_{yy}) \\ &\quad -UU_x - VU_y - WU_z - P_x\,, \qquad (1) \\ W_t &= \frac{1}{Re}(W_{xx} + W_{zz} + W_{yy}) \\ &\quad -UW_x - VW_y - WW_z - P_z\,, \qquad (2) \\ V_t &= \frac{1}{Re}(V_{xx} + V_{zz} + V_{yy}) \\ &\quad -UV_x - VV_y - WV_z - P_y\,, \qquad (3) \end{aligned}$$

$$U_x + W_z + V_y = 0\,, \qquad (4)$$

$$\begin{aligned} U(t,x,z,0) &= W(t,x,z,0) = V(t,x,z,0) = 0\,, \qquad (5) \\ U(t,x,z,1) &= W(t,x,z,1) = V(t,x,z,1) = 0\,, \qquad (6) \end{aligned}$$

where U is the streamwise velocity, W is the spanwise velocity, V is the wall-normal velocity, P is the pressure and Re is the Reynolds number. By combining (1)–(4) we find a Poisson equation for P:

$$\begin{aligned} &P_{xx} + P_{zz} + P_{yy} = \\ &\quad -2V_y^2 - 2U_xW_z - 2U_yV_x - 2U_zW_x - 2V_zW_y\,, \qquad (7) \end{aligned}$$

$$P_y(t,x,z,0) = \frac{V_{yy}(t,x,z,0)}{Re}\,, \qquad (8)$$

$$P_y(t,x,z,1) = \frac{V_{yy}(t,x,z,1)}{Re}\,. \qquad (9)$$

The equilibrium solution to (1)–(4) that we shall stabilize is the parabolic Poiseuille profile

$$U^e = 4y(1-y)\,, \tag{10}$$
$$W^e = V^e = 0\,, \tag{11}$$
$$P^e = P_0 - \frac{8}{Re}x\,, \tag{12}$$

which is unstable for high Reynolds numbers.

Our new equations, after defining the fluctuation variables

$$u = U - U^e\,, \qquad p = P - P^e\,, \tag{13}$$

and linearizing around the equilibrium profile, are

$$\begin{aligned} u_t &= \frac{1}{Re}(u_{xx}+u_{zz}+u_{yy}) + 4y(y-1)u_x \\ &\quad +4(2y-1)V - p_x\,, \end{aligned} \tag{14}$$
$$\begin{aligned} W_t &= \frac{1}{Re}(W_{xx}+W_{zz}+W_{yy}) + 4y(y-1)W_x \\ &\quad -p_z\,, \end{aligned} \tag{15}$$
$$\begin{aligned} V_t &= \frac{1}{Re}(V_{xx}+V_{zz}+V_{yy}) + 4y(y-1)V_x \\ &\quad -p_y\,, \end{aligned} \tag{16}$$

$$p_{xx}+p_{zz}+p_{yy} = 8(2y-1)V_x\,, \tag{17}$$

$$u(t,x,z,0) = W(t,x,z,0) = V(t,x,z,0) = 0\,, \tag{18}$$
$$u(t,x,z,1) = U_c(t,x,z)\,, \tag{19}$$
$$W(t,x,z,1) = W_c(t,x,z)\,, \tag{20}$$
$$V(t,x,z,1) = V_c(t,x,z)\,, \tag{21}$$
$$p_y(t,x,z,0) = \frac{V_{yy}(t,x,z,0)}{Re}\,, \tag{22}$$
$$\begin{aligned} p_y(t,x,z,1) &= \frac{V_{yy}(t,x,z,1)}{Re} - (V_c)_t(t,x,z) \\ &\quad + \frac{(V_c)_{xx}(t,x,z)+(V_c)_{zz}(t,x,z)}{Re}\,. \end{aligned} \tag{23}$$

We define

$$\alpha^2 = 4\pi^2(k_x^2+k_z^2)\,, \qquad \beta = 16\pi k_x i\,, \tag{24}$$

and then transform equations (14)–(23) to Fourier space. This results in an infinite number of 1D systems paramaterized by the wave numbers k_x and k_z,

$$\begin{aligned} u_t &= \frac{1}{Re}(-\alpha^2 u + u_{yy}) + \frac{\beta}{2}y(y-1)u \\ &\quad +4(2y-1)V - 2\pi i k_x p\,, \end{aligned} \tag{25}$$
$$\begin{aligned} W_t &= \frac{1}{Re}(-\alpha^2 W + W_{yy}) + \frac{\beta}{2}y(y-1)W \\ &\quad -2\pi i k_z p\,, \end{aligned} \tag{26}$$
$$\begin{aligned} V_t &= \frac{1}{Re}(-\alpha^2 V + V_{yy}) + \frac{\beta}{2}y(y-1)V \\ &\quad -p_y\,, \end{aligned} \tag{27}$$

$$-\alpha^2 p + p_{yy} = \beta(2y-1)V\,, \tag{28}$$

$$u(t,k_x,k_z,0) = W(t,k_x,k_z,0) = V(t,k_x,k_z,0) = 0\,, \tag{29}$$
$$u(t,k_x,k_z,1) = U_c(t,k_x,k_z)\,, \tag{30}$$
$$W(t,k_x,k_z,1) = 0\,, \tag{31}$$
$$V(t,k_x,k_z,1) = V_c(t,k_x,k_z)\,, \tag{32}$$
$$p_y(t,k_x,k_z,0) = \frac{V_{yy}(t,k_x,k_z,0)}{Re}\,, \tag{33}$$
$$\begin{aligned} p_y(t,k_x,k_z,1) &= \frac{V_{yy}(t,k_x,k_z,1) - \alpha^2 V_c(t,k_x,k_z)}{Re} \\ &\quad -(V_c)_t(t,k_x,k_z)\,. \end{aligned} \tag{34}$$

We divide the wave numbers into two sets. The first set contains the wave numbers $|k_x|, |k_z| \le M$, and is controlled by V_c, U_c and W_c to be designed. The other set, containing all other wave numbers, is left uncontrolled. We separate these sets mathematically using the following function

$$\chi(k_x,k_z) = \begin{cases} 1, & |k_x| < M \text{ and } |k_x| < M \\ 0, & \text{else,} \end{cases} \tag{35}$$

where

$$M = \frac{1}{\pi}\sqrt{\frac{Re}{2}} \tag{36}$$

for the analysis in this paper. For implementation, M will be chosen using the numerical results in [12]. In the next two subsections we will design controllers for stabilization of the new variables

$$Y = k_x u + k_z W \tag{37}$$
$$\omega = k_z u - k_x W\,. \tag{38}$$

The physical meaning of Y is related to V,

$$Y = i\frac{V_y}{2\pi}\,, \tag{39}$$

whereas ω is the vorticity fluctuation. By stabilizing (Y,ω) we stabilize the entire Navier-Stokes system because

$$u = \frac{k_x Y + k_z \omega}{k_x^2+k_z^2} \tag{40}$$
$$W = \frac{k_z Y - k_x \omega}{k_x^2+k_z^2} \tag{41}$$

and

$$V(y) = -2i\pi\int_0^y Y(\eta)d\eta\,. \tag{42}$$

A. Stabilization of Y

If we solve the system defined by equations (28), (33) and (34), we can replace p and p_y in equations (25)–(27). Doing this, we can now find an evolution equation for $Y(t,y)$ that only depends on the pressure boundary conditions, (which in turn depend on V), and not on p:

$$\begin{aligned} Y_t &= \frac{1}{Re}(-\alpha^2 Y + Y_{yy}) + \frac{\beta}{2}y(y-1)Y \\ &\quad +4(2y-1)k_x V \\ &\quad -\frac{i\alpha}{2\pi}\left\{\beta\int_o^y V(t,\eta)(2\eta-1)\sinh(\alpha(y-\eta))d\eta \right. \\ &\quad -\beta\frac{\cosh(\alpha y)}{\sinh(\alpha)}\int_0^1 V(t,\eta)(2\eta-1)\cosh(\alpha(1-\eta))d\eta \\ &\quad -\frac{\cosh(\alpha(1-y))}{\sinh(\alpha)}p_y(t,x,z,0) \\ &\quad \left. +\frac{\cosh(\alpha y)}{\sinh(\alpha)}p_y(t,x,z,1)\right\}\,, \end{aligned} \tag{43}$$

$$Y(t,0) = 0\,, \qquad Y(t,1) = k_x U_c(t) + k_z W_c(t)\,. \quad (44)$$

To continue with our derivations, we can use the continuity equation and the definition of Y to transform (33) and (34) into

$$p_y(t,0) = -\frac{2\pi i Y_y(t,0)}{Re}\,, \quad (45)$$

$$p_y(t,1) = -\frac{2\pi i Y_y(t,1) + \alpha^2 V_c}{Re} - (V_c)_t\,. \quad (46)$$

