Boundary Control of PDEs:

A Course on Backstepping Designs

class slides

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Introduction

- *Fluid flows* in aerodynamics and propulsion applications;

  plasmas in lasers, fusion reactors, and hypersonic vehicles;

  liquid metals in cooling systems for tokamaks and computers, as well as in welding and metal casting processes;

  acoustic waves, water waves in irrigation systems...

- *Flexible structures* in civil engineering, aircraft wings and helicopter rotors, astronomical telescopes, and in nanotechnology devices like the atomic force microscope...

- *Electromagnetic* waves and quantum mechanical systems...

- *Waves* and “ripple” instabilities in thin film manufacturing and in flame dynamics...

- *Chemical* processes in process industries and in internal combustion engines...

Unfortunately, even “toy” PDE control problems like heat and wave equations (neither of which is unstable) require some background in functional analysis.

Courses in control of PDEs rare in engineering programs.

This course: methods which are easy to understand, minimal background beyond calculus.

Boundary Control

Two PDE control settings:

- "in domain" control (actuation penetrates inside the domain of the PDE system or is evenly distributed everywhere in the domain, likewise with sensing);

- "boundary" control (actuation and sensing are only through the boundary conditions).

Boundary control physically more realistic because actuation and sensing are non-intrusive (think, fluid flow where actuation is from the walls).*

*"Body force" actuation of electromagnetic type is also possible but it has low control authority and its spatial distribution typically has a pattern that favors the near-wall region.

Boundary control harder problem, because the “input operator” (the analog of the B matrix in the LTI finite dimensional model $\dot{x} = Ax + Bu$) and the output operator (the analog of the C matrix in $y = Cx$) are unbounded operators.

Most books on control of PDEs either don’t cover boundary control or dedicate only small fractions of their coverage to boundary control.

This course is devoted exclusively to boundary control.

Backstepping

A particular approach to stabilization of dynamic systems with "triangular" structure.

Wildly successful in the area of nonlinear control since

[KKK] Krstic, Kanellakopoulos, Kokotovic
Nonlinear and Adaptive Control Design, 1995.

Other methods:

Optimal control for PDEs requires sol'n of operator Riccati equations (nonlinear and infinite-dimensional algebraic eqns).

Pole placement pursues precise assignment of a finite subset of the PDE's eigenvalues and requires model reduction.

Instead, backstepping achieves Lyapunov stabilization by transforming the system into a stable "target system."

A Short List of Other Books on Control of PDEs

- R. F. CURTAIN AND H. J. ZWART, *An Introduction to Infinite Dimensional Linear Systems Theory*, Springer-Verlag, 1995.
- I. LASIECKA, R. TRIGGIANI, *Control Theory for Partial Differential Equations: Continuous and Approximation Theories*, Cambridge Univ. Press, 2000.
- A. BENSOUSSAN, G. DA PRATO, M. C. DELFOUR AND S. K. MITTER, *Representation and control of infinite–dimensional systems*, Birkhauser, 2006.
- Z. H. LUO, B. Z. GUO, AND O. MORGUL, *Stability and Stabilization of Infinite Dimensional Systems with Applications*, Springer Verlag, 1999.
- J. E. LAGNESE, *Boundary stabilization of thin plates*, SIAM, 1989.
- P. CHRISTOFIDES, *Nonlinear and Robust Control of Partial Differential Equation Systems: Methods and Applications to Transport-Reaction Processes*, Boston: Birkhäuser, 2001.

The Role of Model Reduction

Plays an important role in most methods for control design for PDEs.

They extract a finite dimensional subsystem to be controlled, while showing robustness to neglecting the remaining infinite dimensional dynamics in the design.

Backstepping does not employ model reduction—none is needed, except at the implementation stage.

Control Objectives for PDE Systems

- *Performance improvement*—for stable systems, optimal control.
- *Stabilization*—this course deals almost exclusively w/ unstable plants.
- *Trajectory tracking*—requires stabilizing fbk plus sol'n to trajectory generation probl.
- *Trajectory generation/motion planning*—towards the end of the course.

Classes of PDEs and Benchmark PDEs Dealt With in the Course

In contrast to ODEs, no general methodology for PDEs.

Two basic categories of PDEs studied in textbooks: *parabolic* and *hyperbolic* PDEs, with standard examples being heat and wave equations.

Many more categories.

Categorization of PDEs studied in the course

	∂_t	∂_{tt}
∂_x	transport PDEs, delays	
∂_{xx}	parabolic PDEs, reaction-advection-diffusion systems	hyperbolic PDEs, wave equations
∂_{xxx}	Korteweg-de Vries	
∂_{xxxx}	Kuramoto-Sivashinsky and Navier-Stokes (Orr-Sommerfeld form)	Euler-Bernoulli and shear beams, Schrodinger, Ginzburg-Landau

Timoshenko beam model has four derivatives in both time and space.

Also, complex-valued PDEs (with complex coefficients): Schrodinger and Ginzburg-Landau eqns. They “look” like parabolic PDEs, but behave like oscillatory, hyperbolic PDEs. Schrodinger equivalent to the Euler-Bernoulli beam PDE.

Choices of Boundary Controls

Thermal: actuate heat flux or temperature.

Structural: actuate beam's boundary position, or force, or angle, or moment.

Mathematical choices of boundary control:

Dirichlet control $u(1,t)$—actuate value of a function at boundary

Neumann control $u_x(1,t)$—actuate slope of a function at boundary

The Domain Dimension—1D, 2D, and 3D

PDE control complex enough in 1D: string, acoustic duct, beam, chemical tubular reactor, etc.

Can have finitely- and even infinitely-many unstable eigenvalues.

Some PDEs evolve in 2D and 3D but are dominated by phenomena evolving in one coordinate direction (while the phenomena in the other directions are stable and slow).

Some PDEs are genuinely 3D: Navier-Stokes.

