1. Introduction

Linear (TV) model:

$$\dot{x} = A(t)x + B(t)x$$
$$y = C(t)x + D(t)u$$

Nonlinear model:

$$\dot{x} = f(t,x,u)$$
$$y = h(t,x,u)$$

281-B Control Design $u(t,x)$

281-A Analysis without u or $u(t)$

$\dot{x} = f(t,x)$ - non-autonomous (TV)

$\dot{x} = f(x)$ - autonomous (TI)

$\dot{x} = f(x)$

equilibrium : point $x = x^*$ s.t. $f(x) = \emptyset$

LTI case

$$\dot{x} = Ax$$

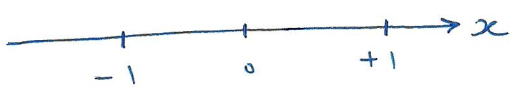
Equilibria : Null (A)

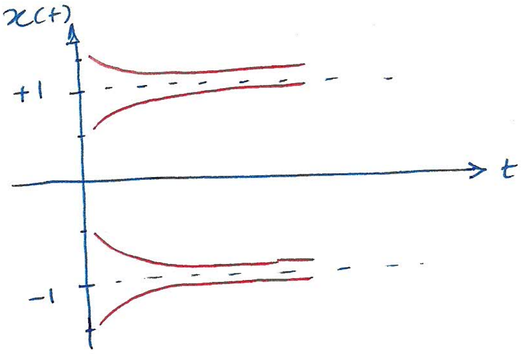
- Multiple Isolated Equilibria

* linear : $\dot{x} = Ax$ Null (A) is eq. manifold

- Continuum of equilibria

* nonlinear : $\dot{x} = x - x^3 = x(x - x^2)$

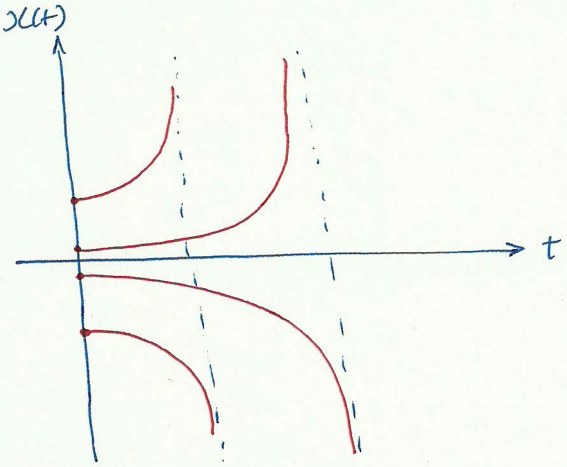




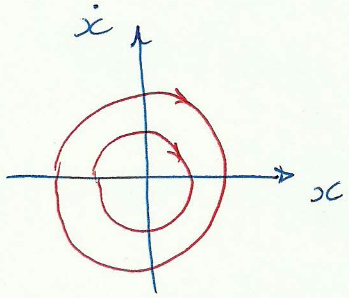
- Finite escape time

* linear unstable : $\dot{x} = x \rightarrow x(t) = x_0 e^t$

* Nonlinear Unstable : $\dot{x} = x^3 \rightarrow x(t) = \dfrac{x_0}{\sqrt{1 - 2x_0^2 t}}$ -> blows up as $t \rightarrow \dfrac{1}{2x_0^2}$



- Limit Cycle



* linear case: $\ddot{x} + x = 0$

$\left\{ \begin{array}{l} \text{mass-spring} \\ \text{LC} \end{array} \right.$

* nonlinear case: $\ddot{x} + \overbrace{(x^2 - 1)\dot{x}} + x = 0$

Van der Pol oscillator

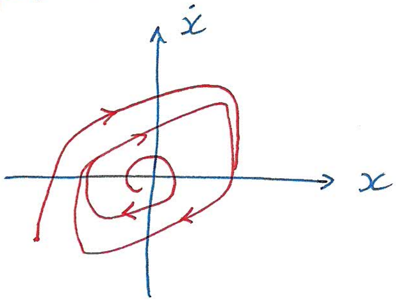
Recap Limit Cycles

* linear (harmonic Oscillator): $\ddot{x} + x = 0$

* nonlinear : $\ddot{x} + (x^2 - 1)\dot{x} + x = 0$ Van der pol Oscillator

pos. damping for large x

neg. damping for small x



isolated periodic orbit

Single amplitude

- human heart
- LC circuit w/ tunnel diode
- Compressor in "surge"

