Solutions, HW 5

- 3.41

(a)

$$x_2 = 1 = \dot{x}_1$$
$$2x_1x_2 + 3t + 2 - 3x_1 - 2(t+1)x_2 = 2t + 3t + 2 - 3t - 2(t+1) = 0 = \dot{x}_2$$

Thus, $x(t) = \begin{bmatrix} t \\ 1 \end{bmatrix}$ is a solution.

(b) Recall from the discussion at the beginning of Section 3.4 that to show asymptotic stability of a solution we shift it to the origin and then show asymptotic stability of the origin. Let $z_1 = x_1 - t$ and $z_2 = x_2 - 1$. Then

$$\begin{aligned} \dot{z}_1 &= z_2 \\ \dot{z}_2 &= 2z_1z_2 - z_1 - 2z_2 \end{aligned}$$

We need to show that the origin $z = 0$ is uniformly asymptotically stable.

$$\frac{\partial f}{\partial z} = \begin{bmatrix} 0 & 1 \\ -1 + 2z_2 & -2 + 2z_1 \end{bmatrix}, \quad A = \left.\frac{\partial f}{\partial z}\right|_{z=0} = \begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix}$$

The matrix A is Hurwitz; hence, the origin is uniformly asymptotically stable.

- **5.5** The closed-loop system is given by

$$\dot{x} = (A - BB^TP)x + Bg(t,x)$$

Rewrite the Riccati equation as

$$P(A - BB^TP) + (A - BB^TP)^TP + Q + PBB^TP + 2\alpha P = 0$$

Consider $V(x) = x^TPx$ as a Lyapunov function candidate.

$$\begin{aligned} \dot{V}(t,x) &= x^T[P(A - BB^TP) + (A - BB^TP)^TP]x + 2x^TPBg(t,x) \\ &= -x^T[Q + PBB^TP + 2\alpha P]x + 2x^TPBg(t,x) \\ &\leq -k^2\|x\|_2^2 - \|w\|_2^2 - 2\alpha\lambda_{min}(P)\|x\|_2^2 + 2k\|w\|_2\|x\|_2, \quad \text{where } w = B^TPx \\ &= -[k\|x\|_2 - \|w\|_2]^2 - 2\alpha\lambda_{min}(P)\|x\|_2^2 \leq -2\alpha\lambda_{min}(P)\|x\|_2^2 \end{aligned}$$

Hence, the origin is globally exponentially stable.

• 5.18

(a) Let $b = 0$. Try $V(x) = \frac{1}{2}x_1^2 + \frac{1}{2}x_2^2$.

$$\dot{V}(x) = -x_1^2 + x_1x_2(x_1 + a) - x_1x_2(x_1 + a) = -x_1^2$$

$$\dot{V}(x) = 0 \Rightarrow x_1(t) \equiv 0 \Rightarrow ax_2(t) \equiv 0 \Rightarrow x_2(t) \equiv 0$$

Thus, the origin is globally asymptotically stable. To investigate exponential stability, linearize at $x = 0$.

$$A = \left.\frac{\partial f}{\partial x}\right|_{x=0} = \left[\begin{array}{cc} -1 + x_2 & x_1 + a \\ -2x_1 - a & 0 \end{array}\right]_{x=0} = \left[\begin{array}{cc} -1 & a \\ -a & 0 \end{array}\right]$$

The characteristic equation of A is $\lambda^2 + \lambda + a^2 = 0$. Hence, A is Hurwitz and the origin is exponentially stable.

(b) Let $b > 0$. The linearization at the origin is given by

$$A = \left.\frac{\partial f}{\partial x}\right|_{x=0} = \left[\begin{array}{cc} -1 & a \\ -a & b \end{array}\right]$$

The characteristic equation of A is $\lambda^2 + (1 - b)\lambda + a^2 - b = 0$. Hence, A is Hurwitz if $b < \min\{1, a^2\}$.