Now, if we set V_c as follows

$$\begin{aligned}(V_c)_t &= \frac{1}{Re}(2\pi i(Y_y(t,0) - Y_y(t,1)) - \alpha^2 V_c)\\ &\quad -\beta\int_0^1 V(t,\eta)(2\eta-1)\cosh(\alpha(1-\eta))d\eta\,, \quad (47)\end{aligned}$$

and plug (45), (46) and (47) into (43), our equation for Y becomes

$$\begin{aligned}Y_t &= \frac{1}{Re}(-\alpha^2 Y + Y_{yy}) + \frac{\beta}{2}y(y-1)Y\\ &\quad +4(2y-1)k_x V\\ &\quad -\frac{i\alpha}{2\pi}\left\{\beta\int_o^y V(t,\eta)(2\eta-1)\sinh(\alpha(y-\eta))d\eta\right.\\ &\quad \left. +2\pi i\frac{\cosh(\alpha(1-y)) - \cosh(\alpha y)}{\sinh(\alpha)}\frac{Y_y(t,0)}{Re}\right\}\,. \quad (48)\end{aligned}$$

We can use the continuity equation and the fact that V is zero at the uncontrolled boundary to express everything in terms of Y as in equation (42) and the evolution equation for Y then becomes

$$\begin{aligned}Y_t &= \frac{1}{Re}(-\alpha^2 Y + Y_{yy}) + \frac{\beta}{2}y(y-1)Y\\ &\quad -8\pi i(2y-1)k_x\int_0^y Y(t,\eta)d\eta\\ &\quad -\alpha\beta\int_0^y(2\eta-1)\sinh(\alpha(y-\eta))\int_0^\eta Y(t,\sigma)d\sigma d\eta\\ &\quad +\frac{\alpha}{Re}\frac{\cosh(\alpha(1-y)) - \cosh(\alpha y)}{\sinh(\alpha)}Y_y(t,0)\,. \quad (49)\end{aligned}$$

By using integration by parts, we see that we have a system which can be stabilized using the backstepping technique. If we set the following for notational convenience,

$$\epsilon = \frac{1}{Re}\,, \quad (50)$$

$$\phi(y) = 8\pi i k_x y(y-1)\,, \quad (51)$$

$$\begin{aligned}f(y,\eta) &= 8i\Big\{\pi k_x(2y-1) - 4\pi\frac{k_x}{\alpha}\sinh(\alpha(y-\eta))\\ &\quad -2\pi k_x(2\eta-1)\cosh(\alpha(y-\eta))\Big\}\,, \quad (52)\end{aligned}$$

$$g(y) = \epsilon\alpha\frac{\cosh(\alpha(1-y)) - \cosh(\alpha y)}{\sinh(\alpha)}\,, \quad (53)$$

$$(54)$$

the system we stabilize is

$$\begin{aligned}Y_t &= \epsilon(-\alpha^2 Y + Y_{yy}) + \phi(y)Y\\ &\quad +g(y)Y_y(t,0) + \int_0^y f(y,\eta)Y(t,\eta)d\eta\,. \quad (55)\end{aligned}$$

We want to link Y to a family of heat equations (which are also parameterized by the wave numbers k_x and k_z)

$$\Psi_t = \epsilon(-\alpha^2\Psi + \Psi_{yy}) + \phi(y)\Psi\,, \quad (56)$$

$$\Psi(t,0) = \Psi(t,1) = 0\,, \quad (57)$$

where

$$\Psi = Y - \int_0^y K(k_x,k_z,y,\eta)Y(t,k_x,k_z,\eta)d\eta\,, \quad (58)$$

$$Y = \Psi + \int_0^y L(k_x,k_z,y,\eta)\Psi(t,k_x,k_z,\eta)d\eta\,, \quad (59)$$

are the direct and inverse transformations. To find $K(y,\eta)$ and thus our control laws, we must find the pde that K solves. To do this, we differentiate (58) with respect to t and plug (55) in for Y_t. We then twice differentiate (58) with respect to y and plug that, along with (58), into (56). We then set these two equations equal to each other and find the following PDE and boundary conditions for K.

$$\begin{aligned}\epsilon K_{yy}(y,\eta) &= \epsilon K_{\eta\eta}(y,\eta) - f(y,\eta)\\ &\quad +(\phi(\eta) - \phi(y))K(y,\eta)\\ &\quad +\int_\eta^y K(y,\xi)f(\xi,\eta)d\xi\,, \quad (60)\end{aligned}$$

$$\epsilon K(y,0) = \int_0^y K(y,\eta)g(\eta)d\eta - g(y)\,, \quad (61)$$

$$\frac{dK(y,y)}{dy} = 0\,. \quad (62)$$

We can find an integral equation for K which can be solved using the method of successive approximations. We first employ a change of variables

$$K(y,\eta) = G(\xi,\zeta) \quad (63)$$

$$\xi = y+\eta \qquad \zeta = y-\eta \quad (64)$$

and find the following pde

$$\begin{aligned}4\epsilon G_{\xi\zeta}(\xi,\zeta) &= -f\Big(\frac{\xi+\zeta}{2},\frac{\xi-\zeta}{2}\Big)\\ &\quad +\Big(\phi\Big(\frac{\xi-\zeta}{2}\Big) - \phi\Big(\frac{\xi+\zeta}{2}\Big)\Big)G(\xi,\zeta)\\ &\quad +\int_{\frac{\xi-\zeta}{2}}^{\frac{\xi+\zeta}{2}} G\Big(\frac{\xi+\zeta}{2}+\tau,\frac{\xi+\zeta}{2}-\tau\Big)\\ &\quad \times f\Big(\tau,\frac{\xi-\zeta}{2}\Big)d\tau \quad (65)\end{aligned}$$

$$\epsilon G(\xi,\xi) = \int_0^\xi G(\xi+\tau,\xi-\tau)g(\tau)d\tau - g(\xi) \quad (66)$$

$$\frac{dG(\xi,0)}{d\xi} = 0\,. \quad (67)$$

We then integrate (65) with respect to ζ from 0 to ζ and use (67) to find

$$\begin{aligned}G_\xi(\xi,\zeta) &= \frac{1}{4\epsilon}\Big\{-\int_0^\zeta f\Big(\frac{\xi+\mu}{2},\frac{\xi-\mu}{2}\Big)d\mu\\ &\quad +\int_0^\zeta\Big(\phi\Big(\frac{\xi-\mu}{2}\Big) - \phi\Big(\frac{\xi+\mu}{2}\Big)\Big)G(\xi,\mu)d\mu\\ &\quad +\int_0^\zeta\int_{\frac{\xi-\mu}{2}}^{\frac{\xi+\mu}{2}} G\Big(\frac{\xi+\mu}{2}+\tau,\frac{\xi+\mu}{2}-\tau\Big)\\ &\quad \times f\Big(\tau,\frac{\xi-\mu}{2}\Big)d\tau d\mu\Big\}\,. \quad (68)\end{aligned}$$

Next we integrate with respect to ξ from ζ to ξ and use (66) to obtain

$$
\begin{aligned}
G(\xi,\zeta) = \frac{1}{4\epsilon}\Bigg\{ & -\int_{\zeta}^{\xi}\int_{0}^{\zeta} f\Big(\frac{s+\mu}{2},\frac{s-\mu}{2}\Big)d\mu ds \\
& +\int_{\zeta}^{\xi}\int_{0}^{\zeta}\Big(\phi\Big(\frac{s-\mu}{2}\Big)-\phi\Big(\frac{s+\mu}{2}\Big)\Big)G(s,\mu)d\mu ds \\
& +\int_{\zeta}^{\xi}\int_{0}^{\zeta}\int_{\frac{s-\mu}{2}}^{\frac{s+\mu}{2}} G\Big(\frac{s+\mu}{2}+\tau,\frac{s+\mu}{2}-\tau\Big) \\
& \times f\Big(\tau,\frac{s-\mu}{2}\Big)d\tau d\mu ds\Bigg\} \\
& +\frac{1}{\epsilon}\int_{0}^{\zeta} G(\zeta+\tau,\zeta-\tau)g(\tau)d\tau-\frac{1}{\epsilon}g(\zeta) \qquad (69)
\end{aligned}
$$

After changing the variables ξ,ζ back to y,η we obtain our successive approximation representation

$$
\begin{aligned}
K_0(y,\eta) &= -\frac{1}{4\epsilon}\int_{y-\eta}^{y+\eta}\int_{0}^{y-\eta} f\Big(\frac{s+\mu}{2},\frac{s-\mu}{2}\Big)d\mu ds \\
&\quad -\frac{1}{\epsilon}g(y-\eta) \qquad (70)\\
&= \alpha\frac{\cosh(\alpha y-\eta)-\cosh(\alpha(y-\eta-1))}{\sinh(\alpha)} \\
&\quad +\frac{2i\pi k_x}{\epsilon\alpha^2}\eta\Big\{4\sinh(\alpha(y-\eta))\alpha(\eta-1) \\
&\quad +12(\cosh(\alpha(y-\eta))-1)\Big\} \\
&\quad -\frac{2i\pi k_x}{\epsilon}\eta(3y-\eta-2)(y-\eta) \qquad (71)\\
K_n &= \frac{1}{4\epsilon}\Bigg\{\int_{y-\eta}^{y+\eta}\int_{0}^{y-\eta}\int_{\frac{s-\mu}{2}}^{\frac{s+\mu}{2}} K_{n-1}\Big(\frac{s+\mu}{2},\tau\Big) \\
&\quad \times f\Big(\tau,\frac{s-\mu}{2}\Big)d\tau d\mu ds \\
&\quad +\int_{y-\eta}^{y+\eta}\int_{0}^{y-\eta}\Big(\phi\Big(\frac{s-\mu}{2}\Big)-\phi\Big(\frac{s+\mu}{2}\Big)\Big) \\
&\quad \times K_{n-1}\Big(\frac{s+\mu}{2},\frac{s-\mu}{2}\Big)d\mu ds\Bigg\} \\
&\quad +\frac{1}{\epsilon}\int_{0}^{y-\eta} K_{n-1}(y-\eta,\tau)g(\tau)d\tau \\
&\quad +K_{n-1} \qquad (72)
\end{aligned}
$$

and

$$K = \lim_{n\to\infty} K_n \qquad (73)$$

We can also derive a similar PDE and boundary conditions for L. Using those PDEs and boundary conditions, we can derive an integral equation for L using the same method of successive approximations. Both K and L are smooth convergent functions [13], [14].

