Problem 1. (7 pts) Study the stability of the origin of the system
\[\begin{align*}
\dot{x}_1 &= -x_1 + x_1 x_3 \tan^{-1}(x_3) \\
\dot{x}_2 &= x_3 \\
\dot{x}_3 &= -x_3 - x_2 - x_1^2 \tan^{-1}(x_3)
\end{align*}\] (1) (2) (3)
by using the Lyapunov function candidate
\[V(x) = \frac{1}{2} (x_1^2 + x_2^2 + x_3^2).\] (4)

Provide the strongest stability result attainable while using (4).

Solution:
We focus on the equilibrium \(x = (0 \ 0 \ 0)^T\). Notice that \(V(x) > 0\) for all \(x \in \mathbb{R}^3 \setminus \{0\}\); notice also that \(V(0) = 0\) and that \(V(x) \to \infty\) as \(\|x\| \to \infty\). The Lie derivative of \(V\) along (1)–(3) satisfies
\[\dot{V} = \dot{x}_1 x_1 + \dot{x}_2 x_2 + \dot{x}_3 x_3 = - (x_1^2 + x_3^2) \leq 0\] (5)
for all \(x \in \mathbb{R}^3\). By Lyapunov’s theorem, the origin is stable; however, we can improve on this result. We have
\[S := \{x \in \mathbb{R}^3 \mid \dot{V} = 0\} = \{x_1 = x_3 = 0\}.\] (6)
In other words, \(\dot{V} = 0\) on the \(x_2\)-axis. However, from (3), \(x_3 \neq 0\) unless \(x_2 = 0\), that is, no solution of (1)–(3) can stay in \(S\) except for \(x \equiv 0\). By Barbashin–Krasovskii’s theorem, the origin is globally asymptotically stable.

Problem 2. For a non–negative, continuously differentiable and bounded function \(f(t)\) satisfying \(\dot{f}(t) \leq f(t) \leq k\) for all \(t \in [0, \infty)\) and \(k > 0\), consider the linear nonautonomous system
\[\begin{align*}
\dot{x}_1 &= -x_1 - f(t) (x_2 - x_3) \\
\dot{x}_2 &= -(x_2 - x_1) \\
\dot{x}_3 &= -(x_1 + x_3)
\end{align*}\] (7) (8) (9)

(a) (7 pts) Prove that the origin is globally uniformly asymptotically stable (GUAS) for (7)–(9).

Hint: For \(x := (x_1, x_2, x_3)^T\), consider a Lyapunov function of the form
\[V(t, x) = \tilde{V}(x) + f(t) \tilde{V}((0, x_2, x_3)^T).\]

Next, employ Theorems 4.8 and 4.9 from Khalil’s book.

(b) (1 pt) Does it follow from GUAS that (7)–(9) is exponentially stable about the origin? Explain.
(c) (7 pts) Prove the inequality
\[ \|x(t)\| \leq Ce^{-\gamma t}\|x(0)\|, \quad (10) \]
providing at least one set of valid constants \{C, \gamma\}, for \(C, \gamma > 0\), explicitly in terms of \(k\), where \(\| \cdot \|\) denotes the Euclidean norm.