See the companion book:

> Vazquez and Krstic, *Control of Turbulent and Magnetohydrodynamic Channel Flows*, Birkhauser, 2007.

Domain Shape in 2D and 3D

Rectangle or annulus much more readily tractable than a problem where the domain has an “amorphous/wiggly” shape.

Beware: literature abounds with abstract control methods for 2D and 3D PDE systems on general domains, where the complexities are hidden behind neatly written Riccati eqns.

Genuinely 2D or 3D systems, particularly if unstable and on oddly shaped domains (e.g., turbulent fluids in 3D around irregularly shaped bodies), truly require millions of differential equations to simulate and tens of thousands of equations to do control design for them.

Reasonable set up: boundary control of an endpoint of a line interval; edge of a rectangle; side of e parallelepiped.
(Dimension of actuation domain lower by one than dimension of PDE domain.)

Observers

Observer design using boundary sensing, dual to full-state fbk boundary control design.

Observer error system is exponentially stabilized.

Separation principle holds.

Adaptive Control of PDEs

Parameter estimators—system *identifiers*—for PDEs.

Unstable PDEs with unknown parameters controlled using parameter estimators supplied by identifiers and using state estimators supplied by adaptive observers.

See the companion book

> Smyshlyaev and Krstic, *Adaptive Control of Parabolic PDEs*, Princeton University Press, 2010.

Nonlinear PDEs

At present, virtually no methods exist for boundary control of nonlinear PDEs.

Several results are available that apply to nonlinear PDEs that are neutrally stable and where the nonlinearity plays no destabilizing role.

No advanced control designs exist for broad classes of nonlinear PDEs that are open-loop unstable and where a sophisticated control Lyapunov function of non-quadratic type needs to be constructed to achieve closed-loop stability.

Though the focus of the course is on linear PDEs, we introduce basic ideas for stabilization of nonlinear PDEs at the end.

Delay Systems

A special class of ODE/PDE systems.

Delay is a transport PDE. (One derivative in space and one in time. First-order hyperbolic.)

Specialized books by Gu, Michiels, Niculescu.

A book focused on input delays, nonlinear plants, and unknown delays:

> M. Krstic, *Delay Compensation for Nonlinear, Adaptive, and PDE Systems*, Birkhauser, 2009.

Organization of the Course

1. Basic Lyapunov stability ideas for PDEs. Backstepping transforms a PDE into a desirable “target PDE” within the same class. General Lyapunov thms for PDEs not very useful. We learn how to calculate *stability estimates* for a basic stable PDE and highlight the roles of spatial norms (L_2, H_1, and so on), the role of the Poincare, Agmon, and Sobolev inequalities, the role of integration by parts in Lyapunov calculations, and the distinction between energy boundedness and pointwise (in space) boundedness.

2. Eigenvalues, eigenfunctions, and basics of finding solutions of PDEs analytically.

3. Backstepping method. Our main “tutorial tool” is the reaction-diffusion PDE example

$$u_t(x,t) = u_{xx}(x,t) + \lambda u(x,t)\,,$$

on the spatial interval $x \in (0,1)$, with one uncontrolled boundary condition at $x = 0$,

$$u(0,t) = 0$$

and with a control applied through Dirichlet boundary actuation at $x = 1$.

4. Observer design. Develop a dual of backstepping for finding observer gain functions. Use reaction-diffusion PDE as an example.

5. Schrodinger and Ginzburg-Landau PDEs. Complex-valued but a backstepping design for parabolic PDEs easily extended. GL models vortex shedding.

6. Hyperbolic and “hyperbolic-like” equations—*wave equations, beams, transport equations, and delay equations*.

7. “Exotic” PDEs, with just one time derivative but with three and even four spatial derivatives—Kuramoto-Sivashinsky and Korteweg-de Vries eqns.

8. 3D Navier-Stokes eqn at high Reynolds number.

9. Motion planning/trajectory generation for PDEs. For example, how to find the time function for the input force for one end of a flexible beam to produce precisely the desired time-varying motion with the tip of the free end of the beam.

10. Adaptive control for parametrically uncertain PDEs.

11. Nonlinear PDEs.

Why We Don’t State Theorems

Focus on tools that allow to solve many problems, rather than on developing complete theorem statements for a few problems.

Want to move fast and cover many classes of PDEs and control/estimation topics.

Want to maintain physical intuition.

Want to make the material accessible to any control engineering grad student.

Focus on Unstable PDEs in 1D and Feedback Design Challenges

Unstable parabolic and hyperbolic PDEs in 1D with terms causing instability unmatched by the boundary control.

Feedback design challenges greater than the existence/uniqueness challenges, which are well addressed in analysis-oriented PDE books.

The Main Idea of Backstepping Control

Backstepping is a robust[†] extension of the "feedback linearization" approach for nonlinear finite-dimensional systems.

[†]Backstepping provides design tools that endow the controller with robustness to uncertain parameters and functional uncertainties in the plant nonlinearities, and robustness to external disturbances, robustness to other forms of modeling errors.

Feedback linearization entails two steps:

1. Construction of an *invertible change of variables* such that the system appears as linear in the new variables, except for a nonlinearity which is “in the span” of the control input vector;

2. *Cancellation of the nonlinearity*[‡] and the assignment of desirable linear exponentially stable dynamics on the closed-loop system.

[‡]In contrast to the standard feedback linearization, backstepping allows the flexibility to not necessarily cancel the nonlinearity. A nonlinearity may be kept if it is useful or it may be dominated (rather than cancelled non-robustly) if it is potentially harmful and uncertain.

Backstepping for PDEs:

1. Identify the undesirable terms in the PDE.

2. Choose a target system in which the undesirable terms are to be eliminated by state transformation and feedback, as in feedback linearization.