B. Stabilization of ω

The variable ω satisfies the system:

$$
\begin{aligned}
\omega_t &= \epsilon(-\alpha^2\omega+\omega_{yy}) \\
&\quad +\phi(y)\omega+h(y)\int_0^y Y(\eta)d\eta \qquad (74)\\
\omega(t,0) &= 0\,,\ \omega(t,1)=k_z u(t,1)-k_x W(t,1) \qquad (75)
\end{aligned}
$$

where

$$h(y) = 8\pi k_z i(2y-1)\,. \qquad (76)$$

Note that we have again used equation (42). To fully define our controllers, we use a double backstepping transformation to stabilize ω:

$$
\begin{aligned}
\Omega &= \omega-\int_0^y \Gamma(k_x,k_z,y,\eta)Y(t,k_x,k_z,\eta)d\eta, \qquad (77)\\
\omega &= \Omega+\int_0^y \Theta(k_x,k_z,y,\eta)\Psi(t,k_x,k_z,\eta)d\eta. \qquad (78)
\end{aligned}
$$

where Y and Ψ are defined above in equations (58) and (59), and Ω satisfies the following heat equation:

$$
\begin{aligned}
\Omega_t &= \epsilon(-\alpha^2\Omega+\Omega_{yy})+\phi(y)\Omega \qquad (79)\\
\Omega(t,0) &= \Omega(t,1)=0, \qquad (80)
\end{aligned}
$$

similar to Ψ. Note that, using equations (78), (77) and (59) we can deduce Θ from Γ and L:

$$\Theta = \Gamma+\int_\eta^y \Gamma(y,\sigma)L(\sigma,\eta)d\sigma\,. \qquad (81)$$

We find the PDE that Γ satisfies, following the same steps we took to find the PDE that K solved.

$$
\begin{aligned}
\epsilon\Gamma_{yy} &= \epsilon\Gamma_{\eta\eta}-h(y)+(\phi(\eta)-\phi(y))\Gamma \\
&\quad +\int_\eta^y \Gamma(y,\sigma)f(\sigma,\eta)d\sigma \qquad (82)\\
\epsilon\Gamma(y,y) &= 0,\ \epsilon\Gamma(y,0)=\int_0^y \Gamma(y,\eta)g(\eta)d\eta \qquad (83)
\end{aligned}
$$

Again, using these equations we can derive an integral representation of Γ using the method of successive approximations. Following the same steps we took to find (71), (72) we find

$$
\begin{aligned}
\Gamma_0(y,\eta) &= -\frac{1}{4\epsilon}\int_{y-\eta}^{y+\eta}\int_0^{y-\eta} h\Big(\frac{s+\mu}{2}\Big)d\mu ds \qquad (84)\\
&= -\frac{2i\pi k_z}{\epsilon}\eta(3y-\eta-2)(y-\eta) \qquad (85)\\
\Gamma_n(y,\eta) &= \frac{1}{4\epsilon}\Bigg\{\int_{y-\eta}^{y+\eta}\int_0^{y-\eta}\int_{\frac{s-\mu}{2}}^{\frac{s+\mu}{2}} \Gamma_{n-1}\Big(\frac{s+\mu}{2},\tau\Big) \\
&\quad \times f\Big(\tau,\frac{s-\mu}{2}\Big)d\tau d\mu ds \\
&\quad +\int_{y-\eta}^{y+\eta}\int_0^{y-\eta}\Big(\phi\Big(\frac{s-\mu}{2}\Big)-\phi\Big(\frac{s+\mu}{2}\Big)\Big) \\
&\quad \times\Gamma_{n-1}\Big(\frac{s+\mu}{2},\frac{s-\mu}{2}\Big)d\mu ds\Bigg\} \\
&\quad +\frac{1}{\epsilon}\int_0^{y-\eta}\Gamma_{n-1}(y-\eta,\tau)g(\tau)d\tau \\
&\quad +\Gamma_{n-1} \qquad (86)
\end{aligned}
$$

and $\Gamma=\lim_{n\to\infty}\Gamma_n$. Both Γ and Θ are smooth convergent functions.

C. Control design summary and stability guarantees

With the transformations Y to Ψ and ω to Ω, we can now state the equations for the controllers U_c and W_c in wavespace:

$$U_c = \frac{4\pi^2}{\alpha^2}\Big(k_x Y(t,1) + k_z\omega(t,1)\Big) \quad (87)$$
$$= 4\pi^2\int_0^1 \frac{k_x K(1,\eta) + k_z\Gamma(1,\eta)}{\alpha^2} Y(t,\eta)d\eta \quad (88)$$
$$W_c = \frac{4\pi^2}{\alpha^2}\Big(k_z Y(t,1) - k_x\omega(t,1)\Big) \quad (89)$$
$$= 4\pi^2\int_0^1 \frac{k_z K(1,\eta) - k_x\Gamma(1,\eta)}{\alpha^2} Y(t,\eta)d\eta. \quad (90)$$

Given the previous derivations, we can now state the controllers in physical space:

$$\begin{aligned} V_c = & \int_0^t\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-M}^{M}\int_{-M}^{M}\frac{2\pi i}{Re} \\ & \times\Big\{k_x\Big(u_y(\tau,\tilde{x},\tilde{z},0) - u_y(\tau,\tilde{x},\tilde{z},1)\Big) \\ & +k_z\Big(W_y(\tau,\tilde{x},\tilde{z},0) - W_y(\tau,\tilde{x},\tilde{z},1)\Big)\Big\} \\ & \times e^{\frac{\alpha^2}{Re}(t-\tau)} e^{2\pi i\left(k_x(x-\tilde{x})+k_z(z-\tilde{z})\right)} dk_x dk_z d\tilde{x}d\tilde{z}d\tau \\ & -\int_0^t\int_0^1\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-M}^{M}\int_{-M}^{M} V(\tau,\tilde{x},\tilde{z},\eta)(2\eta-1) \\ & \times 16\pi k_x i\cosh(\alpha(1-\eta))e^{\frac{\alpha^2}{Re}(t-\tau)} \\ & \times e^{2\pi i\left(k_x(x-\tilde{x})+k_z(z-\tilde{z})\right)} dk_x dk_z d\tilde{x}d\tilde{z}d\eta d\tau\,, \end{aligned} \quad (91)$$

$$\begin{aligned} U_c = & \int_0^1\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-M}^{M}\int_{-M}^{M}\frac{4\pi^2}{k_x^2+k_z^2} \\ & \times\Big(k_x K(k_x,k_z,1,\eta) + k_z\Gamma(k_x,k_z,1,\eta)\Big) \\ & \times\Big(k_x u(t,\tilde{x},\tilde{z},\eta) + k_z W(t,\tilde{x},\tilde{z},\eta)\Big) \\ & \times e^{2\pi i\left(k_x(x-\tilde{x})+k_z(z-\tilde{z})\right)} dk_x dk_z d\tilde{x}d\tilde{z}d\eta\,, \end{aligned} \quad (92)$$

$$\begin{aligned} W_c = & \int_0^1\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-M}^{M}\int_{-M}^{M}\frac{4\pi^2}{k_x^2+k_z^2} \\ & \times\Big(k_z K(k_x,k_z,1,\eta) - k_x\Gamma(k_x,k_z,1,\eta)\Big) \\ & \times\Big(k_x u(t,\tilde{x},\tilde{z},\eta) + k_z W(t,\tilde{x},\tilde{z},\eta)\Big) \\ & \times e^{2\pi i\left(k_x(x-\tilde{x})+k_z(z-\tilde{z})\right)} dk_x dk_z d\tilde{x}d\tilde{z}d\eta\,, \end{aligned} \quad (93)$$

where K and Γ are defined by the systems (60)–(62) and (82)–(83) respectively. These controllers guarentee the following result.