**Hint:** Use the comparison principle together with some of the inequalities derived in (a).

**Solution:**
(a) Consider the Lyapunov function
\[ V(t, x) = \frac{1}{2} \left( x_1^2 + (1 + f(t)) \left( x_2^2 + x_3^2 \right) \right); \quad (11) \]
it follows from the conditions on \(f(t)\) that for all \(t \in [0, \infty)\),
\[ \frac{1}{2} \|x\|^2 \leq V \leq \frac{1 + k}{2} \|x\|^2; \quad (12) \]
notice also that \(V(t, x) \to \infty\) as \(\|x\| \to \infty\) (here, \(W_1(\|x\|) = \frac{1}{2}\|x\|^2 \in \mathcal{K}\) and \(W_2(\|x\|) = \frac{1+k}{2}\|x\|^2 \in \mathcal{K}\)). The Lie derivative of \(V\) along (7)–(9) satisfies
\[
\dot{V}(t, x) = -x_1^2 - \frac{1}{2} \left( 2(1 + f(t)) - \dot{f}(t) \right) \left( x_2^2 + x_3^2 \right) + x_1(x_2 - x_3) \\
\leq - \left( x_1^2 + x_2^2 + x_3^2 \right) + x_1(x_2 - x_3) \quad (13)
\]
for all \(t \in [0, \infty)\), since \(2f(t) \geq \dot{f}(t)\). For \(a, b \in \mathbb{R}_{>0}\), we have from Young’s inequality that
\[ |x_1 x_2| \leq \frac{x_1^2}{2a} + \frac{ax_2^2}{2} \quad (14) \]
and
\[ |x_1 x_3| \leq \frac{x_1^2}{2b} + \frac{bx_3^2}{3}. \quad (15) \]
Choosing \(a, b\) such that \(\frac{1}{a} + \frac{1}{b} < 2\) and \(a < 2, b < 2\), together with (14) and (15), ensure that \(\dot{V} \leq -W_3(\|x\|)\) uniformly in \(t\): for example, choosing \(a = b = \frac{3}{2}\) yields
\[
\dot{V} \leq -\frac{1}{3} x_1^2 - \frac{2}{3} (x_2^2 + x_3^2) \\
\leq - \frac{1}{3}\|x\|^2 \quad := W_3(\|x\|) \quad (16)
\]
for all \(t \in [0, \infty)\) (here, \(\alpha_3(\|x\|) = \frac{1}{3}\|x\|^2 \in \mathcal{K}\)). In general, we recover
\[
\dot{V} \leq - \left( 1 - \frac{1}{2a} - \frac{1}{2b} \right) x_1^2 - \left( 1 - \frac{1}{2a} \right) x_2^2 - \left( 1 - \frac{1}{2b} \right) x_3^2 \\
\leq - \left( 1 - \frac{1}{2a} - \frac{1}{2b} \right) \|x\|^2 \quad := W_3(\|x\|) \quad (17)
\]
Thus, (7)–(9) is GUAS.
(b) While asymptotic stability \(\implies\) exponential stability for linear time–invariant systems, for LTV systems this implication cannot be directly deduced from the definition of GUAS. An answer utilizing Theorem 4.10 in Khalil’s text is also acceptable.
(c) It follows from (12) and (17) that for \(a = b = \frac{3}{2}\),
\[
\dot{V}(t, x) \leq - \frac{2}{3(1+k)} V(t, x). \quad (18)
\]
An application of the comparison lemma yields
\[ V(t, x) \leq e^{-\frac{2t}{M(1+\kappa)}} V(0, x(0)). \] (19)

Finally, we deduce from (12) that
\[
\|x(t)\|^2 \leq 2V(t, x) \\
\leq 2e^{-\frac{2t}{M(1+\kappa)}} V(0, x(0)) \\
\leq (1 + k)e^{-\frac{2t}{M(1+\kappa)}} \|x(0)\|^2;
\] (20)

taking the square root yields
\[
\|x(t)\| \leq \sqrt{(1 + k)e^{-\frac{2t}{M(1+\kappa)}}} \|x(0)\|. 
\] (21)

All inequalities of the form
\[
\|x(t)\| \leq \sqrt{(1 + k)e^{-\frac{2t}{M(1+\kappa)}}(1 - \frac{1}{a} - \frac{1}{b})} \|x(0)\| 
\] (22)
for \( \frac{1}{a} + \frac{1}{b} < 2, a < 2 \) and \( b < 2 \) also follow from the above treatment.