(c) For $b > 0$, the equilibrium points are

$$(0, 0), \quad \left(-a + \sqrt{b}, \frac{-a + \sqrt{b}}{\sqrt{b}}\right), \quad \left(-a - \sqrt{b}, \frac{a + \sqrt{b}}{\sqrt{b}}\right)$$

Since the system has multiple equilibria, the origin is not globally asymptotically stable.

• 5.21 (1) Let $V(x) = \frac{1}{2}x^2$.

$$\dot{V} = -x^4 + x^4u$$

For $|u| \leq r_u < 1$, we have

$$\dot{V} \leq -(1 - r_u)x^4, \quad \forall\, x$$

By Theorem 5.2, the system is locally input-to-state stable. It is not input-to-state stable since with $u(t) \equiv c > 1$ and $x(0) > 0$, $x(t) \to \infty$ as $t \to \infty$.

(2) Let $V(x) = \frac{1}{2}x^2$.

$$\dot{V} = -x^4 + ux^4 - x^6 \leq -x^4, \quad \forall\, |x| \geq \sqrt{u}$$

By Theorem 5.2, the system is input-to-state stable.

(3) Let $V(x) = \frac{1}{2}x^2$. For $|u| < r_u$ and $|x| < r < 1$, we have

$$\dot{V} = -x^2 + x^3u \leq -(1 - \theta)x^2 - \theta x^2 + r^2|x|\,|u| \leq -(1 - \theta)x^2, \quad \forall\, |x| \geq \frac{r^2|u|}{\theta}$$

The preceding inequality is valid provided $rr_u < \theta \leq 1$. By Theorem 5.2, the system is locally input-to-state stable. It is not input-to-state stable since with $u(t) \equiv 1$ and $x(0) > 0$, $x(t) \to \infty$ as $t \to \infty$.

• 5.22

(1) The x_1-system is ISS w.r.t. x_2, and the x_2-system is ISS w.r.t. u. Then by [KKK, Lemma C.4], the (x_1, x_2)-system is ISS. To see that the x_1-system is ISS, note that

$$\frac{\dot{\overbrace{x_1^2}}}{2} \le -\frac{1}{2}x_1^2 + \frac{1}{2}x_2^4,$$

which implies that

$$|x_1(t)| \le |x(0)|\, e^{-t/2} + \left(\sup_{[0,t]} |x_2(\tau)|\right)^2.$$

(4) Take $V = \frac{1}{2}x_1^2 + \frac{1}{2}x_2^2$. Then

$$\begin{aligned}
\dot V &= -x_1^2 - x_2^4 + x_1 u_1 + x_2 u_2 \\
&\le -x_1^2 - x_2^4 + \tfrac{1}{2}x_1^2 + \tfrac{1}{2}u_1^2 + \tfrac{1}{4}x_2^4 + \tfrac{3}{4}u_2^{4/3} \quad \text{(Young)} \\
&= -\tfrac{1}{2}\left(x_1^2 + \tfrac{3}{2}x_2^4\right) + \tfrac{1}{2}\left(u_1^2 + \tfrac{3}{2}u_2^{4/3}\right) \\
&= -\tfrac{1}{2}\left(|x_1|^2 + \tfrac{3}{2}|x_2|^4\right) + \tfrac{1}{2}\left(|u_1|^2 + \tfrac{3}{2}\left(\sqrt[3]{|u_2|}\right)^4\right) \\
&\triangleq -Q(|x_1|, |x_2|) + Q(|u_1|, \sqrt[3]{|u_2|})
\end{aligned}$$

Note that $Q(y_1, y_2)$ is pdf, smooth on $\mathbb{R}^2$, and radially unbdd. Then there exist class $\mathcal{K}_\infty$ functions q_1 and q_2 s.t.

$$q_1(|y|) \le Q(y_1, y_2) \le q_2(|y|)$$

Hence

$$\begin{aligned}
\dot V &\le -q_1(|x|) + q_2\left(\sqrt{u_1^2 + u_2^{2/3}}\right) \\
&\le -q_2(|x|) + q_3(|u|),
\end{aligned}$$

where $q_3 \in \mathcal{K}_\infty$. Then by [KKK, Thm C.3], the system is ISS w.r.t. u.