Theorem 1: The equilibrium $u(t,x,z,y) \equiv V(t,x,z,y) \equiv W(t,x,z,y) \equiv 0$ of the system (14)–(23),(91),(92),(93) is exponentially stable in the L_2 sense:

$$\begin{aligned} & \int_0^1\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\Big(|V|^2(t,x,z,y) \\ & +|u|^2(t,x,z,y) + |W|^2(t,x,z,y)\Big)dxdzdy \\ & \le Ce^{-\frac{1}{Re}t}\int_0^1\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\Big(|V|^2(0,x,z,y) \\ & +|u|^2(0,x,z,y) + |W|^2(0,x,z,y)\Big)dxdzdy\,, \end{aligned} \quad (94)$$

where

$$C = \max_{m<|k_x|,|k_z|<M}\Big\{(1+\alpha^2+\|\Gamma\|_\infty^2)(1+\|L\|_\infty)^2(1+\|K\|_\infty+\|\Gamma\|_\infty)^2\Big\} \quad (95)$$

III. Stability Proof

This section proves Theorem 1. First, we prove the stability of the set of controlled wave numbers in Fourier space. We then prove the stability of the set of uncontrolled wave numbers, also in Fourier space. Finally, we use these results to prove stability in physical space.

A. Controlled wave numbers

To prove that the controlled wavenumbered system is stable around the equilibrium, we start with V, u, and W in Fourier space and transform them into Ψ and Ω. Next we find an exponential bound on these transformed variables. We then transform these back to the original variables.

To begin, we use equations (40) and (41) to transform u, W into Y, ω and then the following

$$V = -2\pi i\int_0^y\left[1+\int_\eta^y L(\eta,\sigma)d\sigma\right]\Psi(t,\eta)d\eta \quad (96)$$

to transform V into Ψ,

$$\begin{aligned} & \int_0^1\Big(|V|^2(t,y) + |u|^2(t,y) + |W|^2(t,y)\Big)dy \\ & = \int_0^1\left(\left|-2\pi i\int_0^y\left[1+\int_\eta^y L(\eta,\sigma)d\sigma\right]\Psi(t,\eta)d\eta\right|^2\right. \\ & \left. +\left|4\pi^2\frac{k_xY+k_z\omega}{\alpha^2}\right|^2 + \left|4\pi^2\frac{k_zY-k_x\omega}{\alpha^2}\right|^2\right)dy \quad (97) \\ & \le \int_0^1\Big(4\pi^2(1+\|L\|_\infty)^2|\Psi|^2(t,y) \\ & +\frac{4\pi^2}{\alpha^2}\Big(|Y|^2(t,y)+|\omega|^2(t,y)\Big)\Big)dy\,. \quad (98) \end{aligned}$$

We now use (59) and (78) to transform Y, ω into Ψ, Ω,

$$\begin{aligned} & \int_0^1\Big(|V|^2(t,y) + |u|^2(t,y) + |W|^2(t,y)\Big)dy \\ & \le \int_0^1\Big(4\pi^2(1+\|L\|_\infty)^2|\Psi|^2(t,y) \\ & +\frac{4\pi^2}{\alpha^2}\left|\Psi(t,y)+\int_0^y L(y,\eta)\Psi(t,\eta)d\eta\right|^2 \\ & +\frac{4\pi^2}{\alpha^2}\left|\Omega(t,y)+\int_0^y \Theta(y,\eta)\Psi(\eta)d\eta\right|^2\Big)dy \\ & \le \frac{4\pi^2}{\alpha^2}\int_0^1\Big((1+\alpha^2)(1+\|L\|_\infty)^2|\Psi|^2(t,y) \\ & +|\Omega|^2(t,y)+\|\Theta\|_\infty^2|\Psi|^2(t,y)\Big)dy \quad (99) \\ & \le \frac{4\pi^2}{\alpha^2}\int_0^1\Big((1+\alpha^2+\|\Gamma\|_\infty^2)(1+\|L\|_\infty)^2 \\ & \times\Big(|\Psi|^2(t,y)+|\Omega|^2(t,y)\Big)\Big)dy\,. \quad (100) \end{aligned}$$

We use the following L_2 estimates

$$\int_0^1 |\Psi|^2(t,y)dy \leq e^{-\frac{1}{Re}t}\int_0^1 |\Psi|^2(0,y)dy \quad (101)$$
$$\int_0^1 |\Omega|^2(t,y)dy \leq e^{-\frac{1}{Re}t}\int_0^1 |\Omega|^2(0,y)dy \quad (102)$$

that were derived from (56)–(57) and (79)–(80). Note that the $\phi(y)\Psi(t,y)$ and $\phi(y)\Omega(t,y)$ terms do not affect the L_2 estimates as $\phi(y)$ is purely imaginary. Using (101) and (102) and equations (58), (77) to transform Ψ, Ω back into Y, ω we continue as

$$\begin{aligned}
&\int_0^1 \Big(|V|^2(t,y)+|u|^2(t,y)+|W|^2(t,y)\Big)dy \\
&\leq \frac{4\pi^2}{\alpha^2}e^{-\frac{1}{Re}t}\int_0^1 \Big((1+\alpha^2+\|\Gamma\|_\infty^2)(1+\|L\|_\infty)^2 \\
&\quad\times\Big(|\Psi|^2(0,y)+|\Omega|^2(0,y)\Big)\Big)dy \quad (103)\\
&\leq \frac{4\pi^2}{\alpha^2}e^{-\frac{1}{Re}t}\int_0^1 \Big((1+\alpha^2+\|\Gamma\|_\infty^2)(1+\|L\|_\infty)^2 \\
&\quad\times\Big\{\Big|Y(0,y)-\int_0^y K(y,\eta)Y(0,\eta)d\eta\Big|^2 \\
&\quad+\Big|\omega(0,y)-\int_0^y \Gamma(y,\eta)Y(0,\eta)d\eta)\Big|^2\Big\}\Big)dy \quad (104)\\
&\leq \frac{4\pi^2}{\alpha^2}Ce^{-\frac{1}{Re}t}\int_0^1 \Big(|Y|^2(0,y)+|\omega|^2(0,y)\Big)dy \quad (105)
\end{aligned}$$

where C is defined in equation (95) above. We now use equations (37) and (38) to transform Y, ω back into u, W.

$$\begin{aligned}
&\int_0^1 \Big(|V|^2(t,y)+|u|^2(t,y)+|W|^2(t,y)\Big)dy \\
&\leq \frac{4\pi^2}{\alpha^2}Ce^{-\frac{1}{Re}t}\int_0^1 \Big(\Big|k_xu(0,y)+k_zW(0,y)\Big|^2 \\
&\qquad+\Big|k_zu(0,y)-k_xW(0,y)\Big|^2\Big)dy \quad (106)\\
&\leq Ce^{-\frac{1}{Re}t}\int_0^1\Big(|V|^2(0,y)+|u|^2(0,y)+|W|^2(0,y)\Big)dy\,. \quad (107)
\end{aligned}$$

This shows an exponential stability bound for the system containing controlled wavenumbers,

$$\begin{aligned}
&\int_0^1\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\chi(k_x,k_z) \\
&\quad\times\Big(|V|^2(t,y)+|u|^2(t,y)+|W|^2(t,y)\Big)dk_xdk_zdy \\
&\leq Ce^{-\frac{1}{Re}t}\int_0^1\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\chi(k_x,k_z) \\
&\quad\times\Big(|V|^2(0,y)+|u|^2(0,y)+|W|^2(0,y)\Big)dk_xdk_zdy\,. \quad (108)
\end{aligned}$$

B. Uncontrolled wave numbers

To prove the stability of the uncontrolled system, we define a new Lyapunov functional

$$\Lambda_{ucw}(t) = \frac{1}{2}\int_0^1\Big(|u|^2+|V|^2+|W|^2\Big)dy \quad (109)$$
$$\begin{aligned}
\dot\Lambda_{ucw} &= -2\epsilon\alpha^2\Lambda_{ucw} \\
&\quad -\epsilon\int_0^1\Big(|u_y|^2+|V_y|^2+|W_y|^2\Big)dy \\
&\quad +\int_0^1 4(2y-1)\frac{(V\bar u+\bar Vu)}{2}dy\,. \quad (110)
\end{aligned}$$

By using the Poincare inequality

$$-\int_0^1\Big(|u_y|^2+|V_y|^2+|W_y|^2\Big)dy \leq -\Lambda_{ucw}\,, \quad (111)$$

we find

$$\dot\Lambda_{ucw}\leq-2\epsilon\alpha^2\Lambda_{ucw}-\epsilon\Lambda_{ucw}+\int_0^1 2\Big(V\bar u+\bar Vu\Big)dy\,. \quad (112)$$

By noting that

$$\begin{aligned}
\int_0^1 2|V||u|dy &\leq \int_0^1\Big(|V|^2+|u|^2\Big)dy \\
&\leq \int_0^1\Big(|V|^2+|u|^2+|W|^2\Big)dy \quad (113)
\end{aligned}$$

we see that

$$\dot\Lambda_{ucw}\leq-2\epsilon\alpha^2\Lambda_{ucw}-\epsilon\Lambda_{ucw}+4\Lambda_{ucw}\,, \quad (114)$$

and if $\alpha^2\geq 2/\epsilon$, (which is equivalent to $(k_x^2+k_z^2)\geq\frac{Re}{2\pi^2}$), then

$$\dot\Lambda_{ucw} \leq -\epsilon\Lambda_{ucw}\,. \quad (115)$$

We obtain an exponential stability bound for the uncontrolled system:

$$\begin{aligned}
&\int_0^1\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(1-\chi(k_x,k_z)) \\
&\quad\times\Big(|V|^2(t,y)+|u|^2(t,y)+|W|^2(t,y)\Big)dk_xdk_zdy \\
&\leq e^{-\frac{1}{Re}t}\int_0^1\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(1-\chi(k_x,k_z)) \\
&\quad\times\Big(|V|^2(0,y)+|u|^2(0,y)+|W|^2(0,y)\Big)dk_xdk_zdy \quad (116)
\end{aligned}$$

Adding together (108) and (116) and applying Parseval's identity to both sides of the inequality, we get (94).