**Problem 3.** For \( \lambda \in \mathbb{R} \) and \( t \in [0, \infty) \), consider the nonlinear autonomous system
\[
\dot{x}_1 = x_2 + x_1 \left( \lambda - x_1^2 - x_2^2 \right) \\
\dot{x}_2 = -x_1 + x_2 \left( \lambda - x_1^2 - x_2^2 \right)
\] (23)
(24)

(a) (4 pts) For \( \lambda \neq 0 \), discuss the stability of the origin by linearizing (23)–(24).

**Hint:** Consider the polar coordinate transformation \( r = \sqrt{x_1^2 + x_2^2} \) and \( \phi = \tan^{-1} \left( \frac{x_1}{x_2} \right) \).

(b) (4 pts) Determine the limit cycle of (23)–(24) in terms of \( \lambda \).

**Hint:** Consider the polar coordinate transformation \( r = \sqrt{x_1^2 + x_2^2} \) and \( \phi = \tan^{-1} \left( \frac{x_1}{x_2} \right) \).

(c) (7 pts) For \( \lambda > 0 \), show that any trajectory of (23)–(24) converges to the limit cycle as \( t \to \infty \) (except the trivial one, \( x_1(x_1, x_2) \equiv 0 \)).

**Hint:** Use Lasalle's invariance principle and consider a Lyapunov–like function which incorporates the square of the distance from the limit cycle found in (b).

**Solution:**
(a) We linearize (23)–(24) around the origin to obtain the linearized system
\[
\begin{pmatrix}
\dot{x}_1 \\
\dot{x}_2
\end{pmatrix} =
\begin{pmatrix}
\lambda & 1 \\
-1 & \lambda
\end{pmatrix}
\begin{pmatrix}
x_1 \\
x_2
\end{pmatrix},
\] (25)
where the characteristic equation of the matrix is given by \( \chi(s) = s^2 - 2\lambda s + (1 + \lambda^2) \), yielding the eigenvalues \( \lambda + j \) and \( \lambda - j \), where \( j \) denotes the complex unit. Hence, for \( \lambda < 0 \), the origin is stable for (23)–(24), whereas it is unstable for \( \lambda > 0 \).

(b) Defining \( x_1 := r \cos \theta \) and \( x_2 := r \sin \theta \) (i.e., switching to polar coordinates), (23)–(24) rewrites as
\[
\begin{align*}
\dot{r} &= r \left( \lambda - r^2 \right), \\
\dot{\theta} &= 1.
\end{align*}
\] (26) (27)

It follows that the limit cycle of (23)–(24) is the circle with radius \( \sqrt{\lambda} \) centered at the origin, given by the set of points \( \{x_1^2 + x_2^2 = \lambda\} \).

(c) Consider the function
\[
V(x) = \frac{1}{4} \left( \lambda - (x_1^2 + x_2^2) \right)^2.
\] (28)
Notice that $V$ is continuously differentiable; moreover, the Lie derivative of $V$ along (23)–(24) is given by

$$
\dot{V} = \frac{1}{2} (\lambda - (x_1^2 + x_2^2)) (-2x_1\dot{x}_1 - 2x_2\dot{x}_2)
= - (\lambda - (x_1^2 + x_2^2)) (x_1 (x_2 + 1 (\lambda - x_1^2 - x_2^2)) + x_2 (-x_1 + 2 (\lambda - x_1^2 + x_2^2)))
= - (x_1^2 + x_2^2) (\lambda - (x_1^2 + x_2^2))^2.
$$

Hence, $V$ is negative semidefinite for all $x \in \mathbb{R}^2$. Defining the positively invariant compact set $\Omega_c := \{x \in \mathbb{R}^2 \mid V(x) \leq c\}$ for $c > 0$, we have

$$
E := \{x \in \Omega_c \mid \dot{V}(x) = 0\} = \{x_1^2 + x_2^2 = \lambda\} \cup \{0\}. \quad (30)
$$

Notice that the largest invariant subset contained in $E$ is itself (i.e., $M = E$). Hence, by Lasalle’s invariance principle, for any $x(0) \in \Omega_c$, $x(t) \to M$ as $t \to \infty$. For $\lambda > 0$ we have established in (a) that the origin is unstable; moreover, we have established in (b) that $\{x_1^2 + x_2^2 = \lambda\}$ described the limit cycle for (23)–(24). Thus, any trajectory of (23)–(24) aside from the trivial one converges to the limit cycle.