3. Find the state transformation as *identity minus a Volterra operator* (in x).
   Volterra operator = integral operator from 0 up to x (rather than from 0 to 1).
   Volterra transformation is “triangular” or “spatially causal.”

4. Obtain boundary feedback from the Volterra transformation. The transformation alone cannot eliminate the undesirable terms, but the transformantion brings them to the boundary, so control can cancel them.

Gain fcn of boundary controller = kernel of Volterra transformation.

Volterra kernel satisfies a *linear* PDE.

Backstepping is not “one-size-fits-all.” Requires structure-specific effort by designer.

Reward: elegant controller, clear closed-loop behavior.

Unique to This Course—Elements of Adaptive and Nonlinear Designs for PDEs

Prior to backstepping, state-of-the-art in *adaptive and nonlinear control* for PDEs comparable to the state-of-the-art for ODEs in the 1960s.

A wide range of PDE structures with nonlinearities, unknown parameters, and boundary control require backstepping.

Origins of This Course

Developed out of research results and papers by the instructor and his PhD students.

First taught as MAE 287 Distributed Parameter Systems at University of California, San Diego, in Fall 2005.

Lyapunov Stability

Recall some basics of stability analysis for linear ODEs.

An ODE

$$\dot{z} = Az, \qquad z \in \mathbb{R}^n \tag{1}$$

is exponentially stable (e.s.) at $z = 0$ if $\exists M > 0$ (overshoot coeff.) and $\alpha > 0$ (decay rate) s.t.

$$\boxed{\|z(t)\| \leq M\mathrm{e}^{-\alpha t}\|z(0)\|, \quad \text{for all } t \geq 0} \tag{2}$$

$\|\cdot\|$ denotes one of the equivalent vector norms, e.g., the 2-norm.

This is a *definition* of stability. If all the eigenvalues of the matrix A have negative real parts, this guarantees e.s., but this test is not always practical.

An alternative (iff) test which is more useful in state-space/time-domain and robustness studies:

> $\forall$ positive definite $n \times n$ matrix Q, $\exists$ a positive definite and symmetric matrix P s.t.
>
> $$\boxed{PA + A^T P = -Q\,.} \tag{3}$$

Lyapunov function:

$$V = x^T P x\,, \qquad \text{positive definite} \tag{4}$$

$$\dot{V} = -x^T Q x\,, \qquad \text{negative definite}\,. \tag{5}$$

For PDEs, an (infinite-dimensional) operator equation like (3) is hard to solve.

Key question for PDEs: not Lyapunov functions but system norms!

In finite dimension, vector norms are "equivalent." No matter which norm $\|\cdot\|$ one uses in (2) (for example, the 2-norm, 1-norm, or ∞-norm) one gets e.s. in the sense of any other vector norm. What changes are the constants M and α in (2).

For PDEs, the state space is not a Euclidean space but a function space, and likewise, the state norm is not a vector norm but a function norm.

Unfortunately, norms on function spaces are not equivalent. Bounds on the state in terms of the L_1, L_2, or L_∞ norm in x do not follow from one another.

To make matters more complicated, other natural choices of state norms for PDEs exist which are not equivalent with L_p norms. Those are the *Sobolev* norms, examples of which are the H_1 and H_2 norms (not to be confused with Hardy space norms in robust control for ODE systems), which, roughly, are the L_2 norms of the first and second derivative, respectively, of the PDE state.

With such a variety of choice, dictated by idiosyncracies of the PDE classes, general Lyapunov stability theory for PDEs is hopeless, though some efforts are made in

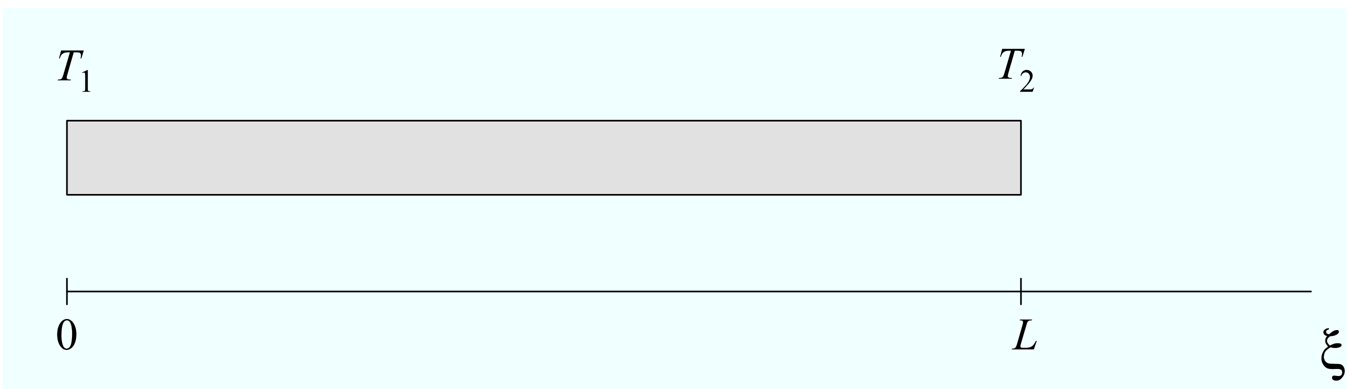
1. J. A. WALKER, *Dynamical Systems and Evolution Equations*, Plenum, 1980.

2. D. HENRY, *Geometric Theory of Semilinear Parabolic Equations*, Springer, 1993.

Instead, one is better off learning how to derive, from scratch, “energy estimates” (one’s own Lyapunov theorems) in different norms.

A Basic PDE Model

Before introducing stability concepts, we develop a basic "non-dimensionalized" PDE model, a 1D heat equation, which will help introduce the idea of energy estimates now, and be used as a target system for some backstepping designs later.



A thermally conducting rod.