IV. The Case $k_x=0$

We examine the special case of $k_x=0$ in this section. This case if often considered the"ultimate problem" in control of channel flow turbulence because it is the case where the transient growth is the largest [9], [7], [12]. Setting $k_x=0$ allows us to explicity solve for K and Γ, which gives us explicit formulae U_c and W_c. We derive these solutions and then discuss their properties.

In the case of $k_x=0$, the variables Y and ω turn into:

$$Y=k_zW\,,\qquad \omega=k_zu\,. \quad (117)$$

Denoting

$$\kappa = \alpha\Big|_{k_x=0}=2\pi k_z\,, \quad (118)$$

the plant is now

$$u_t = \epsilon\Big(- \kappa^2 u + u_{yy} + \bar{h}(y) \int_0^y W(\eta) d\eta\Big) \quad (119)$$

$$W_t = \epsilon\Big(- \kappa^2 W + W_{yy} + \bar{g}(y) W_y(0)\Big) \quad (120)$$

$$V_t = \epsilon\Big(- \kappa^2 V + V_{yy}\Big) - p_y \quad (121)$$

$$u(0) = W(0) = V(0) = 0 \quad (122)$$

$$u(1) = U_c \qquad W(1) = W_c \qquad V(1) = V_c \quad (123)$$

where

$$\bar{g}(y) = \kappa \frac{\cosh(\kappa(1-y)) - \cosh(\kappa y)}{\sinh(\kappa)}, \quad (124)$$

$$\bar{h}(y) = \frac{4\kappa}{\epsilon} i(2y-1), \quad (125)$$

and the controllers become

$$U_c = \int_0^1 \Gamma(0, k_z, 1, \eta) W(t, 0, k_z, \eta) d\eta \quad (126)$$

$$W_c = \int_0^1 K(0, k_z, 1, \eta) W(t, 0, k_z, \eta) d\eta \quad (127)$$

$$\dot{V}_c(t, 0, k_z) = \kappa\epsilon\Big[- \kappa V_c(t, 0, k_z) + i\left(W_y(t, 0, k_z, 0) - W_y(t, 0, k_z, 1)\right)\Big]. \quad (128)$$

The transformations (77), (58) reduce to

$$\hat{u} = u - \int_0^y \Gamma(0, k_z, y, \eta) W(t, 0, k_z, \eta) d\eta, \quad (129)$$

$$\hat{W} = W - \int_0^y K(0, k_z, y, \eta) W(t, 0, k_z, \eta) d\eta, \quad (130)$$

where $\hat{u}$ and $\hat{W}$ are the target variables for $k_x = 0$ with the following systems

$$\hat{u}_t = \epsilon\Big(- \kappa^2 \hat{u} + \hat{u}_{yy}\Big) \quad (131)$$

$$\hat{W}_t = \epsilon\Big(- \kappa^2 \hat{W} + \hat{W}_{yy}\Big) \quad (132)$$

$$\hat{u}(0) = \hat{u}(1) = \hat{W}(0) = \hat{W}(1) = 0. \quad (133)$$

Taking a look at the gain kernel PDE (60)–(61) for $k_x = 0$ we get

$$K_{yy} = K_{\eta\eta} \quad (134)$$

$$K(y, y) = -\bar{g}(0) \quad (135)$$

$$K(y, 0) = -\left(\bar{g}(y) - \int_0^y K(y, \eta) \bar{g}(\eta) d\eta\right). \quad (136)$$

Proposition 4.1: The solution $K(y, \eta)$ to equations (134)–(136) is

$$K(y, \eta) = \Big(\frac{\kappa^2}{\bar{g}(0)} - \bar{g}(0)\Big) \mathrm{e}^{\bar{g}(0)(y-\eta)} - \frac{\kappa^2}{\bar{g}(0)}, \quad (137)$$

where

$$\bar{g}(0) = \kappa \frac{\cosh(\kappa) - 1}{\sinh(\kappa)} \quad (138)$$

Proof: This explicit solution is found by postulating $K(y, \eta) = F(y - \eta)$, which yields a Volterra equation

$$F(s) = -\bar{g}(s) + \int_0^s \bar{g}(s - \sigma) F(\sigma) d\sigma. \quad (139)$$

The equation for $F(s)$ can be reduced to a second order ODE by using the fact that $\bar{g}'' = \kappa^2 \bar{g}$,

$$F'(s) = -\bar{g}'(s) + \bar{g}(0) F(s) + \int_0^s \bar{g}'(s - \sigma) F(\sigma) d\sigma \quad (140)$$

$$F''(s) = -\bar{g}''(s) + \bar{g}(0) F'(s) + \bar{g}'(0) F(s) + \int_0^s \bar{g}''(s - \sigma) F(\sigma) d\sigma \quad (141)$$

$$F''(s) = -\bar{g}''(s) + \bar{g}(0) F'(s) + \bar{g}'(0) F(s) + \kappa^2 \int_0^s \bar{g}(s - \sigma) F(\sigma) d\sigma \quad (142)$$

$$F''(s) = -\bar{g}''(s) + \bar{g}(0) F'(s) + \bar{g}'(0) F(s) + \kappa^2 \{F(s) + \bar{g}(s)\} \quad (143)$$

$$F''(s) - \bar{g}(0) F'(s) = 0 \quad (144)$$

$$F'' - \bar{g}(0) F' = 0, \quad (145)$$

$$F(0) = -\bar{g}(0), \quad (146)$$

$$F'(0) = -\bar{g}'(0) + F(0)\bar{g}(0) \quad (147)$$

$$= \kappa^2 - \bar{g}^2(0) \quad (148)$$

We now see

$$F(s) = A_1 \mathrm{e}^{\bar{g}(0) s} + A_2 \quad (149)$$

(150)

and therefore

$$A_1 + A_2 = -\bar{g}(0) \quad (151)$$

$$\bar{g}(0) A_1 = \kappa^2 - \bar{g}^2(0). \quad (152)$$

Solving these gives $K(y, \eta)$ defined above. ■

By taking a Taylor series expansion of $K(y, \eta)$ around $\kappa = 0$, it is not hard to see that the first term is quadratic in k_z. Note that K vanishes for $k_z = 0$. Thus the gain $K(0, k_z, 1, \eta)$ is independent of ϵ, grows quadratically in k_z for k_z small, and exponentially when k_z is large.

Next we turn out attention to the gain kernel PDE (82)–(83) for Γ with $k_x = 0$:

$$\Gamma_{yy} = \Gamma_{\eta\eta} - \bar{h}(y) \quad (153)$$

$$\Gamma(y, y) = 0 \quad (154)$$

$$\Gamma(y, 0) = \int_0^y \Gamma(y, \eta) \bar{g}(\eta) d\eta. \quad (155)$$

Proposition 4.2: The solution $\Gamma(y, \eta)$ to equations (155)–(157) is:

$$\Gamma = \frac{1}{\epsilon}\Bigg\{ - \kappa i \eta (y - \eta)(3y - \eta - 2) - \frac{\kappa^2 i}{\sinh(\kappa)} \Bigg\{ B_1 + B_2 (y - \eta) + B_3 (y - \eta)^2 + B_4 \mathrm{e}^{-\kappa(y-\eta)} + B_5 \mathrm{e}^{\kappa(y-\eta)} + B_6 (y - \eta) \mathrm{e}^{-\kappa(y-\eta)} + B_7 (y - \eta) \mathrm{e}^{\kappa(y-\eta)} + B_8 \mathrm{e}^{\lambda_1 (y-\eta)} + B_9 \mathrm{e}^{\lambda_2 (y-\eta)} \Bigg\}\Bigg\} \quad (156)$$

where

$$B_1 = \frac{1}{4\kappa^6}\Big\{(\cosh(\kappa)-1)(6\kappa^6 - 2\kappa^2\bar{g}(0) + 3\bar{g}(0)) + 8\kappa\sinh(\kappa)(\kappa^2 - \bar{g}(0)\Big\} \quad (157)$$

$$B_2 = -\frac{1}{2\kappa^4}\Big\{8\sinh(\kappa)\kappa + (2\kappa^2 - 3\bar{g}(0))(\cosh(\kappa)-1)\Big\} \quad (158)$$

$$B_3 = \frac{3}{2\kappa^2}(\cosh(\kappa)-1) \quad (159)$$

$$B_4 = 2\frac{\mathrm{e}^{\kappa}-1}{(\kappa+\bar{g}(0))^2\kappa^3}(\kappa(4-\kappa) + \bar{g}(0)(2-\kappa)) \quad (160)$$

$$B_5 = 2\frac{1-\mathrm{e}^{-\kappa}}{(\kappa-\bar{g}(0))^2\kappa^3}(-\kappa(4+\kappa) + \bar{g}(0)(2+\kappa)) \quad (161)$$