**Problem 4.** (15 pts) For small $\epsilon > 0$, study the stability properties of the “weakly” nonlinear system

\begin{align*}
\dot{x}_1 &= 5\epsilon x_1 + x_2 \\
\dot{x}_2 &= \epsilon (1 - x_1^2) x_2 - x_1
\end{align*}

using averaging theory.

**Hint:** When $\epsilon = 0$, (31)–(32) exhibits a simple behavior. This behavior motivates the change of coordinates $r^2 := x^2 + y^2$ and $\phi := \tan^{-1}\left(\frac{x_1}{x_2}\right)$. While this change of coordinates may not directly yield a system in the form $\dot{x} = \epsilon f(t, x, \epsilon)$, consider utilizing the chain rule to combine the resulting equations to obtain a new periodic variable satisfying an equation for which averaging directly applies. If you are still stuck, consult Chapter 10.5 in Khalil’s book for more assistance.

**Solution:**

When $\epsilon = 0$, system (31)–(32) describes a rotation in the $\mathbb{R}^2$–plane. This motivates the polar coordinate transformation

$$
r^2 := x_1^2 + x_2^2 \quad \text{and} \quad \phi := \tan^{-1}\left(\frac{x_1}{x_2}\right). \quad (33)
$$

We compute the derivatives

\begin{align*}
\dot{r} &= \frac{1}{2r} \left(2r \sin \phi (r \cos \phi + 5\epsilon r \sin \phi) + 2r \cos \phi (\epsilon(1 - r^2 \sin^2 \phi) r \cos \phi - r \sin \phi)\right) \\
&= \epsilon r (1 + (4 - r^2 \cos^2 \phi) \sin^2 \phi)
\end{align*}

and

\begin{align*}
\dot{\phi} &= \frac{1}{1 + \sin^2 \phi \cos^2 \phi} \left(\frac{\dot{x}_1 x_2 - \dot{x}_2 x_1}{x_2^2}\right) \\
&= \frac{1}{r} (\dot{x}_1 \cos \phi - \dot{x}_2 \sin \phi) \\
&= \cos^2 \phi + 5\epsilon \sin \phi \cos \phi + \sin^2 \phi - \epsilon (1 - r^2 \sin^2 \phi) \sin \phi \cos \phi \\
&= 1 + \epsilon (4 + r^2 \sin^2 \phi) \sin \phi \cos \phi. \quad (35)
\end{align*}

Notice that we have not recovered $\left(\dot{r}, \dot{\phi}\right)^T = \epsilon f(t, (r, \phi)^T, \epsilon)$, and hence cannot apply the averaging theory directly. Upon closer inspection of (35), one notices that $\dot{\phi}$ is a perturbation of 1; it follows that $\phi$ is nothing but
a perturbation of time itself. From this viewpoint, we treat $\phi$ as a time–like variable, and we utilize the chain rule to obtain

$$\frac{dr}{d\phi} = \frac{\epsilon r (1 + (4 - r^2 \cos^2 \phi) \sin^2 \phi)}{1 + \epsilon (4 + r^2 \sin^2 \phi) \sin \phi \cos \phi} =: \epsilon f(\phi, r, \epsilon), \quad (36)$$

where $f$ is $2\pi$–periodic in $\phi$. We compute $f_{av}(r) := \frac{1}{2\pi} \int_0^{2\pi} f(\phi, r, 0) d\phi$:

$$f_{av}(r) = \frac{1}{2\pi} \int_0^{2\pi} r (1 + (4 - r^2 \cos^2 \phi) \sin^2 \phi) \ d\phi = r \left(3 - \frac{r^2}{8}\right). \quad (37)$$