The evolution of the temperature profile $T(\xi,\tau)$, as a function of the spatial variable ξ and time τ, is described by the heat equation[§]

$$\begin{aligned} T_\tau(\xi,\tau) &= \varepsilon T_{\xi\xi}(\xi,\tau)\,, \qquad x \in (0,L) && (6)\\ T(0,\tau) &= T_1\,, \qquad \text{left end of rod} && (7)\\ T(L,\tau) &= T_2\,, \qquad \text{right end of rod} && (8)\\ T(\xi,0) &= T_0(\xi)\,, \qquad \text{initial temperature distribution}\,. && (9) \end{aligned}$$

ε = thermal diffusivity

T_τ, $T_{\xi\xi}$ = partial derivatives with respect to time and space.

[§]While in physical heat conduction problems it is more appropriate to assume that the heat flux T_ξ is held constant at the boundaries (rather than the temperature T itself), for simplicity of our introductory exposition we proceed with the boundary conditions as in (7), (8) Our objective is to write this equation in nondimensional variables that describe the error between the unsteady temperature and the equilibrium profile of the temperature:

1. Scale ξ to normalize length:

$$x = \frac{\xi}{L}, \tag{10}$$

   which gives

$$\begin{aligned} T_\tau(x,\tau) &= \frac{\varepsilon}{L^2} T_{xx}(x,\tau) && (11) \\ T(0,\tau) &= T_1 && (12) \\ T(1,\tau) &= T_2 \,. && (13) \end{aligned}$$

2. Scale time to normalize thermal diffusivity:

$$t = \frac{\varepsilon}{L^2}\tau\,, \tag{14}$$

   which gives

$$\begin{aligned} T_t(x,t) &= T_{xx}(x,t) && (15) \\ T(0,t) &= T_1 && (16) \\ T(1,t) &= T_2 \,. && (17) \end{aligned}$$

3. Introduce new variable

$$w = T - \bar{T} \tag{18}$$

where

$$\bar{T}(x) = T_1 + x(T_2 - T_1)$$

is the steady-state profile and is found as a solution to the two-point boundary-value ODE

$$\bar{T}''(x) = 0 \tag{19}$$

$$\bar{T}(0) = T_1 \tag{20}$$

$$\bar{T}(1) = T_2 . \tag{21}$$

We obtain

$$
\begin{aligned}
w_t &= w_{xx} && (22)\\
w(0) &= 0 && (23)\\
w(1) &= 0\,, && (24)
\end{aligned}
$$

with initial distribution $w_0 = w(x,0)$.

Note that here and throughout the rest of the course for compactness and ease of the presentation we drop the dependence on time and spatial variable where it does not lead to a confusion, i.e. by w, $w(0)$ we always mean $w(x,t)$, $w(0,t)$, respectively, unless specifically stated.

The following are the basic types of boundary conditions for PDEs in dimension one:

- Dirichlet: $\boxed{w(0)=0}$ (fixed temperature at $x=0$)
- Neumann: $\boxed{w_x(0)=0}$ (fixed heat flux at $x=0$)
- Robin (mixed): $\boxed{w_x(0)+qw(0)=0}$

Lyapunov Analysis for a Heat Equation in Terms of 'L_2 Energy'

$$w_t = w_{xx} \tag{25}$$
$$w(0) = 0 \tag{26}$$
$$w(1) = 0\,. \tag{27}$$

Obviously stable for physical reasons and stability can also be shown by finding explic. soln.

But we want to learn a *method* for analyzing stability.

Lyapunov function candidate ("energy")[¶]

$$V(t) = \frac{1}{2}\int_0^1 w^2(x,t)\,dx = \frac{1}{2}\|w(t)\|^2 \tag{28}$$

where $\|\cdot\|$ denotes the L_2 norm of a function of x: $\boxed{\|w(t)\| = \left(\int_0^1 w(x,t)^2\,dx\right)^{1/2}.}$

[¶]Strictly speaking, this is a functional, but we refer to it simply as a "Lyapunov function."

Time derivative of V:

$$
\begin{aligned}
\dot{V} = \frac{dV}{dt} &= \int_0^1 w(x,t)w_t(x,t)dx && \text{(applying the chain rule)} \\
&= \int_0^1 ww_{xx}dx && \text{(from (25))} \\
&= \cancel{ww_x|_0^1} - \int_0^1 w_x^2 dx && \text{(integration by parts)} \\
&= -\int_0^1 w_x^2 dx . && (29)
\end{aligned}
$$

Since $\dot{V} \leq 0$, V is bounded. However, it is not clear if V goes to zero because (29) depends on w_x and not on w, so one cannot express the right hand side of (29) in terms of V.

Recall two useful inequalities:

Young's Inequality (special case)

$$\boxed{ab \leq \frac{\gamma}{2}a^2 + \frac{1}{2\gamma}b^2} \tag{30}$$

Cauchy-Schwartz Inequality

$$\boxed{\int_0^1 uw\,dx \leq \left(\int_0^1 u^2\,dx\right)^{1/2} \left(\int_0^1 w^2\,dx\right)^{1/2}} \tag{31}$$

The following lemma establishes the relationship between the L_2 norms of w and w_x.

Lemma 1 (Poincare Inequality) *For any w, continuously differentiable on $[0,1]$,*

$$\boxed{\begin{aligned}\int_0^1 w^2\,dx &\le 2w^2(1) + 4\int_0^1 w_x^2\,dx\\ \int_0^1 w^2\,dx &\le 2w^2(0) + 4\int_0^1 w_x^2\,dx\end{aligned}} \tag{32}$$

Remark 1 The inequalities (32) are conservative. A tighter version of (32) is

$$\int_0^1 w^2\,dx \le w^2(0) + \frac{8}{\pi^2}\int_0^1 w_x^2\,dx\,, \tag{33}$$

which is called "a variaton of Wirtinger's inequality." The proof of (33) is far more complicated than the proof of (32) and is given in the classical book on inequalities by Hardy, Littlewood, and Polya. When $\underline{w(0) = 0 \text{ or } w(1) = 0}$, one can even get $\boxed{\|w\| \le \frac{2}{\pi}\|w_x\|}$.