Chaos

"random" yet deterministic ($\dot{x} = f(x)$)

- continuous time : order 3 or higher (order 2 or higher $\dot{x} = f(t, x)$)
- discrete time : order 1

The Lorenz Equation :

$$\begin{cases} y = x \\ z = \dfrac{y^2}{\beta} \\ (\rho - 1) y - y \dfrac{y^2}{\beta} = 0 \end{cases} \rightarrow \quad y = 0, \quad y = \pm\sqrt{\beta(\rho - 1)}$$

Equilibria:

$$(x, y, z) = \begin{cases} (0, 0, 0) \\ (\sqrt{\beta(\rho-1)}, \sqrt{\beta(\rho-1)}, \rho - 1) \\ (-\sqrt{\beta(\rho-1)}, -\sqrt{\beta(\rho-1)}, \rho - 1) \end{cases}$$

2.3. The Lorenz Equations

In [1963] Lorenz, a meteorologist working on the basis of earlier results due to Salzmann [1962], presented an analysis of a coupled set of three quadratic ordinary differential equations representing three modes (one in velocity and two in temperature) of the Oberbeck–Boussinesq equations for fluid convection in a two-dimensional layer heated from below. The papers cited above provide more details. The equations are

$$\left.\begin{aligned} \dot{x} &= \sigma(y - x), \\ \dot{y} &= \rho x - y - xz, \\ \dot{z} &= -\beta z + xy, \end{aligned}\right\} \begin{aligned} &(x, y, z) \in \mathbb{R}^3 \\ &\sigma, \rho, \beta > 0, \end{aligned} \tag{2.3.1}$$

and contain the three parameters σ (the Prandtl number), ρ (the Rayleigh number), and β (an aspect ratio). This three mode truncation accurately reflects the dominant convective properties of the fluid for Rayleigh numbers ρ near 1. In particular, when $\rho = 1$, the pure conductive solution of the partial differential fluid equations, having zero velocity and linear temperature gradient, becomes unstable to a solution containing steady convective rolls or cells. With stress free boundary conditions, the Lorenz equations are a minimal truncation of the fluid equations which embody the essential features of this bifurcation. In Chapter 3 we present more details on this example as an illustration of computations using the center manifold theorem.

Lorenz's [1963] analysis examines the behavior of this equation well outside the parameter domain $\rho \approx 1$, and work of Curry [1978] and Francheschini [1982] demonstrates that, for large ρ, seven and fourteen mode truncations display significantly different behavior.* Thus, as the Rayleigh number increases, higher-order modes become important and predictions made on the basis of the three mode truncation are of doubtful physical relevance, in contrast to the beam problem, in which a single mode truncation does capture significant physical behavior over a wide parameter range. Nonetheless, the equations have become of great interest to mathematicians and physicists in recent years. In the remainder of this section we outline some significant features of the flow of the Lorenz system. For more information, see the papers of Guckenheimer [1976], Guckenheimer and Williams [1979], Williams [1977], Rand [1978], and the book of Sparrow [1982].