$$B_6 = 4\frac{\mathrm{e}^{\kappa}-1}{\kappa^2(\kappa+\bar{g}(0))} \qquad B_7 = 4\frac{1-\mathrm{e}^{-\kappa}}{\kappa^2(\kappa-\bar{g}(0))} \quad (162)$$

$$B_8 = \frac{AA-BB}{8\kappa^6(\kappa^2-\bar{g}^2(0))^2} \qquad B_9 = \frac{AA+BB}{8\kappa^6(\kappa^2-\bar{g}^2(0))^2} \quad (163)$$

and

$$AA = \frac{4\kappa^6}{(\cosh(\kappa)+1)^3}\Big\{6\kappa\sinh(\kappa) + 10\kappa\sinh(\kappa)\cosh(\kappa) \quad (164)$$

$$-49\cosh^2(\kappa) - 48\cosh^3(\kappa)\Big\} \quad (165)$$

$$BB = \frac{4\kappa^7(\cosh(\kappa)-1)^4}{\sinh^7(\kappa)\sqrt{\bar{g}^2(0)+8\kappa^2}}\Big\{14\kappa\sinh(\kappa) + 18\kappa\sinh(\kappa)\cosh(\kappa) - 170\cosh(\kappa) - 317\cosh^2(\kappa) - 144\cosh^3(\kappa) - 9\Big\} \quad (166)$$

and $\lambda_{1,2}$ are as in (**??**).

Proof: We start with a change of variables:

$$\xi = y + \eta \quad (167)$$
$$\zeta = y - \eta \quad (168)$$
$$\Gamma(y,\eta) = \Gamma\left(\frac{\xi+\zeta}{2}, \frac{\xi-\zeta}{2}\right) = \Sigma(\xi,\zeta)\,. \quad (169)$$

This turns the PDE (155)–(157) into the following PDE:

$$\Sigma_{\xi\zeta} = -\frac{1}{4}\bar{h}\Big(\frac{\xi+\zeta}{2}\Big) \quad (170)$$
$$\Sigma(\xi,0) = 0 \quad (171)$$
$$\Sigma(\xi,\xi) = \int_0^{\xi}\Sigma(\xi+\tau,\xi-\tau)\bar{g}(\tau)d\tau\,. \quad (172)$$

Integrating (172) with respect to ζ from 0 to ζ, we see

$$\Sigma_\xi(\xi,\zeta) = -\int_0^{\zeta}\frac{1}{4}\bar{h}\Big(\frac{\xi+\tau}{2}\Big)d\tau + \Sigma_\xi(\xi,0) \quad (173)$$
$$= -\int_0^{\zeta}\frac{1}{4}\bar{h}\Big(\frac{\xi+\tau}{2}\Big)d\tau\,. \quad (174)$$

We now integrate (176) with respect to ξ from ζ to ξ

$$\Sigma = \int_{\zeta}^{\xi}\int_0^{\zeta} -\frac{1}{4}\bar{h}\Big(\frac{s+\tau}{2}\Big)d\tau ds + \Sigma(\zeta,\zeta) \quad (175)$$
$$= -\frac{\kappa i}{2\epsilon}(\xi-\zeta)\zeta(2\zeta+\xi-2) + \int_0^{\zeta}\Sigma(\zeta+\tau,\zeta-\tau)\bar{g}(\tau)d\tau\,. \quad (176)$$

Postulating

$$\Sigma(\xi,\zeta) = -\frac{\kappa i}{2\epsilon}(\xi-\zeta)\zeta(2\zeta+\xi-2) + \frac{1}{\epsilon}\Delta(\zeta) \quad (177)$$

we are left with an equation for Δ that depends only on ζ

$$\frac{1}{\epsilon}\Delta = \int_0^{\zeta}\Big(-\frac{\kappa i}{\epsilon}\sigma(\zeta-\sigma)(3\zeta-\sigma-2) + \frac{1}{\epsilon}\Delta(z-\sigma)\Big)\bar{g}(\sigma)d\sigma = \frac{1}{\epsilon}\Big\{\Upsilon(\zeta) - \int_0^{\zeta}\Delta(\sigma)\bar{g}(\zeta-\sigma)d\sigma\Big\} \quad (178)$$

where $\Upsilon = -\kappa i\int_0^{\zeta}\sigma(\zeta-\sigma)(3\zeta-\sigma-2)\bar{g}(\sigma)d\sigma$. We can turn equation (180) into an ODE by again recalling that $\bar{g}'' = \kappa^2\bar{g}$

$$\Delta'' - \bar{g}(0)\Delta' = \Upsilon'' - \kappa^2\Upsilon \quad (179)$$
$$\Delta(0) = \Upsilon(0) = 0 \quad (180)$$
$$\Delta'(0) = \Upsilon'(0) - \Upsilon(0)\bar{g}(0) = 0\,. \quad (181)$$

Note that the homogeneous part of (181) has the same coefficients as equation (147). Therefore we know already that the solution to (181) will contain terms of the form $\mathrm{e}^{\lambda_1\zeta}$ and $\mathrm{e}^{\lambda_2\zeta}$, where $\lambda_{1,2}$ were defined in (**??**). To find the solution to Δ we use the Laplace Transform and find

$$s^2\Delta(s) - \Delta(0) - \Delta'(0) + \bar{g}(0)\Big(s\Delta(s) - \Delta(0)\Big) - 2\kappa^2\Delta(s) = s^2\Upsilon(s) - \Upsilon(0) - \Upsilon'(0) - \kappa^2\Upsilon(s) \quad (182)$$

$$\Delta(s) = \frac{s^2-\kappa^2}{s^2+\bar{g}(0)s-2\kappa^2}\Upsilon(s) \quad (183)$$

where $\Upsilon(s) = \mathcal{L}\Big(\Upsilon(\zeta)\Big)$. To find $\Upsilon(s)$ we take a look at $\Upsilon(\zeta)$ and rearrange it,

$$\Upsilon(\zeta) = -\kappa i\int_0^{\zeta}\sigma(\zeta-\sigma)(3\zeta-\sigma-2)\bar{g}(\sigma)d\sigma = -\kappa i\left\{\int_0^{\zeta}\Big(3(\zeta-\sigma)^2 + 2(\zeta-\sigma)(\sigma-1)\Big)\sigma\bar{g}(\sigma)d\sigma\right\} = -\kappa i\left\{3\,\zeta^2 * \Big(\zeta\,\bar{g}(\zeta)\Big) + 2\,\zeta * \Big((\zeta-1)\,\zeta\,\bar{g}(\zeta)\Big)\right\} \quad (184)$$

then use the convolution property of the Laplace transform

$$\Upsilon(s) = -\kappa i\left\{\frac{6}{s^3}\mathcal{L}\Big(\zeta\,\bar{g}(\zeta)\Big) + \frac{2}{s^2}\mathcal{L}\Big((\zeta-1)\,\zeta\,\bar{g}(\zeta)\Big)\right\} \quad (185)$$

where $\bar{g}(s) = \mathcal{L}\Big(\bar{g}(\zeta)\Big)$. After utilizing the differentiation property we find

$$\Upsilon(s) = -\kappa i\left\{-\frac{6}{s^3}\bar{g}'(s) + \frac{2}{s^2}\Big(\bar{g}''(s) + \bar{g}'(s)\Big)\right\}. \quad (186)$$

Therefore

$$\Delta(\zeta) = \frac{-\kappa^2 i}{\sinh(\kappa)}\Big\{B_1 + B_2\zeta + B_3\zeta^2 + B_4\mathrm{e}^{-\kappa\zeta} + B_5\mathrm{e}^{\kappa\zeta} + B_6\zeta\mathrm{e}^{-\kappa\zeta} + B_7\zeta\mathrm{e}^{\kappa\zeta} + B_8\mathrm{e}^{\lambda_1\zeta} + B_9\mathrm{e}^{\lambda_2\zeta}\Big\} \quad (187)$$

where B_1 through B_9 are defined in (159)–(165). Substituting (189) into (179) and then into (171) we get (158). ■

As equation (158) shows, Γ is linearly dependent on $1/\epsilon$, the Reynolds number. Similar to the solution for K, Γ grows linearly in k_z for k_z small, and exponentially when k_z is large.

Finally, we point out that the "peak-to-peak" gain of the dynamic controller in (128) from the skin friction sensor $W_y(t,0,k_z,0) - W_y(t,0,k_z,1)$ to the actuated variable $V_c(t,0,k_z)$ is

$$\frac{\|V_c(\cdot,0,k_z)\|_\infty}{\|W_y(\cdot,0,k_z,0) - W_y(\cdot,0,k_z,1)\|_\infty} \le \frac{1}{2\pi k_z}, \quad (188)$$

which means that it is independent of the Reynolds number $1/\epsilon$ and that this controller is nearly inactive for large k_z, whereas its effort is significant for small k_z.

Theorem 2: The closed loop system (119)–(123), (126)–(128), (137)–(140), (158)–(168) is exponentially stable for any finite k_z.

Proof: As in Section III-A, however in this case the dependence of C on k_z comes only from the norms of K, L, Γ, and Θ. ■

V. Small k_x, k_z Analysis

In this section we go through the small wavenumber analysis of the controllers.