Proof.

$$\begin{aligned}\int_0^1 w^2\,dx &= xw^2|_0^1 - 2\int_0^1 xww_x\,dx \quad \text{(integration by parts)} \\ &= w^2(1) - 2\int_0^1 xww_x\,dx \\ &\leq w^2(1) + \frac{1}{2}\int_0^1 w^2dx + 2\int_0^1 x^2w_x^2\,dx\,.\end{aligned}$$

Subtracting the second term from both sides we get the first inequality in (32):

$$\begin{aligned}\frac{1}{2}\int_0^1 w^2dx &\leq w^2(1) + 2\int_0^1 x^2w_x^2dx \\ &\leq w^2(1) + 2\int_0^1 w_x^2dx\,. \end{aligned} \tag{34}$$

The second inequality in (32) is obtained in a similar fashion. **QED**

We now return to

$$\dot{V} = -\int_0^1 w_x^2 dx .$$

Using Poincare inequality along with boundary conditions $w(0) = w(1) = 0$, we get

$$\dot{V} = -\int_0^1 w_x^2 dx \le -\frac{1}{4}\int_0^1 w^2 \le -\frac{1}{2}V \tag{35}$$

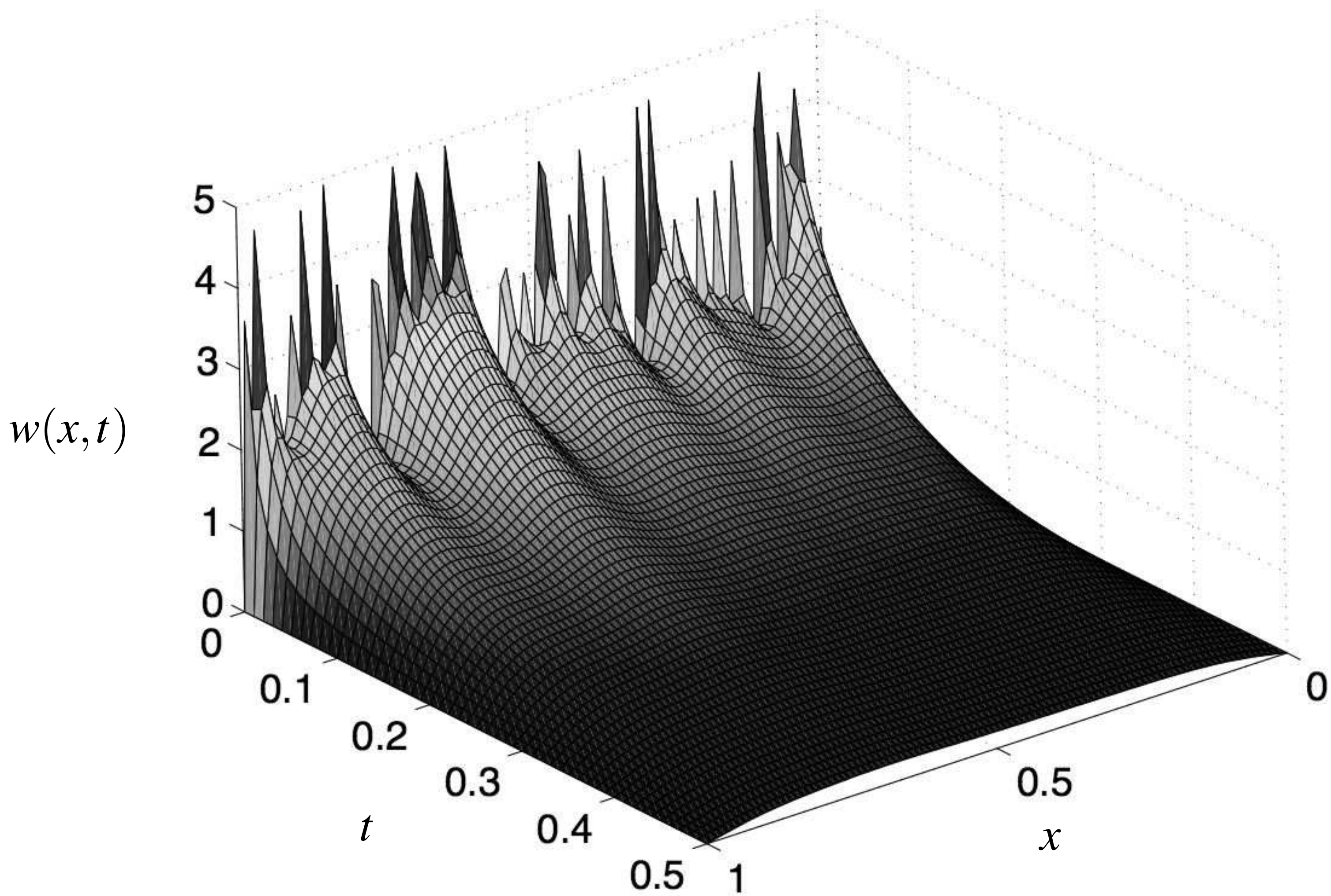
which, by the basic comparison principle for first order differential inequalities, implies that

$$V(t) \le V(0)e^{-t/2}, \tag{36}$$

or

$$\boxed{\|w(t)\| \le e^{-t/4}\|w_0\|} \tag{37}$$

Thus, the system (25)–(27) is exponentially stable in L_2.



Response of a heat equation to a non-smooth initial condition.

The “instant smoothing” effect is the characteristic feature of the diffusion operator that dominates the heat equation.

Pointwise-in-Space Boundedness and Stability in Higher Norms

We established that

$$\|w\| \to 0 \quad \text{as} \quad t \to \infty$$

but this does not imply that $w(x,t)$ goes to zero for each $x \in (0,1)$.

