# AMES 207

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## NONLINEAR SYSTEMS

## Solutions, HW 4

• 3.3

(1)

$$\begin{aligned} \dot{x}_1 &= -x_1 + x_2^2 \\ \dot{x}_2 &= -x_2 \end{aligned}$$

Try $V(x) = \frac{1}{2}(x_1^2 + x_2^2)$.

$$\dot{V}(x) = x_1(-x_1 + x_2^2) - x_2^2 = -x_1^2 - x_2^2 + x_1 x_2^2$$

In the neighborhood of the origin, the term $-(x_1^2 + x_2^2)$ dominates. Hence, the origin is asymptotically stable. Moreover

$$x_2(t) = e^{-t} x_{20}$$

$$\begin{aligned} \Rightarrow x_1(t) &= e^{-t} x_{10} + \int_0^t e^{-(t-s)} e^{-2s}\, ds\, x_{20}^2 \\ &= e^{-t} x_{10} + \left[e^{-t} - e^{-2t}\right] x_{20}^2 \end{aligned}$$

For all $x_0$, $x(t) \to 0$ as $t \to \infty$, which implies that the origin is globally asymptotically stable.

(2)

$$\begin{aligned} \dot{x}_1 &= (x_1 - x_2)(x_1^2 + x_2^2 - 1) \\ \dot{x}_2 &= (x_1 + x_2)(x_1^2 + x_2^2 - 1) \end{aligned}$$

$$\begin{aligned} V(x) &= a x_1^2 + b x_2^2, \quad a > 0,\ b > 0 \\ \dot{V}(x) &= 2a x_1(x_1 - x_2)(x_1^2 + x_2^2 - 1) + 2b x_2(x_1 + x_2)(x_1^2 + x_2^2 - 1) \\ &= 2[a x_1(x_1 - x_2) + b x_2(x_1 + x_2)](x_1^2 + x_2^2 - 1) \end{aligned}$$

Let $a = b$.

$$\dot{V}(x) = -2a(x_1^2 + x_2^2)[1 - (x_1^2 + x_2^2)]$$

For $x_1^2 + x_2^2 < 1$, $\dot{V}(x)$ is negative definite. Hence, the origin is asymptotically stable. It is not globally asymptotically stable since there other equilibrium points on the unit circle.

• 3.12 Take $V(x) = -\frac{1}{6}x_1^6 + \frac{1}{4}x_2^4$.

$$\dot{V}(x) = -x_1^5 \dot{x}_1 + x_2^3 \dot{x}_2 = x_1^6 + x_2^6 - x_1^5 x_2^6 + x_2^3 x_1^6$$

Near the origin

$$\left|-x_1^5 x_2^6 + x_2^3 x_1^6\right| \le k\left(x_1^6 + x_2^6\right)$$

for some $k > 0$. Hence

$$\dot{V}(x) \ge (1 - k)\left(x_1^6 + x_2^6\right)$$

which shows that $\dot{V}(x)$ is positive definite. Application of Chetaev's theorem shows that the origin is unstable.

• **3.17**

(a)

$$\begin{aligned} 0 &= -x_1 + g(x_3) \\ 0 &= -g(x_3) \\ 0 &= -ax_1 + bx_2 - cx_3 \end{aligned}$$

From the properties of $g(\cdot)$ we know that $g(x_3) = 0$ has an isolated root $x_3 = 0$. Substituting $x_3 = 0$ in the foregoing equations we obtain $x_1 = x_2 = 0$. Hence, the origin is an isolated equilibrium point.

(b)

$$\begin{aligned} V(x) &= \frac{a}{2}x_1^2 + \frac{b}{2}x_2^2 + \int_0^{x_3} g(y)\ dy \\ \dot{V}(x) &= ax_1[-x_1 + g(x_3)] - bx_2 g(x_3) + g(x_3)[-ax_1 + bx_2 - cg(x_3)] \\ &= -ax_1^2 - cg^2(x_3) \ \leq \ 0 \end{aligned}$$

$$\dot{V}(x) = 0 \Rightarrow x_1(t) \equiv 0 \text{ and } x_3(t) \equiv 0 \Rightarrow \dot{x}_3(t) \equiv 0$$

From the third state equation we see that $x_2(t) \equiv 0$. Hence, by LaSalle's theorem (Corollary 3.1), the origin is asymptotically stable.

(c) To conclude that the origin is globally asymptotically stable, we need to know that $V(x)$ is radially unbounded. But this is not guaranteed since

$$yg(y) > 0,\ \forall\ |y| \neq 0 \not\Rightarrow \int_0^x g(y)\ dy \to \infty \text{ as } |x| \to \infty.$$

Consider, for example, $g(y) = (1 - e^{-|y|})e^{-|y|}\text{sgn}(y)$. For $x > 0$, we have

$$\int_0^x (1 - e^{-y})e^{-y}\ dy = 1 - e^{-x} - \tfrac{1}{2}(1 - e^{-2x}) \to \tfrac{1}{2} \text{ as } x \to \infty$$

Thus we cannot conclude that the origin is globally asymptotically stable.