Theorem 3: For small wavenumbers k_x and k_z, the kernels K and Γ are defined as

$$\begin{aligned} K(y,\eta) &= k_x a_2(y,\eta) + k_x^2 a_4(y,\eta) + k_z^2 a_6(y,\eta) \\ &\quad + k_x^3 a_7(y,\eta) + k_x k_z^2 a_9(y,\eta) + O(k_x^4, k_z^4) \quad (189) \\ \Gamma(y,\eta) &= k_z b_3(y,\eta) + k_x k_z b_5(y,\eta) + k_x^2 k_z b_8(y,\eta) \\ &\quad + k_z^3 b_{10}(y,\eta) + O(k_x^4, k_z^4) \quad (190) \end{aligned}$$

where

$$a_2 = \frac{2i\pi}{\epsilon}\eta(y-\eta)(3y-\eta-2) \quad (191)$$

$$\begin{aligned} a_4 &= \frac{\pi^2}{30\epsilon^2}\Big\{60\epsilon^2\Big(2y-2\eta-1\Big) \\ &\quad +\eta(y-\eta)^3\Big(51\,y^3 - 102\,y^2 + 57\,y^2\eta - 126\,y\eta \\ &\quad +33\,y\eta^2 + 50\,y - 12\,\eta^2 + 70\,\eta - 21\,\eta^3\Big)\Big\} \quad (192) \end{aligned}$$

$$a_6 = 2\pi^2(2y-2\eta-1) \quad (193)$$

$$\begin{aligned} a_7 &= -\frac{i\pi^3}{12600\epsilon^3}(y-\eta)^2\Big\{840\epsilon^2\Big(20\,y + 100\,\eta + 24\,y^3 \\ &\quad -14\,\eta^3 - 45\,y^2 - 5\,\eta^2 - 22\,y^2\eta + 12\,y\eta^2 - 70\,y\eta\Big) \\ &\quad +\eta(y-\eta)^3\Big(1939\,y^5 - 889\,\eta^5 - 6660\,y^4 + 930\,\eta^4 \\ &\quad +7635\,y^3 + 4245\,\eta^3 + 35\,y\eta^4 + 3290\,y^2\eta^3 \\ &\quad +17065\,y\eta^2 - 18990\,y^2\eta^2 + 21455\,y^2\eta - 7410\,y\eta^3 \\ &\quad -18270\,y^3\eta + 5215\,\eta\,y^4 + 7210\,y^3\eta^2 - 5440\,\eta^2 \\ &\quad -8440\,y\eta - 2920\,y^2\Big)\Big\} \quad (194) \end{aligned}$$

$$\begin{aligned} a_9 &= -\frac{i\pi^3}{15\epsilon}\Big\{20\,y + 100\,\eta + 24\,y^3 - 14\,\eta^3 - 45\,y^2 - 5\,\eta^2 \\ &\quad -22\,y^2\eta + 12\,y\eta^2 - 70\,y\eta\Big\} \quad (195) \end{aligned}$$

and

$$b_3 = -\frac{2i\pi}{\epsilon}\eta(y-\eta)(3y-\eta-2) \quad (196)$$

$$\begin{aligned} b_5 &= \frac{\pi^2}{30\epsilon^2}\eta(y-\eta)^3\Big\{-51\,y^3 - 33\,y\eta^2 - 57\,y^2\eta + 21\,\eta^3 \\ &\quad +102\,y^2 + 12\,\eta^2 + 126\,y\eta - 50\,y - 70\,\eta\Big\} \quad (197) \end{aligned}$$

$$\begin{aligned} b_8 &= \frac{i\pi^3}{12600\epsilon^3}(y-\eta)^3\Big\{840\epsilon^2\Big(20 - 48\,y\eta + 24\,(y^2+\eta^2) \\ &\quad -45\,(y-\eta)\Big) \\ &\quad +\eta(y-\eta)^3\Big(35\,y\eta^4 + 3290\,y^2\eta^3 - 7410\,y\eta^3 \\ &\quad +1939\,y^5 - 889\,\eta^5 - 6660\,y^4 + 930\,\eta^4 + 7635\,y^3 \\ &\quad +4245\,\eta^3 + 17065\,y\eta^2 - 18270\,\eta\,y^3 + 5215\,y^4\eta \\ &\quad +21455\,y^2\eta - 2920\,y^2 - 8440\,y\eta - 5440\,\eta^2 \\ &\quad -18990\,y^2\eta^2 + 7210\,\eta^2 y^3\Big)\Big\} \quad (198) \end{aligned}$$

$$\begin{aligned} b_{10} &= \frac{i\pi^3}{15\epsilon}(y-\eta)^3\Big\{20 - 48\,y\eta + 24\,(y^2+\eta^2) \\ &\quad -45\,(y-\eta)\Big\} \quad (199) \end{aligned}$$

Proof: We start by examining the original equations for K, equations (60)–(62) and Γ, equations (82)–(83). If we take a Taylor series expansion of $f(y,\eta)$ and $g(y)$ around $k_x = k_z = 0$ we see:

$$\begin{aligned} f(y,\eta) &= 8i\Big\{\pi k_x(2y-1) - 4\pi\frac{k_x}{\alpha}\sinh(\alpha(y-\eta)) \\ &\quad -2\pi k_x(2\eta-1)\cosh(\alpha(y-\eta))\Big\} \\ &= 8i\Big\{\pi k_x(2y-1) \\ &\quad -4\pi\frac{k_x}{\alpha}\Big(\alpha(y-\eta) + \frac{\alpha^3}{6}(y-\eta)^3 + O(\alpha^5)\Big) \\ &\quad -2\pi k_x(2\eta-1)\Big(1 + \frac{\alpha^2}{2}(y-\eta)^2 + O(\alpha^4)\Big)\Big\} \\ &= 8i\pi\Big\{k_x(1-2y) \\ &\quad -\frac{4\pi^2}{3}k_x(k_x^2+k_z^2)(2y+4\eta-3)(y-\eta)^2 \\ &\quad +O(k_x\alpha^4)\Big\} \quad (200) \\ g(y) &= \epsilon\alpha\frac{\cosh(\alpha(1-y)) - \cosh(\alpha y)}{\sinh(\alpha)}. \\ &= \epsilon\alpha\frac{1 + \frac{\alpha^2}{2}(1-y)^2 - 1 - \frac{\alpha^2}{2}y^2 + O(\alpha^4)}{\alpha + \frac{\alpha^3}{6} + O(\alpha^5)} \\ &= \epsilon\Big(2\pi^2(k_x^2+k_z^2)(1-2y) + O(\alpha^4)\Big) \quad (201) \end{aligned}$$

We now assume K and Γ are of the form

$$\begin{aligned} K &= a_1(y,\eta) + k_x a_2(y,\eta) + k_z a_3(y,\eta) + k_x^2 a_4(y,\eta) \\ &\quad +k_x k_z a_5(y,\eta) + k_z^2 a_6(y,\eta) + k_x^3 a_7(y,\eta) + k_x^2 k_z a_8(y,\eta) \\ &\quad +k_x k_z^2 a_9(y,\eta) + k_z^3 a_{10}(y,\eta) + O(k_x^4, k_z^4) \quad (202) \\ \Gamma &= b_1(y,\eta) + k_x b_2(y,\eta) + k_z b_3(y,\eta) + k_x^2 b_4(y,\eta) \\ &\quad +k_x k_z b_5(y,\eta) + k_z^2 b_6(y,\eta) + k_x^3 b_7(y,\eta) + k_x^2 k_z b_8(y,\eta) \\ &\quad +k_x k_z^2 b_9(y,\eta) + k_z^3 b_{10}(y,\eta) + O(k_x^4, k_z^4) \quad (203) \end{aligned}$$

and substitute (202), (203) and (204), (205) into (60)–(62) and (82)–(83). After matching the like powers of k_x and k_z we arrive at

$$a_1 = a_3 = a_5 = a_8 = a_{10} = 0 \quad (204)$$

$$b_1 = b_2 = b_4 = b_6 = b_7 = b_9 = 0 \quad (205)$$

and the PDEs

$$\epsilon(a_2)_{yy} = \epsilon(a_2)_{\eta\eta} - 8i\pi(1-2y) \quad (206)$$
$$a_2(y,0) = 0 \quad (207)$$
$$\frac{da_2(y,y)}{dy} = 0 \quad (208)$$

$$\epsilon(a_4)_{yy} = \epsilon(a_4)_{\eta\eta} + 8i\pi(\eta(\eta-1) - y(y-1)a_2(y,\eta) + \int_\eta^y 8i\pi a_2(y,\xi)(1-2\xi)d\xi \quad (209)$$
$$\epsilon a_4(y,0) = -\epsilon 2\pi^2(1-2y) \quad (210)$$
$$\frac{da_4(y,y)}{dy} = 0 \quad (211)$$

$$(a_6)_{yy} = (a_6)_{\eta\eta} \quad (212)$$
$$\epsilon a_6(y,0) = -\epsilon 2\pi^2(1-2y) \quad (213)$$
$$\frac{da_6(y,y)}{dy} = 0 \quad (214)$$