> Are there “unbounded spikes” for some x along the spatial domain (on a set of measure zero) which do not contribute to the L_2 norm?

It would be desirable to show that

$$\max_{x\in[0,1]} |w(x,t)| \leq e^{-\frac{t}{4}} \max_{x\in[0,1]} |w(x,0)| \,, \tag{38}$$

namely, stability in the spatial L_∞ norm. But this is possible only in some special cases and not worth our attention in a course that focuses on basic but generally applicable tools.

However, it is easy to show a more restrictive result than (38), given by

$$\max_{x\in[0,1]} |w(x,t)| \leq Ke^{-\frac{t}{2}} \|w_0\|_{H_1} \tag{39}$$

for some $K > 0$, where the H_1 norm is defined by

$$\boxed{\|w\|_{H_1}^2 := \sqrt{\int_0^1 w^2 dx + \int_0^1 w_x^2 dx}} \tag{40}$$

Remark 2 The H_1 norm can be defined in different ways, the definition given above suits our needs. Note also that by using the Poincare inequality, it is possible to drop the first integral in (40) for most problems.

Before we proceed to prove (39), we need the following result.

Lemma 2 (Agmon's Inequality) *For a function $w \in H_1$, the following inequalities hold*

$$\boxed{\begin{aligned} \max_{x\in[0,1]} |w(x,t)|^2 \leq w(0)^2 + 2\|w(t)\|\|w_x(t)\| \\ \max_{x\in[0,1]} |w(x,t)|^2 \leq w(1)^2 + 2\|w(t)\|\|w_x(t)\| \end{aligned}} \tag{41}$$

Proof.

$$\begin{aligned} \int_0^x w w_x dx &= \int_0^x \partial_x \frac{1}{2} w^2 dx \\ &= \frac{1}{2} w^2 |_0^x \\ &= \frac{1}{2} w(x)^2 - \frac{1}{2} w(0)^2. \end{aligned} \tag{42}$$

Taking the absolute value on both sides and using the triangle inequality gives

$$\frac{1}{2}|w(x)^2| \leq \int_0^x |w||w_x| dx + \frac{1}{2} w(0)^2. \tag{43}$$

Using the fact that an integral of a positive function is an increasing function of its upper limit, we rewrite the last inequality as

$$|w(x)|^2 \leq w(0)^2 + 2\int_0^1 |w(x)||w_x(x)|dx. \tag{44}$$

The right hand side of this inequality does not depend on x and therefore

$$\max_{x\in[0,1]} |w(x)|^2 \leq w(0)^2 + 2\int_0^1 |w(x)||w_x(x)|dx. \tag{45}$$

Using the Cauchy-Schwartz Inequality we get the first inequality of (41). The second inequality is obtained in a similar fashion. **QED**

The simplest way to prove $\max_{x\in[0,1]}|w(x,t)| \leq Ke^{-\frac{t}{2}}\|w_0\|_{H_1}$ is to use the following Lyapunov function

$$V_1 = \frac{1}{2}\int_0^1 w^2\,dx + \frac{1}{2}\int_0^1 w_x^2\,dx\,. \tag{46}$$

The time derivative of (46) is given by

$$\begin{aligned}
\dot{V}_1 &\leq -\|w_x\|^2 - \|w_{xx}\|^2 \leq -\|w_x\|^2 \\
&\leq -\frac{1}{2}\|w_x\|^2 - \frac{1}{2}\|w_x\|^2 \\
&\leq -\frac{1}{8}\|w\|^2 - \frac{1}{2}\|w_x\|^2 \qquad \text{(using (35))} \\
&\leq -\frac{1}{4}V_1.
\end{aligned}$$

Therefore,

$$\|w\|^2 + \|w_x\|^2 \leq e^{-t/2}\left(\|w_0\|^2 + \|w_{0,x}\|^2\right), \tag{47}$$

and using Young's and Agmon's inequalities we get

$$\begin{aligned}
\max_{x\in[0,1]}|w(x,t)|^2 &\leq 2\|w\|\|w_x\| \\
&\leq \|w\|^2 + \|w_x\|^2 \\
&\leq e^{-t/2}\left(\|w_0\|^2 + \|w_{x,0}\|^2\right).
\end{aligned} \tag{48}$$

We have thus showed that

$$w(x,t) \longrightarrow 0 \text{ as } t \longrightarrow \infty$$

for all $x \in [0,1]$.

Example 1 Consider the *diffusion-* *advection* equation

$$\begin{aligned} w_t &= w_{xx} + \boxed{w_x} && (49) \\ w_x(0) &= 0 && (50) \\ w(1) &= 0\,. && (51) \end{aligned}$$

Using the Lyapunov function (28) we get

$$\begin{aligned} \dot{V} &= \int_0^1 w w_t dx \\ &= \int_0^1 w w_{xx} dx + \int_0^1 w w_x dx \\ &= w w_x|_0^1 - \int_0^1 w_x^2 dx + \boxed{\int_0^1 w w_x dx} && \text{(integration by parts)} \\ &= -\int_0^1 w_x^2 dx + \boxed{\frac{1}{2} w^2|_0^1} \\ &= -\int_0^1 w_x^2 dx + \boxed{\cancel{\frac{1}{2} w^2(0)} - \frac{1}{2} w^2(0)} \\ &= -\int_0^1 w_x^2 dx - \frac{1}{2} w^2(0)\,. \end{aligned}$$

Finally, using the Poincare inequality (32) we get

$$\dot{V} \le -\frac{1}{4}\|w\|^2 \le -\frac{1}{2}V\,, \tag{52}$$

proving the exponential stability in L_2 norm,

$$\|w(t)\| \le \mathrm{e}^{-t/4}\|w_0\|.$$

Summary on Lyapunov function calculations so far

> It might appear that we are not constructing any non-trivial Lyapunov functions but only using the "diagonal" Lyapunov functions that do not involve any "cross-terms."