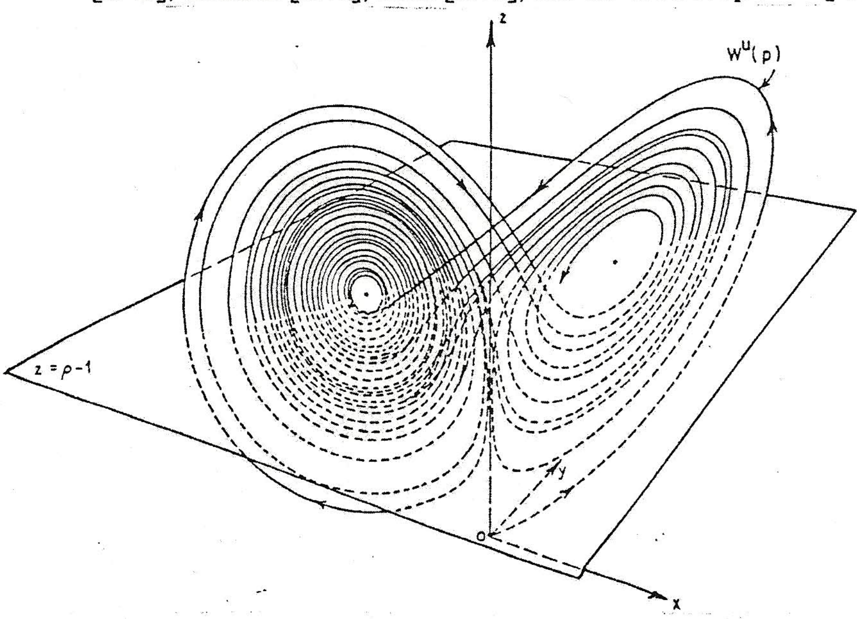


Figure 2.3.2. Numerical solution of the Lorenz equation, $\sigma = 10$, $\beta = \frac{8}{3}$, $\rho = 28$. The initial condition is chosen arbitrarily close to the saddle at $p(0, 0, 0)$, so that the solution approximates $W^u(p)$, defining the boundary of the apparent surface S. After Lanford [1977]. See text for description of Σ, etc.

Qualitative Behavior Near Equilibria

$\dot{x}_t = f(x)$, $f(p) = \phi$

Taylor expansion at $x = p$:

$$\dot{x} = \underbrace{f(p)}_{\phi} + \underbrace{\frac{\partial f(x)}{\partial x}\Big|_{x=p}}_{\text{Const. Square Matrix}} \underbrace{(x-p)}_{y} + H.O.T. \quad , A = \frac{\partial f(x)}{\partial x}\Big|_{x=p} , x - p = y$$

Linearization : $\dot{y} = Ay$

$$\frac{\partial f}{\partial x} = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & \ddots & \\ \frac{\partial f_n}{\partial x_1} & \cdots & \frac{\partial f_n}{\partial x_n} \end{bmatrix}$$

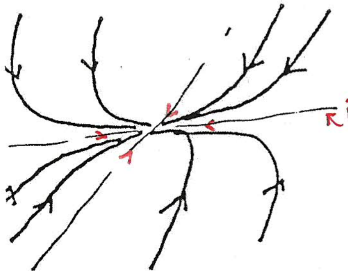
Qualitative Behavior of Second-Order Linear Systems

$\dot{x} = Ax$

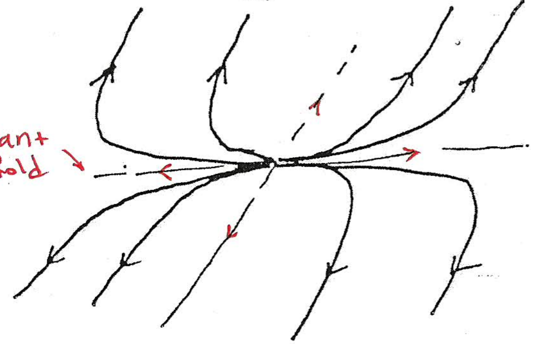
Three possible Jordan forms of A:

$\begin{bmatrix} \lambda_1 & \\ & \lambda_2 \end{bmatrix}$, $\begin{bmatrix} \lambda & k \\ 0 & \lambda \end{bmatrix}$, $\begin{bmatrix} \alpha & -\beta \\ \beta & \alpha \end{bmatrix}$

Case 1: $\lambda_1 \neq \lambda_2 \neq 0$ $\quad \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$ $\quad$ 12



invariant manifold

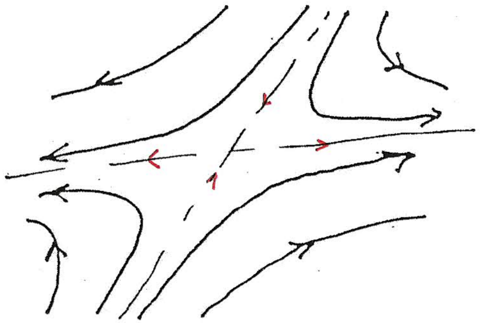


$\lambda_1, \lambda_2 < 0$

stable node

$\lambda_1, \lambda_2 > 0$

unstable node

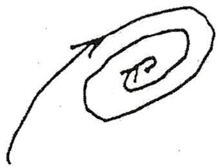


$\lambda_1 > 0, \; \lambda_2 < 0$

saddle point

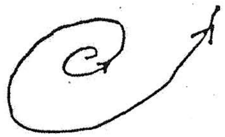
invariant manifolds — eigenspaces

Case 2: $\lambda_{1,2} = \alpha \pm j\beta \quad \begin{bmatrix} \alpha & -\beta \\ \beta & \alpha \end{bmatrix}$