$$\epsilon(a_7)_{yy} = \epsilon(a_7)_{\eta\eta} + \frac{32i\pi^3}{3}(2y+4\eta-3)(y-\eta)^2 + 8i\pi(\eta(\eta-1) - y(y-1))a_4(y,\eta) + \int_\eta^y 8i\pi a_4(y,\xi)(1-2\xi)d\xi \quad (215)$$
$$\epsilon a_7(y,0) = \int_0^y a_2(y,\eta)\epsilon 2\pi^2(1-2\eta)d\eta \quad (216)$$
$$\frac{da_7(y,y)}{dy} = 0 \quad (217)$$

$$\epsilon(a_9)_{yy} = \epsilon(a_9)_{\eta\eta} + \frac{32i\pi^3}{3}(2y+4\eta-3)(y-\eta)^2 + 8i\pi(\eta(\eta-1) - y(y-1))a_6(y,\eta) + \int_\eta^y 8i\pi a_6(y,\xi)(1-2\xi)d\xi \quad (218)$$
$$\epsilon a_9(y,0) = \int_0^y a_2(y,\eta)\epsilon 2\pi^2(1-2\eta)d\eta \quad (219)$$
$$\frac{da_9(y,y)}{dy} = 0 \quad (220)$$

$$\epsilon(b_3)_{yy} = \epsilon(b_3)_{\eta\eta} - 8i\pi(2y-1) \quad (221)$$
$$\epsilon b_3(y,0) = 0 \quad (222)$$
$$b_3(y,y) = 0 \quad (223)$$

$$\epsilon(b_5)_{yy} = \epsilon(b_5)_{\eta\eta} + 8i\pi(\eta(\eta-1) - y(y-1))b_3(y,\eta) + \int_\eta^y b_3(y,\sigma)8i\pi(1-2\sigma)d\sigma \quad (224)$$
$$b_5(y,0) = 0 \quad (225)$$
$$b_5(y,y) = 0 \quad (226)$$

$$\epsilon(b_8)_{yy} = \epsilon(b_8)_{\eta\eta} + 8i\pi(\eta(\eta-1) - y(y-1))b_5(y,\eta) + \int_\eta^y b_5(y,\sigma)8i\pi(1-2\sigma)d\sigma \quad (227)$$
$$\epsilon b_8(y,0) = \int_0^y b_3(y,\eta)\epsilon 2\pi^2(1-2\eta)d\eta \quad (228)$$
$$b_8(y,y) = 0 \quad (229)$$

$$(b_{10})_y y = (b_{10})_\eta \eta \quad (230)$$
$$\epsilon b_{10}(y,0) = \int_0^y b_3(y,\eta)\epsilon 2\pi^2(1-2\eta)d\eta \quad (231)$$
$$b_{10}(y,y) = 0 \quad (232)$$

We note that the systems a_2, a_6, and b_3 are autonomous, whereas the dependencies of the other equations are as follows: $a_4(a_2)$, $a_7(a_2,a_4)$, $a_9(a_2,a_6)$, $b_5(b_3)$, $b_8(b_3,b_5)$, $b_{10}(b_3)$, which describes the order in which the PDEs are solved to obtain (191)–(201). We use (64) to tranform each system, and then integrate up — similarly to how we found (71), (72) and (85), (86). ■

Note that if we take a Taylor series expansion of (71) and (85) we find

$$\begin{aligned}K_0(y,\eta) &= \frac{1+\frac{\alpha^2(y-\eta)^2}{2} - 1 - \frac{\alpha^2(y-\eta-1)^2}{2} + O(\alpha^4)}{1+\frac{\alpha^2}{6}+O(\alpha^4)} \\ &\quad + \frac{2i\pi k_x}{\epsilon}\eta\Bigg(4\Big((y-\eta)+\frac{\alpha^2}{6}(y-\eta)^3 \\ &\quad + O(\alpha^4)\Big)(\eta-1) \\ &\quad + 12(1+\frac{\alpha^2}{2}(y-\eta)^2 + O(\alpha^4) - 1)\Bigg) \\ &\quad - \frac{2i\pi k_x}{\epsilon}\eta(3y-\eta-2)(y-\eta) \\ &= k_x\frac{2i\pi}{\epsilon}\Bigg(\eta\,(3\,y-\eta-2)\,(y-\eta)\Bigg) \\ &\quad + \frac{\alpha^2}{2}(2y-2\eta-1) \\ &\quad + k_x\frac{\alpha^2 2i\pi}{\epsilon}\left(\frac{2}{3}\eta(y-\eta)^3(\eta-1)\right) \\ &\quad + O(\alpha^4) \end{aligned} \quad (233)$$

$$\Gamma_0(y,\eta) = -\frac{2i\pi k_z}{\epsilon}\eta(3y-\eta-2)(y-\eta)\,. \quad (234)$$

The expansions of K_0 and Γ_0 capture the linear terms in k_x, k_z — a_2 and b_3 — but do not capture the second or third order terms in their entirety. For instance, the second order part of (235), $2\pi^2(2y-2\eta-1)$ can be seen in a_4 and a_6, though it does not complete a_4. It is very hard to see the third order part of (235), $\frac{8i\pi^3}{\epsilon}\frac{2}{3}\eta(y-\eta)^3(\eta-1)$ in a_7 and a_9. Also, (236) only contains a first order part, and not higher orders that are seen in b_5 through b_{10}. We note that computing K_n and Γ_n and afterwards taking a Taylor series expansion would indeed provide accurate $(n-1)^{th}$ order expressions for K and Γ.

Plugging (191)–(201) into (88) and (90), we find third order approximations to the controllers U_c and W_c

$$\begin{aligned}U_c = \int_0^1 \Bigg\{&(k_x^2-k_z^2)\frac{2i\pi}{\epsilon}(\eta-1)\eta \\ &+ k_x(k_x^2+k_z^2)2\pi^2(1-2\eta) \\ &+ k_x(k_x^2-k_z^2)\frac{\pi^2}{30\epsilon^2}\eta\,\left(21\,\eta^2-1\right)(\eta-1)^4 \\ &- k_x^4\frac{4i\pi^3}{3}\eta(\eta-1)^4 \\ &- k_z^4\frac{i\pi^3}{15\epsilon}(\eta-1)^3(24\eta^2-3\eta-1) \\ &- k_x^2k_z^2\frac{i\pi^3}{12600\epsilon^3}\eta(\eta-1)^4\Big\{8400\epsilon^2 \\ &- \left(889\,\eta^4-76\,\eta^3-201\,\eta^2-46\,\eta-6\right)(\eta-1)^2\Big\}\Bigg\} \\ &\times\frac{k_x u(t,\eta)+k_z W(t,\eta)}{k_x^2+k_z^2}d\eta \end{aligned} \quad (235)$$

$$
\begin{aligned}
W_c = \int_0^1 \Bigg\{ & k_x k_z \frac{4i\pi}{\epsilon}\eta(\eta-1)^2 \\
& + k_x^2 k_z \frac{\pi^2}{15\epsilon^2}\Big\{30\epsilon^2(1-2\eta) + \eta\,\left(21\,\eta^2-1\right)(\eta-1)^4\Big\} \\
& + k_z^3 2\pi^2(1-2\eta) \\
& + k_x^3 k_z \frac{i\pi^3}{12600\epsilon^3}(\eta-1)^3\Big\{16800\epsilon^3\eta(1-\eta) \\
& + 840\epsilon^2(24\eta^2-3\eta-1) \\
& - \eta\,\left(889\,\eta^4-76\,\eta^3-201\,\eta^2-46\,\eta-6\right)(\eta-1)^3\Big\} \\
& + k_x k_z^3 \frac{2i\pi^3}{15\epsilon}(\eta-1)^2(19\eta^3-17\eta^2-3\eta+1)\Bigg\} \\
& \times \frac{k_x u(t,\eta) + k_z W(t,\eta)}{k_x^2+k_z^2} d\eta
\end{aligned} \tag{236}
$$

VI. Discussion

We have shown the derivation for controllers which stabilize the 3D Navier-Stokes equations linearized around a Poiseuille profile equilibrium. We have also shown that the controllers induce stability around the equilibrium solution in the L_2 sense. These results can easily be extended to a periodic channel flow by substituting the Fourier transform by a Fourier series.

We also looked at the special case of $k_x = 0$. The system (119), (120) displays the cascade connection commonly regarded as the cause for non-orthogonality that leads to transient growth [9], [7], [12]. With our transformations (129), (130) and boundary feedback (126)–(128) we cut the coupling and reduce the system to two heat equations (131)–(133). The controllers in this case depend, at most, linearly on the Reynolds number. However, the gain kernals have an exponential dependence on k_z for large k_z.

We provided a small wavenumber expansion of the kernels K and Γ showing the cubic approximation of the controllers U_c and W_c around $k_x = k_z = 0$. This result involves an exact solution of a series of PDE problems. Taking a Taylor expansion of the original PDE system is more coherent than a Taylor expansion of K_n and Γ_n as the first method provides an exact Taylor series approximation for each order, whereas the second provides exact solutions up to order $(n-1)$ and then a partial selection of higher order terms.

In the future, we plan to extend the obsevers developed in [15] to 3D, and use these along with the results from this paper to perform DNS simulations showing the performance of an output feedback compensator using measurments and actuation only along the walls.

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