This is actually not the case with the remainder of the course. While so far we have studied only Lyapunov functions that are plain spatial norms of functions, in the sequel we are going to be constructing changes of variables for the PDE states. The Lyapunov functions will be employing the norms of the *transformed* state variables, which means that in the original PDE state our Lyapunov functions will be complex, sophisticated constructions that include 'non-diagonal' and 'cross-term' effects.

Homework

1. Prove the second inequalities in (32) and (41).

2. Consider the heat equation

$$w_t = w_{xx}$$

for $x \in (0,1)$ with the initial condition $w_0(x) = w(x,0)$ and boundary conditions

$$\begin{aligned} w_x(0) &= 0 \\ w_x(1) &= -\frac{1}{2}w(1)\,. \end{aligned}$$

Show that

$$\|w(t)\| \leq e^{-\frac{t}{4}}\|w_0\|\,.$$

3. Consider the Burgers equation

$$w_t = w_{xx} - w w_x$$

for $x \in (0,1)$ with the initial condition $w_0(x) = w(x,0)$ and boundary conditions

$$\begin{aligned} w(0) &= 0 \\ w_x(1) &= -\frac{1}{6}\left(w(1) + w^3(1)\right). \end{aligned}$$

Show that

$$\|w(t)\| \leq e^{-\frac{t}{4}} \|w_0\|.$$

Hint: complete the squares.

Exact Solutions to PDEs

In general, seeking explicit solutions to partial differential equations is a hopeless pursuit.

But closed-form solutions can be found for some linear PDE systems with constant coefficients.

The solution does not only provide us with an exact formula for a given initial condition, but also gives insight into the spatial structure (smooth or ripply) and the temporal evolution (monotonic or oscillating) of the PDE.

Separation of Variables

Consider the reaction-diffusion equation

$$u_t = u_{xx} + \lambda u \tag{53}$$

with boundary conditions

$$u(0) = 0 \tag{54}$$

$$u(1) = 0 \tag{55}$$

and initial condition $u(x,0) = u_0(x)$.

The most frequently used method to obtain solutions to PDEs with constant coefficients is the method of separation of variables (the other common method employs Laplace transform).

Let us assume that the solution $u(x,t)$ can be written as

$$u(x,t) = X(x)T(t)\,. \tag{56}$$

If we substitute the solution (56) into the PDE (53), we get

$$X(x)\dot{T}(t) = X''(x)T(t) + \lambda X(x)T(t). \tag{57}$$

Gathering the like terms on the opposite sides yields

$$\frac{\dot{T}(t)}{T(t)} = \frac{X''(x) + \lambda X(x)}{X(x)}. \tag{58}$$

Since the function on the left depends only on time and the function on the right depends only on the spatial variable, the equality can only hold if both functions are constant. Let us denote this constant by σ.

We then get two ODEs:

$$\dot{T} = \sigma T \tag{59}$$

with initial condition $T(0) = T_0$, and

$$X'' + (\lambda - \sigma)X = 0 \tag{60}$$

with boundary conditions $X(0) = X(1) = 0$ (they follow from the PDE boundary conditions).

The solution to (59) is given by

$$T = T_0 e^{\sigma t}. \tag{61}$$

The solution to (60) has the form

$$X(x) = A\sin(\sqrt{\lambda - \sigma}x) + B\cos(\sqrt{\lambda - \sigma}x), \tag{62}$$

where A and B are constants that should be determined from the boundary conditions.

We have:

$$X(0) = 0 \Rightarrow B = 0\,,$$
$$X(1) = 0 \Rightarrow A\sin(\sqrt{\lambda - \sigma}) = 0\,.$$

The last equality can only hold true if $\sqrt{\lambda - \sigma} = \pi n$ for $n = 0, 1, 2, \ldots$, so that

$$\sigma = \lambda - \pi^2 n^2, \qquad n = 0, 1, 2, \ldots \tag{63}$$

Substituting (61), (62) into (56) yields

$$u_n(x,t) = T_0 A_n e^{(\lambda - \pi^2 n^2)t} \sin(\pi n x), \;\; n = 0, 1, 2, \ldots \tag{64}$$

For linear PDEs the sum of particular solutions is also a solution (the principle of superposition). Therefore the formal general solution of (53)–(55) is given by

$$u(x,t) = \sum_{n=0}^{\infty} C_n e^{(\lambda - \pi^2 n^2)t} \sin(\pi n x) \tag{65}$$

where $C_n = A_n T_0$.

To determine the constants C_n, let us set $t = 0$ in (65) and multiply both sides of the resulting equality with $\sin(\pi m x)$:

$$u_0(x)\sin(\pi m x) = \sin(\pi m x)\sum_{n=1}^{\infty} C_n \sin(\pi n x)\,. \tag{66}$$

Then, using the identity

$$\int_0^1 \sin(\pi m x)\sin(\pi n x)dx = \left\{ \begin{array}{c} 1/2 \;\; n = m \\ 0 \;\; n \neq m \end{array} \right\} \tag{67}$$

we get

$$C_n = \frac{1}{2}\int_0^1 u_0(x)\sin(\pi n x)dx\,. \tag{68}$$

Substituting this expression into (65), we get

$$\boxed{u(x,t) = 2\sum_{n=1}^{\infty} e^{(\lambda-\pi^2 n^2)t}\sin(\pi n x)\int_0^1 \sin(\pi n x)u_0(x)dx} \tag{69}$$

Even though we obtained this solution formally, it can be proved that this is indeed a well defined solution in a sense that it is unique, has continuous spatial derivatives up to a second order, and depends continuously on the initial data.