Hence, $\dot{r} = \epsilon f_{av}(r)$ has equilibria $r = 0$ and $r = 2\sqrt{6}$. Notice that

$$\left. \epsilon \frac{\partial f_{av}}{\partial r} \right|_{r=0} = 3\epsilon \quad \text{and} \quad \left. \epsilon \frac{\partial f_{av}}{\partial r} \right|_{r=2\sqrt{6}} = -6\epsilon. \quad (38)$$

Thus, by averaging theory, there exists a unique exponentially stable $2\pi$–periodic (in $\phi$) solution in an $O(\epsilon)$–neighborhood of $r = 2\sqrt{6}$ (or of $x^2 + y^2 = 24$).

**Problem 5.** (10 pts) Show that the system

$$\begin{align*}
\dot{x}_1 &= -x_1 \left(\frac{1}{2} + x_1^2\right) + x_2 \sin x_1 \\
\dot{x}_2 &= -x_2 \left(\frac{1}{2} + x_2^2\right) - x_1^2 x_2 + x_1 u
\end{align*} \quad (39)$$

is input–to–state stable. Moreover, when using the Lyapunov function $V(x) = \frac{1}{2} (x_1^2 + x_2^2)$, show that the system has a gain function $\gamma(r) = \sqrt{r}$. Verify any assumptions of theorems that you use.

**Hint:** Consult Theorem 4.19 in Khalil’s book. You may need to use Young’s inequality several times.

**Solution:** Notice that

$$\alpha_1(||x||) \leq V(x) \leq \alpha_2(||x||), \quad (41)$$

for $\alpha_1(||x||) = \alpha_2(||x||) := \frac{1}{2} ||x||^2 \in \mathcal{K}$. The Lie derivative of $V(x)$ along (39)–(40) satisfies

$$\begin{align*}
\dot{V} &= -\frac{1}{2} (x_1^2 + x_2^2) - (x_1^4 + x_2^4) - x_1^2 x_2^2 + x_1 x_2 \sin x_1 + u \\
&\leq - (x_1^4 + x_2^4) - x_1^2 x_2^2 + ||x_1|| ||x_2|| u \\
&\leq -\frac{1}{2} (x_1^4 + x_2^4) - \frac{1}{2} (x_1^2 + x_2^2)^2 + \frac{1}{2} (x_1^2 + x_2^2) ||u|| \\
&= -\alpha_3(||x||) + \frac{1}{2} ||x||^2 (-||x||^2 + ||u||), \quad (42)
\end{align*}$$

for $\alpha_3(||x||) := \frac{1}{2} ||x||^4 - x_1^2 x_2^2 \in \mathcal{K}$ (e.g., by Young’s inequality), where we’ve employed Young’s inequality in the second and third lines. It follows that

$$\dot{V} \leq -\alpha_3(||x||) \quad (43)$$

provided that $||x|| \geq \sqrt{||u||} =: \rho(||u||)$. It follows that (39)–(40) is input–to–state stable with gain

$$\gamma(r) := \alpha_1^{-1} \circ \alpha_2 \circ \rho(r) = \sqrt{r}. \quad (44)$$
**Bonus Problem.** (3 pts) For any \( r = 2n + 1, n = 0, 1, 2, \ldots \), prove that the origin \( x = (x_1, x_2) = (0, 0) \) for the system

\[
\begin{align*}
\dot{x}_1 &= -x_1^r + x_2 \quad \text{(45)} \\
\dot{x}_2 &= -x_2^r \quad \text{(46)}
\end{align*}
\]

is globally asymptotically stable using Lyapunov’s second method. Provide an explicit Lyapunov function for the case \( r = 3 \).

**Hint:** search for a polynomial in \( x_1, x_2 \) (allowing the coefficients and integer exponents to depend on \( r \)) as a candidate Lyapunov function \( V(x) \). State all of the necessary conditions which must be imposed on \( V(x) \) for it to satisfy Lyapunov’s theorem.

