MAE 281A Prof. M. Krstic

NONLINEAR SYSTEMS Winter 2011

FINAL EXAM

Take home exam.

Due in Professor Krstic's office (slide under the door if the door is locked) by 5 pm on Thursday, March 17, 2011.

Absolutely no collaboration allowed.

Total points: 70

Problem 1. (25 pts) The "bouncing ball" system (imagine a ball bouncing on a floor or table) is not a conventional dynamical system like those that we studied in this class. The bouncing ball is a "hybrid system," which behaves as a regular continuous system while the ball is in the air and immediately after the impact, but at the time of the impact, it behaves as a discrete system that undergoes a jump in velocity, with a change in sign of the velocity. Between the impacts, the ball behaves like a continuous-time linear system, with a constant input (the force of gravity). At the instant of impact, the ball acts as a discrete-time linear system, where the velocity immediately after the impact is equal to the velocity immediately before the impact times a "coefficient of restitution" (which is smaller than unity) times -1 to account for the reversal in the direction of travel of the ball.

Mathematically, the model of the bouncing ball is given as follows. Denote by x_1 the height of the ball above the floor and by x_2 the velocity of the ball in the upward direction. The state space of the ball is $\{x_1 \geq 0\}$ (meaning that the ball is above or on the floor, with an arbitrary velocity. The dynamic equations of the ball are given as one differential equation (during the non-impact phase of the motion) and one difference equation (during the impact phase):

$$\begin{aligned} \dot{x} &= F(x), \quad \text{while the state is in the set } C = \{x_1 \geq 0\} && (1) \\ x^+ &= G(x), \quad \text{while the state is in the set } D = \{x_1 = 0, x_2 \leq 0\} && (2) \end{aligned}$$

where

$$\begin{aligned} F(x) &= \begin{bmatrix} x_2 \\ -g \end{bmatrix} && (3) \\ G(x) &= \begin{bmatrix} 0 \\ -\gamma x_2 \end{bmatrix}, && (4) \end{aligned}$$

where g is the acceleration of gravity and $\gamma \in [0, 1)$ is the coefficient of restitution. The superscript $^+$ in (2) indicates that the state undergoes a jump from the value x to the value $G(x)$ when $x \in D$.

When $0 < \gamma < 1$, we know that the ball's height gradually decays with each bounce. At each impact, the ball looses a part of its kinetic energy. In the extreme case when $\gamma = 0$, which would be a ball made out of a completely non-elastic material, the ball doesn't bounce even once—it stays on the floor upon the first impact.

Your task in this problem is related to the intuitive knowledge that the ball will eventually stop bouncing. This is the completely obvious part. A slightly less obvious part is that the ball will stop bouncing in finite time—not only in practice, but also according to its mathematical model. The fact that the ball will stop bouncing is related to the asymptotic stability property of the ball at the equilibrium position on the floor. However, the ball is not like a standard asymptotically stable system whose solution decays as the time goes to infinity. The ball's solution decays to zero in finite time, which is for standard systems sometime referred to as "finite-time asymptotic stability." In addition, as a hybrid system, the bouncing ball has the so-called "Zeno property," which is roughly the property that the ball will, in the mathematically idealized model, bounce infinitely many times, with decaying heights of each successive bounce, and with decaying times between bounces, as it comes to rest in finite time. This Zeno property has a long history and it has fascinated people since antiquity (Zeno was a Greek philosopher).

To study stability of the rest state of the bouncing ball, you need to know a basic Lyapunov theory for the so-called "uniform Zeno asymptotic stability" (UZAS) of hybrid systems. Adapted to the bouncing ball problem, the Lyapunov theorem goes as follows.

The equilibrium $x = 0$ of the buncing ball is USAZ if there exists a constant $c > 0$ and a positive definite, radially unbounded, continuously differentiable Lyapunov function $V(x)$ defined on the set $C \cup D = \{x_1 \geq 0\}$ such that

$$\frac{\partial V}{\partial x}(x)F(x) \leq -c\,, \qquad \forall x \in C \setminus 0 \tag{5}$$

$$V(G(x)) \leq V(x)\,, \qquad \forall x \in D \setminus 0\,. \tag{6}$$

Note that the construction of a Lyapunov function for a hybrid system is trickier than for the problems that we have studied in the class because two (rather than one) conditions have to be satisfied by V, in addition to the usual conditions of positive definiteness, radial unboundedness, and continuous differentiability. The Lyapunov function needs to simultaneously satisfy a continuous-time requirement (5) and a discrete-time requirement (6) for a non-increase as the time progresses. Note also that the right-hand side of (5) is unusual, as it is not a negative definite function, but a negative constant. This requirement is related to the finite-time (Zeno) property that the ball possesses.

Task (a) Consider the Lyapunov function candidate

$$V(x) = qx_2 + \sqrt{\frac{1}{2}x_2^2 + gx_1}\,, \tag{7}$$

where

$$q = \frac{1}{\sqrt{2}}\frac{1-\gamma}{1+\gamma} \tag{8}$$

and show that $V(x)$ is positive definite on $C \cup D$.

Task (b) Show that condition (5) is satisfied and clearly state the value of c with which you have satisfied it.

Task (c) Show that condition (6) is satisfied. (This is the trickiest part of the problem.)

Task (d) Noting that, during impact, $V(x(t))$ jumps downward, whereas during the continuous motion $V(x(t))$ decays linearly in time according to the decay function $-ct$, provide an estimate for the largest time in which the ball will come to rest starting from a general initial condition $(x_1(0), x_2(0))$. Then give an estimate for the time in which the ball will come to rest starting from height H with a zero initial speed.

Problem 2. (30 pts) Consider the system

$$\dot{x}_1 = -x_1(1 + x_2) \tag{9}$$

$$\dot{x}_2 = -x_2(1 + x_2)\,. \tag{10}$$

Study the stability of this system at the origin using the Lyapunov function

$$V(x) = \frac{1}{2}\ln\left(1 + x_1^2\right) + x_2 - \ln(1 + x_2)\,. \tag{11}$$

Tasks: Show that this function is positive definite on the set $\{x_2 > -1\}$. Show that $\dot{V}$ is negative definite on the same set. What is the region of attraction of the origin? Using Simulink or any other software, produce a phase portrait of the system in the region $-5 < x_1 < 5, -1 < x_2 < 5$.

Bonus: You can actually find an explicit solution of the system (9), (10). If you can do that, then you don't have to produce the phase portrait numerically. You can simply plot the phase portrait of the analytically obtained solutions via Matlab.

Problem 3. (15 pts) Consider the system

$$\dot{y}_1 = -y_2 \left(y_1 + \sin \omega t\right)^2 \sin \omega t \tag{12}$$

$$\dot{y}_2 = y_2 + 4y_2^2 \left(y_1 + \sin \omega t\right)^2 \cos(2\omega t)\,. \tag{13}$$

<u>Task:</u> Study stability of this system for large ω using averaging theory.