Let us look at the structure of this solution. It consists of the following elements:

- eigenvalues (all real): $\lambda - \pi^2 n^2, \quad n = 1, 2, \ldots$
- eigenfunctions: $\sin(\pi n x)$
- effect of initial conditions: $\int_0^1 \sin(\pi n x) u_0(x) dx$

The largest eigenvalue $\lambda - \pi^2$ $(n = 1)$ indicates the rate of growth or decay of the solution. We can see that the plant is stable for $\lambda \leq \pi^2$ and is unstable otherwise.

After the transient response due to the initial conditions, the profile of the state will be proportional to the first eigenfunction $\sin(\pi x)$, since other modes decay much faster.

Sometimes it is possible to use the method of separation of variables to determine the stability properties of the plant even though the complete closed form solution cannot be obtained.

Example 2 Let us find the values of the parameter g for which the system

$$u_t = u_{xx} + gu(0) \tag{70}$$

$$u_x(0) = 0 \tag{71}$$

$$u(1) = 0 \tag{72}$$

is unstable.

This example is motivated by the model of thermal instability in solid propellant rockets, where the term $gu(0)$ is roughly the burning of the propellant at one end of the fuel chamber.

Using the method of separation of variables we set $u(x,t) = X(x)T(t)$ and (70) gives:

$$\frac{\dot{T}(t)}{T(t)} = \frac{X''(x) + gX(0)}{X(x)} = \sigma\,. \tag{73}$$

Hence, $T(t) = T(0)e^{\sigma t}$, whereas the solution of the ODE for X is given by

$$X(x) = A\sinh(\sqrt{\sigma}x) + B\cosh(\sqrt{\sigma}x) + \frac{g}{\sigma}X(0)\,. \tag{74}$$

Here the last term is a particular solution of a nonhomogeneous ODE (73).

Now we find the constant B in terms of $X(0)$ by setting $x = 0$ in the above equation. This gives $B = X(0)(1 - g/\sigma)$.

Using the boundary condition (71) we get $A = 0$ so that

$$X(x) = X(0)\left[\frac{g}{\sigma} + \left(1 - \frac{g}{\sigma}\right)\cosh(\sqrt{\sigma}x)\right]\,. \tag{75}$$

Using the other boundary condition (72), we get the eigenvalue relationship

$$\frac{g}{\sigma} = \left(\frac{g}{\sigma} - 1\right)\cosh(\sqrt{\sigma})\,. \tag{76}$$

The above equation has no closed form solution. However, in this particular example we can still find the stability region by finding values of g for which there are eigenvalues with zero real parts. First we check if $\sigma = 0$ satisfies (76) for some values of g.

Using the Taylor expansion for $\cosh(\sqrt{\sigma})$, we get

$$\frac{g}{\sigma} = \left(\frac{g}{\sigma} - 1\right)\left(1 + \frac{\sigma}{2} + O(\sigma^2)\right) = \frac{g}{\sigma} - 1 + \frac{g}{2} - \frac{\sigma}{2} + O(\sigma), \tag{77}$$

which gives

$$\boxed{g \to 2 \quad \text{as} \quad \sigma \to 0}$$

To show that there are no other eigenvalues on the imaginary axis, we set $\sigma = 2jy^2$, $y > 0$. Equation (76) then becomes

$$\cosh((j+1)y) = \frac{g}{g - 2jy^2}$$
$$\cos(y)\cosh(y) + j\sin(y)\sinh(y) = \frac{g^2 + 2jgy^2}{g^2 + 4y^4}\,.$$

Taking the absolute value, we get

$$\sinh(y)^2 + \cos(y)^2 = \frac{g^4 + 4g^2y^4}{(g^2 + 4y^4)^2}\,. \tag{78}$$

The only solution to this equation is $y = 0$, which can be seen by computing derivatives of both sides of (78):

$$\frac{d}{dy}(\sinh(y)^2 + \cos(y)^2) = \sinh(2y) - \sin(2y) > 0 \quad \text{for all } y > 0 \tag{79}$$
$$\frac{d}{dy}\frac{g^4 + 4g^2y^4}{(g^2 + 4y^4)^2} = -\frac{16g^2y^3}{(g^2 + 4y^4)^2} < 0 \quad \text{for all } y > 0\,. \tag{80}$$

Therefore, both sides of (78) start at the same point at $y = 0$ and for $y > 0$ the left hand side monotonically grows while the right hand side monotonically decays.

We thus proved that the plant (70)–(72) is neutrally stable only when $g = 2$.

SUMMARY: the plant is stable for $g < 2$ and unstable for $g > 2$.

Notes and References

The method of separation of variables is discussed in detail in classical PDE texts

R. Courant and D. Hilbert, *Methods of mathematical physics*, New York, Interscience Publishers, 1962.

E. Zauderer, *Partial differential equations of applied mathematics*, New York : Wiley, 2nd ed., 1998.

The exact solutions for many problems can be found in

H. S. Carslaw and J. C. Jaeger, *Conduction of Heat in Solids*, Oxford, Clarendon Press, 1959.

A. D. Polianin, *Handbook of Linear Partial Differential Equations for Engineers and Scientists*, Boca Raton, Fla, Chapman and Hall/CRC, 2002.

Transform methods for PDEs are studied extensively in

D. G. DUFFY, *Transform methods for solving partial differential equations*, Boca Raton, FL : CRC Press, 1994.

Homework

1. Consider the Reaction-Diffusion equation

$$u_t = u_{xx} + \lambda u$$

for $x \in (0,1)$ with the initial condition $u_0(x) = u(x,0)$ and boundary conditions

$$u_x(0) = 0$$
$$u(1) = 0\,.$$

1) Find the solution of this PDE.

2) For what values of the parameter λ is this system unstable?

2. Consider the heat equation

$$u_t = u_{xx}$$

with Robin's boundary conditions

$$u_x(0) = -qu(0)$$
$$u(1) = 0\,.$$

Find the range of values of the parameter q for which this system is unstable.