MAE 281A Prof. M. Krstic

NONLINEAR SYSTEMS Winter 2020

FINAL EXAM

Open book and class notes. **Collaboration not allowed.**

Total points: 62

Due Thursday, March 19 at 11:00 AM in EBU1 Rm. 2101

Problem 1. (7 pts) Study the stability of the origin of the system

$$\dot{x}_1 = -x_1 + x_1 x_3 \tan^{-1}(x_3) \tag{1}$$
$$\dot{x}_2 = x_3 \tag{2}$$
$$\dot{x}_3 = -x_3 - x_2 - x_1^2 \tan^{-1}(x_3) \tag{3}$$

by using the Lyapunov function candidate

$$V(x) = \frac{1}{2}\left(x_1^2 + x_2^2 + x_3^2\right). \tag{4}$$

Provide the strongest stability result attainable while using (4).

Problem 2. For a non–negative, continuously differentiable and bounded function $f(t)$ satisfying $\dot{f}(t) \leq f(t) \leq k$ for all $t \in [0, \infty)$ and $k > 0$, consider the linear nonautonomous system

$$\dot{x}_1 = -x_1 - f(t)(x_2 - x_3) \tag{5}$$
$$\dot{x}_2 = -(x_2 - x_1) \tag{6}$$
$$\dot{x}_3 = -(x_1 + x_3) \tag{7}$$

(a) (7 pts) Prove that the origin is globally uniformly asymptotically stable (GUAS) for (5)–(7).

Hint: For $x := (x_1, x_2, x_3)^T$, consider a Lyapunov function of the form

$$V(t, x) = \tilde{V}(x) + f(t)\tilde{V}((0, x_2, x_3)^T).$$

Next, employ Theorems 4.8 and 4.9 from Khalil's book.

(b) (1 pt) Does it follow from GUAS that (5)–(7) is exponentially stable about the origin? Explain.

(c) (7 pts) Prove the inequality

$$\|x(t)\| \leq Ce^{-\gamma t}\|x(0)\|, \tag{8}$$

providing at least one set of valid constants $\{C, \gamma\}$, for $C, \gamma > 0$, explicitly in terms of k, where $\|\cdot\|$ denotes the Euclidean norm.

Hint: Use the comparison principle together with some of the inequalities derived in (a).

Problem 3. For $\lambda \in \mathbb{R}$ and $t \in [0, \infty)$, consider the nonlinear autonomous system

$$\dot{x}_1 = x_2 + x_1\left(\lambda - x_1^2 - x_2^2\right) \tag{9}$$
$$\dot{x}_2 = -x_1 + x_2\left(\lambda - x_1^2 - x_2^2\right) \tag{10}$$

(a) (4 pts) For $\lambda \neq 0$, discuss the stability of the origin by linearizing (9)–(10).

(b) (4 pts) Determine the limit cycle of (9)–(10) in terms of λ.

Hint: Consider the polar coordinate transformation $r = \sqrt{x_1^2 + x_2^2}$ and $\phi = \tan^{-1}\left(\frac{x_1}{x_2}\right)$.

(c) (7 pts) For $\lambda > 0$, show that any trajectory of (9)–(10) converges to the limit cycle as $t \to \infty$ (except the trivial one, $(x_1, x_2) \equiv 0$).

Hint: Use Lasalle's invariance principle and consider a Lyapunov–like function which incorporates the square of the distance from the limit cycle found in (b).

Problem 4. (15 pts) For small $\epsilon > 0$, study the stability properties of the "weakly" nonlinear system

$$\dot{x}_1 = 5\epsilon x_1 + x_2 \tag{11}$$
$$\dot{x}_2 = \epsilon\left(1 - x_1^2\right)x_2 - x_1 \tag{12}$$

using averaging theory.

Hint: When $\epsilon = 0$, (11)–(12) exhibits a simple behavior. This behavior motivates the change of coordinates $r^2 := x^2 + y^2$ and $\phi := \tan^{-1}\left(\frac{x_1}{x_2}\right)$. While this change of coordinates may not directly yield a system in the form $\dot{x} = \epsilon f(t, x, \epsilon)$, consider utilizing the chain rule to combine the resulting equations to obtain a new periodic variable satisfying an equation for which averaging directly applies. If you are still stuck, consult Chapter 10.5 in Khalil's book for more assistance.

Problem 5. (10 pts) Show that the system

$$\dot{x}_1 = -x_1\left(\frac{1}{2} + x_1^2\right) + x_2 \sin x_1 \tag{13}$$
$$\dot{x}_2 = -x_2\left(\frac{1}{2} + x_2^2\right) - x_1^2 x_2 + x_1 u \tag{14}$$

is input–to–state stable. Moreover, when using the Lyapunov function $V(x) = \frac{1}{2}\left(x_1^2 + x_2^2\right)$, show that the system has a gain function $\gamma(r) = \sqrt{r}$. Verify any assumptions of theorems that you use.

Hint: Consult Theorem 4.19 in Khalil's book. You may need to use Young's inequality several times.

Bonus Problem. (3 pts) For any $r = 2n+1$, $n = 0, 1, 2, \ldots$, prove that the origin $x = (x_1, x_2) = (0, 0)$ for the system

$$\dot{x}_1 = -x_1^r + x_2 \tag{15}$$
$$\dot{x}_2 = -x_2^r \tag{16}$$

is *globally* asymptotically stable using Lyapunov's second method. Provide an explicit Lyapunov function for the case $r = 3$.

Hint: search for a polynomial in x_1, x_2 (allowing the coefficients and integer exponents to depend on r) as a candidate Lyapunov function $V(x)$. State *all* of the necessary conditions which must be imposed on $V(x)$ for it to satisfy Lyapunov's theorem.