stable focus

$\alpha < 0$



unstable focus

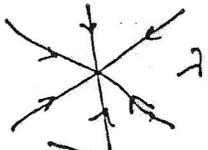
$\alpha > 0$

$\alpha = 0$

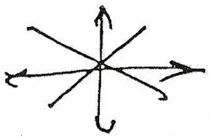
center

Case 3: $\lambda_1 = \lambda_2 = \lambda \neq 0$

$\begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}$



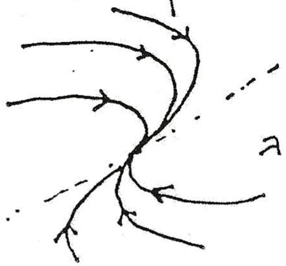
$\lambda < 0$



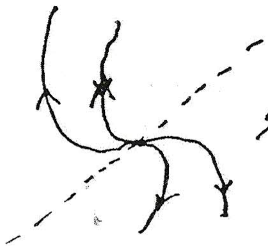
$\lambda >$

$\begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix}$

only one indep. eigenvector



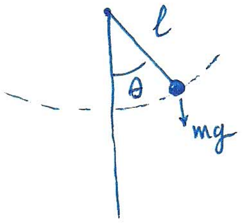
$\lambda < 0$



$\lambda > 0$

Example

Pendulum with friction



$$ml\ddot{\theta} + kl\dot{\theta} + mg\sin\theta = \phi$$

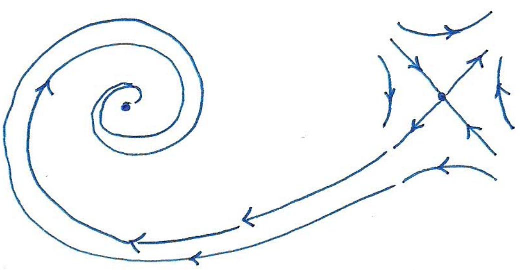
$$\begin{cases} x_1 = \theta \\ x_2 = \dot{\theta} \end{cases} \rightarrow \quad \begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -\frac{g}{l}\sin x_1 - \frac{k}{m}x_2 \end{aligned}$$

Equilibria : $x^{(1)} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ $\quad x^{2} = \begin{bmatrix} \pi \\ \phi \end{bmatrix}$

$$\frac{\partial f}{\partial x} = \begin{bmatrix} \phi & 1 \\ -\frac{g}{l}\cos x_1 & -\frac{k}{m} \end{bmatrix} \qquad A_1 = \begin{bmatrix} 0 & 1 \\ -\frac{g}{l} & -\frac{k}{m} \end{bmatrix} \qquad A_2 = \begin{bmatrix} \phi & 1 \\ \frac{g}{l} & -\frac{k}{m} \end{bmatrix}$$

let $l = g$, $k = \frac{m}{2}$

$\lambda_{1,2} = -.25 \pm j0.97$ — $\lambda_1 = -1.28$, $\lambda_2 = .78$

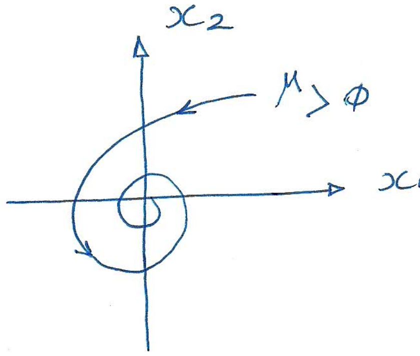


Linearization is a good local approximation if $Re\lambda_i \neq \phi$ (hyperbolic eq.)

Example 2.5

$$\begin{cases} \dot{x}_1 = -x_2 - \mu x_1(x_1^2 + x_2^2) \\ \dot{x}_2 = x_1 - \mu x_2(x_1^2 + x_2^2) \end{cases}$$

Polar Cord; $\begin{cases} x_1 = r\cos\theta \\ x_2 = r\sin\theta \end{cases}$ $\quad \begin{cases} \dot{r} = -\mu r^3 \\ \dot{\theta} = 1 \end{cases}$



$A = \begin{bmatrix} \phi & -1 \\ 1 & \phi \end{bmatrix}$ → Center

Do Not Use Linearization to check the stability when $Re\lambda_i = \phi$