**Solution:** Consider the Lyapunov function

\[
V(x) = \frac{x_1^{a(r)}}{a(r)} + \frac{x_2^{b(r)}}{b(r)}, \quad \text{where } a(r) = 2s \text{ and } b(r) = 2q, \text{ for } s, q \in \mathbb{N}. \]

It follows from these selections of \( a(r) \) and \( b(r) \) that \( V(x) > 0 \) for all \( x \in \mathbb{R}^2 \setminus \{0\} \); notice also that \( V(0) = 0 \) and that \( V(x) \to \infty \) as \( \|x\| \to \infty \). The Lie derivative of \( V \) along (45)–(46) is given by

\[
\dot{V}(x) = -x_1^{a(r)+r-1} - x_2^{b(r)+r-1} + x_1^{a(r)-1}x_2. \quad \text{(48)}
\]

Recall that for \( a, b, p, q \in \mathbb{R} \) with \( \frac{1}{p} + \frac{1}{q} = 1 \), Young’s inequality gives

\[
|ab| \leq \frac{|a|^p}{p} + \frac{|b|^q}{q}. \quad \text{(49)}
\]

Hence, for every fixed \( t \in [0, \infty) \),

\[
\left| x_1^{a(r)-1}x_2 \right| \leq \frac{|x_1|^{(a(r)-1)p}}{p} + \frac{|x_2|^q}{q}. \quad \text{(50)}
\]

In order for cancellation to occur in (48), we impose \( (a(r)-1)p = a(r) + r - 1 \), or equivalently,

\[
p = \frac{a(r) + r - 1}{a(r) - 1}. \quad \text{(51)}
\]

It follows that

\[
q = \frac{a(r) + r - 1}{r}. \quad \text{(52)}
\]

For further cancellation in (48), we also impose \( q = b(r) + r - 1 \), or equivalently,

\[
b(r) = \frac{a(r) - (r-1)^2}{r}. \quad \text{(53)}
\]

But \( r = 2n + 1 \), for \( n \in \mathbb{N} \cup \{0\} \), and \( a(r) = 2s, b(r) = 2q \), for \( s, q \in \mathbb{N} \); hence, we require

\[
2q = \frac{2s - (2n + 1 - 1)^2}{2n + 1} \iff q = \frac{s - 2n^2}{2n + 1}. \quad \text{(54)}
\]

By fixing \( q \in \mathbb{N} \), we require \( s = 2n^2 + q(2n + 1) \in \mathbb{N} \). We obtain the coefficients/integer exponents \( a(r) = 4n^2 + 2q(2n + 1) \) and \( b(r) = 2q \), which are designed to ensure cancellation of the (potentially) positive term in (48). It follows from (48), (50) and these choices that

\[
\dot{V}(x) \leq -\frac{4n^2 + 2q(2n + 1) - 1}{4n^2 + (2n + 1)(2q + 1) - 1} x_1^{4n^2 + (2n + 1)(2q + 1) - 1} - \frac{2n + 1}{4n^2 + (2n + 1)(2q + 1) - 1} x_2^{2(q+n)} \quad \text{(55)}
\]

\[
< 0 \text{ for all } x \in \mathbb{R}^2 \setminus \{0\}, \quad \text{(56)}
\]
since \( n \in \mathbb{N} \cup \{0\} \) and \( q \in \mathbb{N} \).

For \( r = 3 \) (i.e., for \( n = 1 \)), we obtain \( a(3) = 2(2 + 3q) \) and \( b(3) = 2q \). Choosing the smallest integer \( q = 1 \) yields the candidate Lyapunov function

\[
V(x) = \frac{x_1^{10}}{10} + \frac{x_2^2}{2},
\]

whose derivative satisfies

\[
\dot{V}(x) \leq - \left( \frac{3x_1^{12}}{4} + \frac{x_2^4}{4} \right) < 0 \quad \text{for all } x \in \mathbb{R}^2 \setminus \{0\}.
\]