AMES 207 Prof. M. Krstic

NONLINEAR SYSTEMS

Solutions, HW 7

- 8.16

(a)

$$\begin{aligned}
\ddot{y} &= (t^*)^2 \frac{d^2 y}{dt^2} = \frac{m}{ku^*}\frac{d^2 u}{dt^2} \\
&= \frac{1}{ku^*}\left\{-ku + \lambda\left[1-\alpha\left(\frac{du}{dt}\right)^2\right]\frac{du}{dt}\right\} \\
&= -y + \frac{\lambda}{\lambda^*}\left[1-\alpha\left(\frac{u^*}{t^*}\right)^2 \dot{y}^2\right]\dot{y} \\
&= -y + \epsilon\left(1 - \tfrac{1}{3}\dot{y}^2\right)\dot{y}
\end{aligned}$$

(b)

$$f_{av}(r) = \frac{1}{2\pi}\int_0^{2\pi}\left(r\cos\phi - \tfrac{1}{3}r^3\cos^3\phi\right)\cos\phi\, d\phi = \tfrac{1}{2}r - \tfrac{1}{8}r^3$$

Proceed similar to Example 8.11.

- 8.17

(a)

$$\begin{aligned}
\frac{dx_1}{d\tau} &= \epsilon x_2 \\
\frac{dx_2}{d\tau} &= \epsilon\left[-\frac{k}{m}(x_1 + a^2 x_1^3) - \frac{c}{m}x_2 + \frac{A}{m}\cos\tau\right]
\end{aligned}$$

(b) The averaged system is given by

$$\begin{aligned}
\frac{dx_{1av}}{d\tau} &= \epsilon x_{2av} \\
\frac{dx_{2av}}{d\tau} &= \epsilon\left[-\frac{k}{m}(x_{1av} + a^2 x_{1av}^3) - \frac{c}{m}x_{2av}\right]
\end{aligned}$$

The average system has a unique equilibrium point at the origin. Linearization at the origin yields a Hurwitz matrix. Thus, for sufficiently small ϵ (equivalently, sufficiently large ω), the system has a unique exponentially stable periodic solution in an $O(\epsilon)$ (equivalently, $O(1/\omega)$) neighborhood of the origin. The frequency of oscillation is close to ω.

• **9.7** Setting $\epsilon = 0$ yields $h(x) = x^2 + 1$. The reduced model is

$$\dot{x} = 2x^2 + 1, \; x(0) = \xi \;\Rightarrow\; \bar{x}(t) = \frac{1}{\sqrt{2}} \tan\left(t\sqrt{2} + \tan^{-1}\left(\xi\sqrt{2}\right)\right)$$

The boundary-layer model is

$$\frac{dy}{d\tau} = -y, \; y(0) = \eta - (1 + \xi^2) \;\Rightarrow\; y(\tau) = [\eta - (1 + \xi^2)]e^{-\tau}$$

Thus

$$\begin{aligned} x(t, \epsilon) &= \frac{1}{\sqrt{2}} \tan\left(t\sqrt{2} + \tan^{-1}\left(\xi\sqrt{2}\right)\right) + O(\epsilon) \\ z(t, \epsilon) &= 1 + \tfrac{1}{2}\left[\tan\left(t\sqrt{2} + \tan^{-1}\left(\xi\sqrt{2}\right)\right)\right]^2 + [\eta - (1 + \xi^2)]e^{-t/\epsilon} + O(\epsilon) \end{aligned}$$

• **9.11** The manifold equation is

$$-H - x^{4/3} + \tfrac{4}{3}\epsilon x^{16/3} - \epsilon \frac{\partial H}{\partial x} x H^3 = 0$$

It can be easily checked that $H = -x^{4/3}$ satisfies this equation.

• **9.12** The manifold equation is

$$-(H - \sin^2 x)(H - e^{\alpha x})(H - 2e^{2\alpha x}) + \epsilon \frac{\partial H}{\partial x} x H = 0$$

At $\epsilon = 0$ we have three equilibrium manifolds

$$H_1 = \sin^2 x, \quad H_2 = e^{\alpha x}, \quad H_3 = 2e^{2\alpha x}$$

The Jacobian $\partial g / \partial z$ is negative on H_1 and H_3 and positive on H_2. Hence, H_1 and H_3 are attractive.

• **9.29** Setting $\epsilon = 0$ results in $z = \sin t$. The reduced model is $\dot{x} = -x$ and the boundary-layer model is $dy/d\tau = -y$. Clearly all the assumptions of Theorem 9.4 are satisfied. Hence, the $O(\epsilon)$ approximation

$$\begin{aligned} x(t, \epsilon) &= e^{-t} x(0) + O(\epsilon) \\ z(t, \epsilon) &= \sin t + e^{-t} z(0) + O(\epsilon) \end{aligned}$$

holds uniformly in t for all $t \geq 0$.