MAE 281A Prof. M. Krstic

NONLINEAR SYSTEMS Winter 2018

FINAL EXAM

Take home.

Collaboration not allowed.

Total points: 70

Problem 1. (15 pts)

Consider the system

$$\dot{x}_1 = -(1 + |x_2|)\mathrm{sgn}(x_1) \tag{1}$$
$$\dot{x}_2 = -(1 + |x_1|)\mathrm{sgn}(x_2) \tag{2}$$

where the signum function is defined as

$$\mathrm{sgn}(z) = \begin{cases} 1 & z > 0 \\ 0 & z = 0 \\ -1 & z < 0 \end{cases} \tag{3}$$

Moreover, we define the derivative of the absolute value to be

$$\frac{d}{dz}|z| := \mathrm{sgn}(z) \tag{4}$$

(We disregard the issues of smoothness/differentiability at the origin for the sake of this question.)

a) Show $x = (x_1, x_2) = 0$ is g.a.s. using the ℓ_1-norm as a Lyapunov function: $V(x) = |x|_1 := |x_1| + |x_2|$.

b) Using the comparison principle in conjunction with V, show that the ℓ_1 norm converges to 0 a finite time $t_c = \ln(1 + |x(0)|_1/2)$.

Problem 2. (15 pts) Given the system

$$\dot{x}_1 = x_2 \tag{5}$$
$$\dot{x}_2 = -x_1 + (1 - x_1^2 - x_2^2)x_2 \tag{6}$$

a) Using linearization, discuss the stability of $x = 0$.

b) Determine the single limit cycle in this system. (Hint: When does $(1 - x_1^2 - x_2^2)x_2 = 0$? What do the system dyanmics dictate on this manifold?)

c) Using a suitable Lyapunov function, and then applying La Salle's Invariance Principle, show that any trajectory (other than one beginning at the origin) converges to the limit cycle.

Problem 3. (13 pts) Study the stability properties of the system

$$\dot{x}_1 = -x_1(1 - x_2(x_1 + \sin\omega t)^2 \sin\omega t) + x_1(x_2 + \sin\omega t)\sin 2\omega t \tag{7}$$
$$\dot{x}_2 = -x_2 + x_2^2(x_1 + \sin\omega t) + x_2(x_1 + \sin\omega t)^2 \sin 2\omega t \tag{8}$$

using averaging for large ω.

Problem 4. (15 pts) Consider the system

$$\dot{x} = \sin^{-1} y \tag{9}$$
$$\varepsilon \dot{y} = \tan(z) \tag{10}$$
$$\delta \dot{z} = -z - \arctan(y + \sin x) \tag{11}$$

where $0 < \delta \ll \varepsilon$ are small parameters. Assume that $y \in [-1, 1]$ so that the system is well-defined. Use singular pertubation to show that the origin is exponentially stable. Please be explicit in your conclusions, and show the intermediate steps you take to come to the result.

Hint: Use the fact that $\delta \ll \varepsilon$ to treat the z-subsystem as a faster system than the (x, y)-subsystem, leading to the reduced system and boundary layer model. Then apply singular pertubation once again, but now for ε. Make your conclusions about each system you derive clearly.

Problem 5. (12 pts) Show that the system

$$\dot{x} = -x^3 + \sin(x)u \tag{12}$$

is input-to-state stable. Use the Lyapunov function $V(x) = \frac{1}{2}x^2$ to show the system has the gain function $\rho(r) = \sqrt[3]{2r}$.