ENME 621

Robust Linear Control — Prof. M. Krstic

FINAL EXAM

May 14, 1997

Open books and notes. Total points: 50. Time 5:00–6:20.

1. (25 points)
Prove that the LMS estimator is $\mathcal{H}_\infty$ optimal. (This is a surprising, yet simple to prove, recent result due to Hassibi, Sayed, and Kailath, 1993.) To prove the result, complete the following steps. Consider the linear static model

$$y = \eta(t)\theta + v(t)\,, \tag{1}$$

where y is a measured scalar output of the model, $\eta(t)$ is a measured "regressor" row vector signal, $v(t)$ is unmeasured noise scalar signal, and θ is a *constant* but unknown column vector in $\mathcal{R}^p$. The objective is to find an estimator $\widehat{\theta}(t)$ of θ such that the induced $\mathcal{L}_2$ gain from v to $\theta - \widehat{\theta}$ is minimal. The following system is referred to as the LMS (least mean squares) estimator:

$$\dot{\widehat{\theta}} = \mu\eta^{\mathrm{T}}(y - \eta\widehat{\theta})\,, \tag{2}$$

where $\mu > 0$ is referred to as the "adaptation gain."

(a) Pose the estimation problem in the setting of $\mathcal{H}_\infty$ filtering on the finite horizon:

$$\begin{aligned} \dot{x} &= 0\,, \qquad x(0) = \theta && (3)\\ y &= \eta(t)x + v && (4)\\ z &= \eta(t)x\,. && (5) \end{aligned}$$

Show that the $\mathcal{H}_\infty$ filtering problem has a solution for any $\gamma \geq 1$. What do you get for $\gamma = 1$ and with the initial condition of the Riccati differential equation set to $Q(0) = \mu I$?

(b) Show that if the regressor satisfies the following "persistent excitation" condition

$$\underline{\sigma}\left\{\int_0^\infty \eta(t)^{\mathrm{T}}\eta(t)dt\right\} = \infty\,, \tag{6}$$

the value of γ in the $\mathcal{H}_\infty$ filtering problem cannot be less than one, which implies that $\gamma_{\mathrm{opt}} = 1$ is achieved with the LMS estimator.

Finally, note that for $\gamma = \infty$ you get the RLS (recursive least squares) estimator

$$\begin{aligned} \dot{\widehat{\theta}} &= Q\eta^{\mathrm{T}}(y - \eta\widehat{\theta}) && (7)\\ \dot{Q} &= -Q\eta^{\mathrm{T}}\eta Q\,, \qquad Q(0) > 0\,, && (8) \end{aligned}$$

which means that RLS is $\mathcal{H}_2$ optimal.

2. (25 points)
Consider the one-dimensional system

$$\dot{x} = u + w \tag{9}$$

$$z = \begin{bmatrix} x \\ u \end{bmatrix}. \tag{10}$$

(a) Find the optimal full information controller $u = -p(\gamma)x$ that achieves $\|R_{zw}\|_\infty < \gamma$ for any $\gamma > 1$.

(b) Consider the following integral usually referred to as the *entropy*:

$$I(\gamma) = -\frac{\gamma^2}{2\pi}\int_{-\infty}^{\infty} \ln\left|1 - \frac{1}{\gamma^2}R_{zw}(-j\omega)^{\mathrm{T}}R_{zw}(j\omega)\right| d\omega\,. \tag{11}$$

Show that the controller from Part (a) achieves $I(\gamma) = p(\gamma)$.

(c) Show that $I(\infty) = \|R_{zw}\|_2^2$. (This is actually true in general).