Examples of systems that have continuous solutions, despite not having globally Lipschitz right-hand sides and despite the solutions growing to infinity as $t \to \infty$, i.e., nontrivial examples of application of Theorem 3.3

1. Consider the system

$$\dot{x} = 2x(1+y^2) \tag{1}$$
$$\dot{y} = 2y(1-x^2) \tag{2}$$

The right hand side of this system is not globally Lipschitz (though it is locally Lipschitz). But the solution of the system is bounded for all $t \geq 0$, though growing to infinity.

To see this, use the usual polar coordinate change to derive

$$\dot{r} = r \tag{3}$$
$$\dot{\theta} = -r^2 \sin(2\theta) \tag{4}$$

You can see that the solution grows to infinity, yet, it remains bounded for all finite times, so by Theorem 3.3, a unique continuous solution exists. To convince yourselves, with some extra work you can actually find the solution explicitly. It is:

$$x(t) = x_0 e^{t/2} \sqrt{\frac{x_0^2 + y_0^2}{x_0^2 + y_0^2 e^{2(x_0^2+y_0^2)(1-e^{2t})}}} \tag{5}$$

$$y(t) = y_0 e^{2(x_0^2+y_0^2)\left(1-e^{2t}\right)+t/2} \sqrt{\frac{x_0^2 + y_0^2}{x_0^2 + y_0^2 e^{2(x_0^2+y_0^2)(1-e^{2t})}}} \tag{6}$$

Note that $y(t)$ decays to zero (at a super-fast speed, exponential of an exponential), whereas $x(t)$ goes to infinity (exponentially).

2. Consider the system

$$\dot{x} = \frac{1}{x}(1+x^2)\ln(1+x^2) \tag{7}$$

This system is locally Lipschitz, except at the origin. It is not globally Lipschitz. Its solution is continuous and unique for all $x_0 \neq 0$. It is given by

$$x(t) = \mathrm{sgn}(x_0)\sqrt{e^{\ln\left(1+x_0^2\right)e^{2t}} - 1} \tag{8}$$

Note that the solution goes to infinity very fast, at a rate that is exponential of an exponential, but not in finite time.

We have possible non-uniqueness for $x_0 = 0$ but even then continuity is guaranteed by a version of Theorem 3.3 (not stated in Khalil) that weakens the Lipschitzness assumption to continuity, and drops uniqueness from the statement, but does claim continuity.

We can find an example where the solutions are continuous *and unique,* with a locally Lipschitz but not globally Lipschitz right hand side: $\dot{x} = x\ln(1+x^2)$. However, unlike the more complicated looking system (7), this simpler looking system is not solvable in closed form.